# Fragments of the theory of the enumeration degrees

**Mariya I. Soskova**
University of Wisconsin–Madison

Online Logic Seminar 2020, April 23
*Joint work with S. Lempp and T. Slaman*

Supported by the NSF Grant No. DMS-1762648 and FNI-SU Grant No. 80-10-128/16.04.2020

# Degree structures

Reducibilities between sets of natural numbers are used to compare the relative effective content between sets of natural numbers:

## Definition

We use $A \leqslant B$ to denote that a set $A$ is reducible to a set $B$:

1. $A \leqslant_m B$ means that there is a computable function $f$ such that $x \in A$ if and only if $f(x) \in B$.
2. $A \leqslant_T B$ means that one can compute the members of $A$ using an oracle Turing machine with oracle $B$.
3. $A \leqslant_e B$ means that one can (effectively) enumerate the members of $A$ given any enumeration of the members of $B$.
4. $A \leqslant_a B$ if $A$ can be defined arithmetically with parameter $B$.

We say that $A \equiv B$ if $A \leqslant B$ and $B \leqslant A$.

The equivalence class of $A$ is the *degree* $\deg(A)$ of $A$.

The degree structure $\mathcal{D}$ is the induced partial order on degrees.

# The theory of a degree structure

Given a degree structure $\mathcal{D}$ we ask the following natural questions:

## Question

- Is the theory of the structure in the language of partial orders decidable?
- How complicated is the theory?
- How many quantifiers does it take to break decidability?

| Degree structure | Complexity of $Th(\mathcal{D})$ | $\exists\forall\exists$-$Th(\mathcal{D})$ | $\forall\exists$-$Th(\mathcal{D})$ |
|---|---|---|---|
| $\mathcal{D}_T$ | Simpson 77 | Lerman-Schmerl 83 | Shore 78; Lerman 83 |
| $\mathcal{D}_T(\leqslant \mathbf{0}')$ | Shore 81 | Lerman-Schmerl 83 | Lerman-Shore 88 |
| $\mathcal{R}$ | Slaman-Harrington 80s | Lempp-Nies-Slaman 98 | Open |
| $\mathcal{D}_e$ | Slaman-Woodin 97 | Open | Open |
| $\mathcal{D}_e(\leqslant \mathbf{0}')$ | Ganchev-Soskova 12 | Kent 06 | Open |

# Translating the questions in terms of structure

To understand what existential sentences are true $\mathcal{D}$ we need to understand what finite partial orders can be embedded into $\mathcal{D}$;

### Theorem (Sacks 1964)

Every countable partial order can be embedded densely in the c.e. degrees.

The existential theory of $\mathcal{R}$, $\mathcal{D}_T(\leqslant \mathbf{0}')$, $\mathcal{D}_T$, $\mathcal{D}_e(\leqslant \mathbf{0}')$, $\mathcal{D}_e$ is decidable because all of these structures contain an isomorphic copy of $\mathcal{R}$.

# Extension of embeddings

At the next level of complexity is the *extension of embeddings problem*:

**Problem**

We are given a finite partial order $P$ and a finite partial order $Q \supseteq P$. Does every embedding of $P$ extend to an embedding of $Q$?

To understand what $\forall\exists$-sentences are true in $\mathcal{D}$ we need to solve a slightly more complicated problem:

**Problem**

We are given a finite partial order $P$ and finite partial orders $Q_0, \ldots Q_n \supseteq P$. Does every embedding of $P$ extend to an embedding of one of the $Q_i$?

# The Turing degrees and initial segment embeddings

**Theorem (Lerman 71)**

Every finite lattice can be embedded into $\mathcal{D}_T$ as an initial segment.

- Suppose that $P$ is a finite partial order and $Q \supseteq P$ is a finite partial order extending $P$.
- We can extend $P$ to a lattice by adding extra points for joins when necessary.
- The initial segment embedding of the lattice $P$ can be extended to an embedding of $Q$ only if new elements in $Q \setminus P$ are compatible with joins in $P$:
    1. If $q \in Q \setminus P$ is bounded by some element in $P$ then $q$ is one of the added joins.
    2. If $x \in Q \setminus P$ and $u, v \in P$ and $x \geqslant u, v$ then $x \geqslant u \vee v$.

**Theorem (Shore 78; Lerman 83)**

That is the only obstacle.

# A characterization

Let $U$ be an upper semilattice.

## Definition

We say that $U$ *exhibits end-extensions* if for every pair of a finite lattice $P$ and partial order $Q \supseteq P$ such that if $x \in Q \setminus P$ then $x$ is never below any element of $P$ and $x$ respects least upper bounds, every embedding of $P$ into $U$ extends to an embedding of $Q$ into $U$.

## Theorem (Lempp, Slaman, Soskova)

Let $\varphi$ be a $\Pi_2$-sentence in the language of partial orders. The sentence $\varphi$ is true in $\mathcal{D}_T$ if and only if $\varphi$ is true in every upper semilattice $U$ with least element that exhibits end-extensions.

# A characterization

## Theorem (Lempp, Slaman, Soskova)

Let $\varphi$ be a $\Pi_2$-sentence in the language of partial orders. The sentence $\varphi$ is true in $\mathcal{D}_T$ if and only if $\varphi$ is true in every upper semilattice $U$ with least element that exhibits end-extensions.

## Proof.

If $\varphi$ is true in every upper semilattice $U$ with least element that exhibits end-extensions then it is true in $\mathcal{D}_T$ because $\mathcal{D}_T$ is one of these.

Suppose that $\varphi$, given by $P \subseteq Q_1, \ldots Q_n$, is not true in some fixed upper semilattice $U$. So there is an embedding of $P$ into $U$ that does not extend to an embedding of any $Q_i$. Let $P^*$ be the upper semilattice generated by this embedding, taking least upper bounds as in $U$ and adding the least element.

Embed $P^*$ into $\mathcal{D}_T$ as an initial segment. If this embedding of $P^*$ extended to an embedding of $Q_i$ for some $i$ then by the fact that $U$ exhibits end-extensions we can argue that we would be able to pull it back to an embedding of $Q_i$ in $U$ extending the one we started with. □

# The theory of a degree structure

Lets take a look at the table again:

## Question

- Both $\mathcal{R}$ and $\mathcal{D}_e(\leqslant \mathbf{0}')$ are dense structures.
- But what is the case of $\mathcal{D}_e$?

| Degree structure | Complexity of $Th(\mathcal{D})$ | $\exists\forall\exists$-$Th(\mathcal{D})$ | $\forall\exists$-$Th(\mathcal{D})$ |
|---|---|---|---|
| $\mathcal{D}_T$ | Simpson 77 | Lerman-Schmerl 83 | Shore 78; Lerman 83 |
| $\mathcal{D}_T(\leqslant \mathbf{0})$ | Shore 81 | Lerman-Schmerl 83 | Lerman-Shore 88 |
| $\mathcal{R}$ | Slaman-Harrington 80s | Lempp-Nies-Slaman 98 | Open |
| $\mathcal{D}_e$ | Slaman-Woodin 97 | Open | Open |
| $\mathcal{D}_e(\leqslant \mathbf{0}')$ | Ganchev-Soskova 12 | Kent 06 | Open |

# The enumeration degrees

### Theorem (Gutteridge 71)

The enumeration degrees are downwards dense.

A degree $\mathbf{b}$ is a *minimal cover* of a degree $\mathbf{a}$ if $\mathbf{a} < \mathbf{b}$ and the interval $(\mathbf{a}, \mathbf{b})$ is empty.

### Theorem (Slaman, Calhoun 96)

There are degrees $\mathbf{a} < \mathbf{b}$ such that $\mathbf{b}$ is a minimal cover of $\mathbf{a}$.

A degree $\mathbf{b}$ is a *strong minimal cover* of a degree $\mathbf{a}$ if $\mathbf{a} < \mathbf{b}$ and for every degree $\mathbf{x} < \mathbf{b}$ we have that $\mathbf{x} \leqslant \mathbf{a}$.

### Theorem (Kent, Lewis-Pye, Sorbi 12)

There are degrees $\mathbf{a}$ and $\mathbf{b}$ such that $\mathbf{b}$ is a strong minimal cover of $\mathbf{a}$

# The simplest lattice

Consider the lattice $\mathcal{P} = \{a < b\}$. What properties should possible extensions $Q_0, Q_1 \dots Q_n$ have so that every embedding of $\mathcal{P}$ extends to $Q_i$ for some $i$:


1. We can embed $\mathcal{P}$ as degrees $\mathbf{a} < \mathbf{b}$ such that $\mathbf{b}$ is a strong minimal cover of $\mathbf{a}$, blocking extensions to $Q_i$ with new $x$ in the interval $[a, b]$.
2. We can embed $\mathcal{P}$ as degrees $\mathbf{0}_e < \mathbf{b}$, blocking extensions to $Q_i$ with new $x < a$.

**Theorem (Slaman, Sorbi 14)**

Every countable partial order can be embedded below any nonzero enumeration degree.

So these are the only obstacles.

Consider $P = \{a < b\}$, $Q_0 = \{a < x < b\}$ and $Q_1 = \{x < a < b\}$.

# A wild conjecture

Let $U$ be an upper semilattice.

**Definition**

$U$ *exhibits strong downward density* if every finite partial order can be embedded below any nonzero element of $U$.

**Conjecture (Lempp, Slaman, Soskova)**

A $\Pi_2$ sentence $\varphi$ is true in $\mathcal{D}_e$ if and only if $\varphi$ is true in every upper semilattice $U$ with least element that exhibits end-extensions and strong downward density.

- This would imply a decision procedure for the two quantifier theory of $\mathcal{D}_e$.
- This would imply that we can extend the existence of strong minimal covers significantly:

# Strong interval embeddings

**Definition**

Let $\mathcal{L}$ be a lattice. We say that $\mathcal{L}$ *strongly embeds as an interval* in $\mathcal{D}_e$ if there are degrees $\mathbf{a} < \mathbf{b}$ and a bijection $f : \mathcal{L} \to [\mathbf{a}, \mathbf{b}]$ such that for every $\mathbf{x} \leqslant \mathbf{b}$ we have that $\mathbf{x} \in [\mathbf{a}, \mathbf{b}]$ or else $\mathbf{x} < \mathbf{a}$.

- A strong minimal cover induces a strong interval embedding of the 2-element lattice.
- The conjecture implies that every finite lattice has a strong interval embedding in $\mathcal{D}_e$.

# A small victory

**Theorem (Lempp, Slaman, Soskova)**

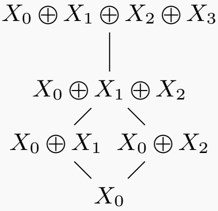
Every finite distributive lattice has a strong interval embedding.

**Proof.**

Fix a finite distributive lattice $\mathcal{L}$ with join irreducible elements $a_0, a_1, \dots a_n$. Every element of the lattice has a unique representation as $a_F = \bigvee_{i \in F} a_i$, where $F$ is downwards closed.

We build $\Pi^0_2$ sets $X_0, \dots X_n$ so that $A_F = \bigoplus_{j \in F} X_j$ represents $a_F$.

- $\mathcal{T}^i_e$: $X_i \neq \Phi_e(A_{F_i})$, where $F_i = \{j \mid a_i \nleqslant_{\mathcal{L}} a_j\}$;
- $\mathcal{M}^{G,F}_e$: Fix $F \subseteq G$ such that $a_G$ is minimal above $a_F$. Note that $G = F \cup \{i\}$ for some fixed number $i$. Denote by $G \smallsetminus F = \{j \in G \mid a_j \leqslant_{\mathcal{L}} a_i\}$. We asks that there is a reduction $\Gamma$ such that $\Psi_e(A_G) = \Gamma(A_F)$ or else $A_{G \smallsetminus F} \leqslant_e \Psi_e(A_G)$.



# The Nies Transfer Lemma

## Definition

Let $\mathcal{C}$ be a class of structures in a finite relational language $L = \{R_1, \ldots, R_n\}$. We say that $\mathcal{C}$ is $\Sigma_k$*-elementarily definable with parameters* in $\mathcal{D}_e$ if there are $\Sigma_k$-formulas $\varphi_U$, $\varphi_{R_i}$, and $\varphi_{\neg R_i}$ for $i \leqslant n$ such that for every $C \in \mathcal{C}$, there are parameters $\vec{\mathbf{p}} \in \mathcal{D}_e$ that make the structure with universe $U = \{\mathbf{x} | \ \mathcal{D}_e \models \varphi_U(\mathbf{x}, \vec{\mathbf{p}})\}$ and relations defined by $\varphi_{R_i}/\varphi_{\neg R_i}$ isomorphic to $C$.

## Lemma (Nies 1996)

Let $r \geqslant 2$ and $k \geqslant 1$. If a class of models $\mathcal{C}$ is $\Sigma_k$-elementarily definable in $\mathcal{D}_e$ with parameters and the $\Pi_{r+1}$-theory of $\mathcal{C}$ is (hereditarily) undecidable, then the $\Pi_{r+k}$-theory of $\mathcal{D}_e$ is (hereditarily) undecidable.

# The three quantifier theory of $\mathcal{D}_e$

### Corollary (Lempp, Slaman, Soskova)

The class of finite distributive lattices is $\Sigma^0_1$-elementary definable with parameters in $\mathcal{D}_e$.

### Theorem (Nies)

The $\Pi_3$ theory of the class of finite distributive lattices is (hereditarily) undecidable.

Applying Nies' Transfer Lemma we get:

### Theorem

The $\forall\exists\forall$-theory of $\mathcal{D}_e$ is undecidable.

# The extension of embeddings problem

## Theorem (Lempp, Slaman, Soskova )

The extension of embeddings problem in $\mathcal{D}_e$ is decidable.

*Proof sketch:*

- Fix partial orders $P \subseteq Q$.
- If $q \in Q \setminus P$ is a point that violates the conditions of the usual algorithm (the one for $\mathcal{D}_T$) then we build a specific embedding that blocks $q$.
- We extend $P$ to $P^*$ by carefully adding points to make $B(A(q)) = \{p \in P^* \mid (\forall s \in P^*)(q \leqslant s \to p \leqslant s)\}$ a distributive lattice and embed that strongly.
- We use generic extensions for the rest of $P$ to make $\bigwedge A(q) = \bigvee B(A(q))$, where $A(q) = \{p \in P^* \mid q < p\}$.
- This leaves $\bigvee B(A(q))$ as the only possible position for $q$.

# The common fragment of the theories of $\mathcal{D}_T$ and $\mathcal{D}_e$

Note that the theories of $\mathcal{D}_e$ and $\mathcal{D}_T$ differ at a $\Sigma_2$ sentence $\varphi$:

$$(\exists \mathbf{a})[\mathbf{a} \neq \mathbf{0} \wedge \forall \mathbf{x}[\mathbf{x} < \mathbf{a} \rightarrow \mathbf{x} = \mathbf{0}]]$$

## Theorem

Let $E$ denote the set of $\Pi_2$-sentences in the language of a partial orders that formalize an instance of the extension of embeddings problem. Then $E \cap Th(\mathcal{D}_e) = E \cap Th(\mathcal{D}_T)$.

*Proof sketch:*

- One direction uses our characterization of the two quantifier theory of $\mathcal{D}_T$ and the fact that $\mathcal{D}_e$ is an upper semilattice that exhibits end extensions.
- The reverse direction follows from the proof of the extension of embedding theorem.

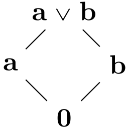
# An unexpected defeat

Recall our conjecture:

**Conjecture (Lempp, Slaman, Soskova)**

A $\Pi_2$ sentence $\varphi$ is true in $\mathcal{D}_e$ if and only if $\varphi$ is true in every upper semilattice $U$ with least element that exhibits end-extensions and strong downward density.

It implies that there are degrees $\mathbf{a}$ and $\mathbf{b}$ such that: $\mathbf{a}$ and $\mathbf{b}$ are a minimal pair and if $\mathbf{x} < \mathbf{a} \vee \mathbf{b}$ then $\mathbf{x} \leqslant \mathbf{a}$ or $\mathbf{x} \leqslant \mathbf{b}$.




This is an instance of a *super minimal pair*: a minimal pair $\{\mathbf{a}, \mathbf{b}\}$ such that every nonzero degree $\mathbf{x} \leqslant \mathbf{a}$ joins $\mathbf{b}$ above $\mathbf{a}$ and every nonzero degree $\mathbf{x} \leqslant \mathbf{b}$ joins $\mathbf{a}$ above $\mathbf{b}$.

# An unexpected defeat

**Theorem (Jacobsen-Grocott, Soskova)**

If $\mathbf{a}$ and $\mathbf{b}$ are enumeration degrees such that every degree $\mathbf{x} \leqslant \mathbf{a} \vee \mathbf{b}$ is bounded by $\mathbf{a}$ or bounded by $\mathbf{b}$, then $\{\mathbf{a}, \mathbf{b}\}$ is not a minimal pair.

**Proof.**

The proof is nonuniform: in the case when neither $\mathbf{a}$ nor $\mathbf{b}$ is $\Delta^0_2$ it uses the special form of the Gutteridge operator to produce a counterexample.

In the case when one of $\mathbf{a}$ or $\mathbf{b}$ is $\Delta^0_2$ then it uses properties of $\mathcal{K}$-pairs. $\square$

**Theorem (Jacobsen-Grocott)**

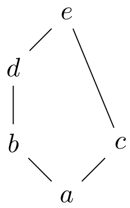
There are degrees $\mathbf{a}$ and $\mathbf{b}$ such that $\{\mathbf{a}, \mathbf{b}\}$ is a minimal pair and every nonzero degree $\mathbf{x} \leqslant \mathbf{a}$ joins $\mathbf{b}$ above $\mathbf{a}$.

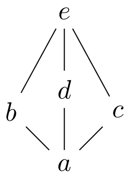
# Questions

## Question

Can we embed all finite lattices in $\mathcal{D}_e$ as strong intervals?

Important test cases are $N_5$ and $M_3$:




## Question

Are there super minimal pairs in $\mathcal{D}_e$ ?

## Question

What property characterizes the two quantifier theory of $\mathcal{D}_e$?

**Thank you!**

**Be safe!**