PA relative to an enumeration oracle

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Enumeration reducibility

Definition (Friedberg and Rogers 1959)

$A \leqslant_e B$ if there is a program that transforms an enumeration of B (a function on the natural numbers with range B) to an enumeration of A.

The program can always be chosen as a c.e. table of axioms of the sort:

$$\textit{If } \{x_1, x_2, \ldots, x_k\} \subseteq B \textit{ then } x \in A.$$

Compare this to the relation "c.e. in" which can be defined as follows:
A is c.e. in B if there is a c.e. table of axioms of the sort:

$$\textit{If } \{x_1, x_2, \ldots, x_k\} \subseteq B \textit{ and } \{y_1, y_2, \ldots, y_n\} \subseteq B^c \textit{ then } x \in A.$$

Proposition. A is c.e. in B if and only if $A \leqslant_e B \oplus B^c$.

Unlike the relation "c.e. in", the relation $\leqslant_e$ is transitive. It gives rise to the structure of the enumeration degrees $\mathcal{D}_e$.

The Turing degrees properly embed into $\mathcal{D}_e$ as the *total degrees*, degrees of sets of the form $A \oplus A^c$.

Relative to an enumeration oracle

When we relativize a class of objects with respect to a Turing oracle A, we usually replace "c.e." by "c.e. in A".

Example

G is A-*generic* if G meets or avoids every set of strings W that is c.e. in A.

U is a $\Sigma^0_1(A)$ *class* if $U = [W] = \{X \in 2^\omega \mid (\exists \sigma \in W)[\sigma \preceq X]\}$ for some c.e. in A set W.

We can extend these properties/relations to enumeration oracles by replacing "c.e. in A" by "$\leqslant_e A$".

Example

G is $\langle A \rangle$-generic if G meets or avoids every set of strings $W \leqslant_e A$.

U is a $\Sigma^0_1\langle A \rangle$ class if $U = [W]$ for some $W \leqslant_e A$.

Today we discuss the extension of the relation "PA above" to enumeration oracles.

The relation "PA above"

Recall that for Turing oracles A and B we say that B *is PA above* A if B computes a member of every nonempty $\Pi^0_1(A)$ class.

Definition

P is a $\Pi^0_1\langle A\rangle$ class if P is the complement of a $\Sigma^0_1\langle A\rangle$ class, i.e. there is some $W \leqslant_e A$ such that $P = 2^\omega \setminus [W]$.

Note that a $\Pi^0_1\langle A \oplus A^c\rangle$ class is just a $\Pi^0_1(A)$-class.

We treat the elements of a $\Pi^0_1\langle A\rangle$ class P as total objects! B enumerates a member of P, if there is some $X \in P$ such that $X \oplus X^c \leqslant_e B$.
If P is a $\Pi^0_1\langle A\rangle$ class then so are $\{X^c \mid X \in P\}$ and $\{X \oplus X^c \mid X \in P\}$.

Definition

$\langle B\rangle$ is PA relative to $\langle A\rangle$ if B enumerates a member of every nonempty $\Pi^0_1\langle A\rangle$ class.

Note that B is PA above A if and only if $\langle B \oplus B^c\rangle$ is PA above $\langle A \oplus A^c\rangle$.

Good oracles: the continuous degrees

The *continuous degrees* were introduced by Miller (2004) to capture the algorithmic content of points in computable Polish spaces. They form a proper (definable) subclass of the enumeration degrees and properly extend the total degrees.

Theorem (Miller 2004).

1. If $\mathbf{a}$ is a nontotal continuous degree then the set total degrees bounded $\mathbf{a}$ is a *Scott set*, i.e. a Turing ideal closed under the relation PA above.
2. For total degrees $\mathbf{y}$ is PA above $\mathbf{x}$ if an only if there is some non-total continuous degree $\mathbf{a}$ with $\mathbf{x} < \mathbf{a} < \mathbf{y}$.

Theorem (Andrews, Igusa, Miller, S 2019). A has continuous degree if and only if A is *codable*—there is a nonempty $\Pi^0_1\langle A\rangle$ class C_A such that every member of C_A uniformly enumerates A.

Good oracles: the continuous degrees

Corollary.

1. If A has continuous degree then $\langle A\rangle$ is not PA relative to $\langle A\rangle$—not *$\langle$self$\rangle$-PA*.
2. If A has continuous degree and $\langle B\rangle$ is PA relative to $\langle A\rangle$ then $A \leqslant_e B$—A is *PA bounded*.
3. There is a *universal $\Pi^0_1\langle A\rangle$-class P*: a nonempty class whose every member is PA relative to $\langle A\rangle$.

Proof:

1. If A enumerates a member of C_A then A is total.
2. If B is PA relative to A then B enumerates a member of C_A and hence by transitivity A.
3. Let P be the $\Pi^0_1\langle A\rangle$ class of all $X \oplus f$ where $X \in C_A$ and f is DNC-2 relative to X. Every nonempty $\Pi^0_1\langle A\rangle$ class is a $\Pi^0_1(X)$ class and f computes a member of it. □

Question. Are there any bad oracles?

Bad oracles: ⟨self⟩-PA oracles

Theorem (Miller, Soskova 2014). There are ⟨self⟩-PA degrees.

Proof: At stage s we have determined finitely many columns of a set A, say $A^{[0]}, \dots A^{[k]}$. Let A_s^* be the set with columns $A^{[0]}, \dots A^{[k]}, \omega, \omega, \dots$. We have that $P_e\langle A\rangle = 2^\omega \setminus \Gamma_e(A)$ is a superset of $P_e\langle A_s^*\rangle$.

1. If $P_e\langle A_s^*\rangle = \varnothing$ then by compactness there is a finite set $E \subseteq A_s^*$ such that $P_e\langle E\rangle$ is empty. Extend to make $E \subseteq A$.
2. Otherwise $P_e\langle A_s^*\rangle$ is a nonempty $\Pi_1^0(\bigoplus_{i<k} A^{[i]})$ class. Extend so that $A^{[k+1]}$ is PA relative to the first $k+1$ columns. □

Proposition. If A is ⟨self⟩-PA then A cannot have a universal class.

Proof: If A is ⟨self⟩-PA and P is universal then A enumerates some $X \in P$. But now every $\Pi_1^0(X)$-class is a $\Pi_1^0\langle A\rangle$ class and X computes a member of it. □

Question.

1. Can ⟨self⟩-PA degrees be PA bounded?
2. Can non-continuous degrees have a universal class?

Continuous = PA bounded

Theorem(Franklin, Lempp, Miller, Schweber, and S 2019). The continuous degrees are exactly the PA bounded enumeration degrees.

Proof idea: If A does not have continuous degree, we use the fact that A is not codable to produce a nested sequence of $\Pi^0_1\langle A\rangle$-classes $\{P_e\}_{e<\omega}$ such that every member of P_e computes a member of each nonempty $\Pi^0_1\langle A\rangle$ indexed by a number less than e but does not enumerate A via Γ_e. We then take $X \in \bigcap P_e$.

Question.

1. Can $\langle$self$\rangle$-PA degree be PA bounded? No!
2. *Can non-continuous degrees have a universal class?*

Other ways to have a universal class

Definition

An enumeration oracle $\langle A \rangle$ is *low for PA* if every set $X \oplus X^c$ that is PA (in the Turing sense) is PA relative to $\langle A \rangle$.

Total non c.e. oracles cannot be low for PA. In fact, low for PA oracles are *quasiminimal* (hence disjoint from continuous degrees).

Low for PA oracles have a universal class (e.g. DNC_2).

Theorem(Goh, Kalimullin, Miller, S). $\langle A \rangle$ is low for PA if and only if every nonempty $\Pi^0_1\langle A \rangle$ class has a nonempty Π^0_1 subclass.

Theorem(GKMS). The following classes of e-oracles are low for PA.

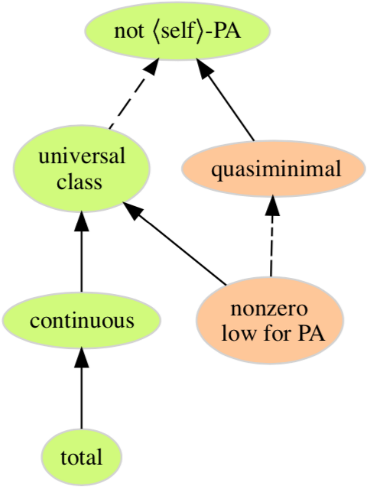
1. The 1-generic degrees.
2. Halves of nontrivial $\mathcal{K}$-pairs.

Proof sketch: Fix a 1-generic G and suppose $P_e\langle G \rangle$ is nonempty. Consider

$$W = \{\tau \mid P_e\langle \tau \rangle = \varnothing\}.$$

Fix $\sigma \preceq G$ with no extension in W. The set $P_e\langle \sigma 111 \dots \rangle$ is nonempty and a subset of $P_e\langle G \rangle$.

The picture so far



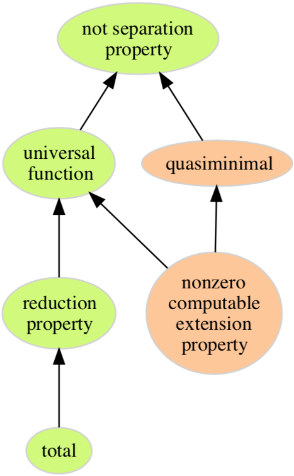
Notions from descriptive set theory

Definition (Kalimullin, Puzarenko 2005)

Let X be an enumeration oracle.

1. X has the *reduction property* if for all pairs of set $A, B \leqslant_e X$ there are sets $A_0, B_0 \leqslant_e X$ such that $A_0 \subseteq A$, $B_0 \subseteq B$, $A_0 \cap B_0 = \varnothing$, and $A_0 \cup B_0 = A \cup B$;
2. X has the *uniformization property* if whenever $R \leqslant_e X$ is a binary relation there is a function f with graph $G_f \leqslant_e X$ such that $dom(f) = dom(R)$.
3. X has the *separation property* if for every pair of disjoint sets $A, B \leqslant_e X$ there is a separator C such that $A \subseteq C$, $B \subseteq C^c$, and $C \oplus C^c \leqslant_e X$.
4. X has the *computable extension property* if every partial function φ with $G_\varphi \leqslant_e X$ has a (partial) computable extension $\psi \subseteq \varphi$.
5. X has a *universal function* if there is a partial function U with $G_U \leqslant_e X$ such that if φ is a partial function with $G_\varphi \leqslant_e X$ then for some e we have that $\varphi = \lambda x.U(e, x)$.

Kalimullin and Puzarenko's theorem



The reduction property

X has the *reduction property* if whenever $A, B \leqslant_e X$ there are disjoint $A_0, B_0 \leqslant_e X$ with $A_0 \subseteq A$, $B_0 \subseteq B$, and $A_0 \cup B_0 = A \cup B$;

Example

Kleene's O has the reduction property because $A \leqslant_e O$ if and only if A is Π^1_1.

We want to construct a $\Pi^0_1\langle X\rangle$ class U such that if $P_e\langle X\rangle \neq \varnothing$ then the e-th column in any member of U codes a member of $P_e\langle X\rangle$.

If X were total we would fix enumerations of $\Gamma_e(X)$ relative to X and let U be the class of separators for

1. The set A of all $\langle e, \sigma\rangle$ such that all extensions of $\sigma 0$ leave $P_e\langle X\rangle$ *first*.
2. The set B of all $\langle e, \sigma\rangle$ such that all extensions of $\sigma 1$ leave $P_e\langle X\rangle$ *first*.

If X is not total then we don't have a notion of *first*!

But then for σ with no extension in P_e we will have $\langle e, \sigma\rangle \in A \cap B$.

The reduction property lets us solve exactly this problem!

Theorem(GKMS). The reduction property implies having a universal class.

The separation property

X has the *separation property* if for every pair of disjoint sets $A, B \leqslant_e X$ there is a separator C such that $A \subseteq C$, $B \subseteq C^c$, and $C \oplus C^c \leqslant_e X$.

Note that the set of all separators C for sets $A, B \leqslant_e X$ is a $\Pi^0_1\langle X\rangle$ class.

Definition

A $\Pi^0_1\langle X\rangle$ class P is a *separation class* if $P = \{C \mid A \subseteq C \;\&\; B \subseteq C^c\}$ for some disjoint $A, B \leqslant_e X$. Call such classes *Sep*$\langle X\rangle$ for short.

Proposition. X has the separation property if and only if X enumerates a path in every Sep$\langle X\rangle$ class.

If X is $\langle$self$\rangle$-PA then X has the separation property.

Computable extension property

X has the *computable extension* property if every partial function φ with $G_\varphi \leqslant_e X$ has a (partial) computable extension $\psi \subseteq \varphi$.

Theorem (GKMS). The following are equivalent:

1. X has the computable extension property.
2. Every $\{0,1\}$-valued function with graph reducible to X has a computable $\{0,1\}$-valued extension.
3. If $A \leqslant_e X$ and $B \leqslant_e X$ are disjoint then there are disjoint c.e sets C and D such that $A \subseteq C$ and $B \subseteq D$.
4. Every set Y with PA degree computes a member of every separation class relative to $\langle X \rangle$.

And so if X is low for PA then X has the computable extension property.

A mystery solved by introducing uniformity

X has a *universal function* if there is a partial function U with $G_U \leqslant_e X$ such that if φ is a partial function with $G_\varphi \leqslant_e X$ then for some e we have that $\varphi = \lambda x.U(e, x)$

Question. This should be an analog of having a universal class, but how?

We defined a *universal* $\Pi^0_1\langle X\rangle$*-class* to be a nonempty class whose every member is PA relative to $\langle X\rangle$, i.e. enumerates a path in every nonempty $\Pi^0_1\langle X\rangle$ class.We will adjust this definition introducing a little uniformity:

Definition

P is a *universal* $\Pi^0_1\langle X\rangle$*-class* if for every nonempty $\Pi^0_1\langle X\rangle$ class Q there is a uniform procedure that produces a path from Q relative to every member of P.

In all cases we looked at so far, that is the case: total degrees, the continuous degrees, the low for PA degrees, the oracles with the reduction property!

Universal for Sep$\langle X \rangle$ classes

Theorem (GKMS). The following are equivalent

1. X has a universal function;
2. X has a $\{0,1\}$-valued universal function U for $\{0,1\}$-valued partial functions φ with $G_\varphi \leqslant_e X$;
3. There is a $\Pi^0_1\langle X \rangle$ class P such that for every Sep$\langle X \rangle$-class Q there is a uniform procedure that produces a path from Q relative to every member of P. (This class can be chosen as a separating class.)

Proof sketch: $2 \Rightarrow 3$ Let P be the separating class for the disjoint sets $\{\langle e, x \rangle \mid U(e,x) = 0\}$ and $\{\langle e, x \rangle \mid U(e,x) = 1\}$. If $A, B \leqslant_e X$ are disjoint then $A \times \{0\} \cup B \times \{1\}$ is the graph of a partial function $\lambda x.U(e,x)$ for some e. The e-th column of any path in P is a separator for A, B.

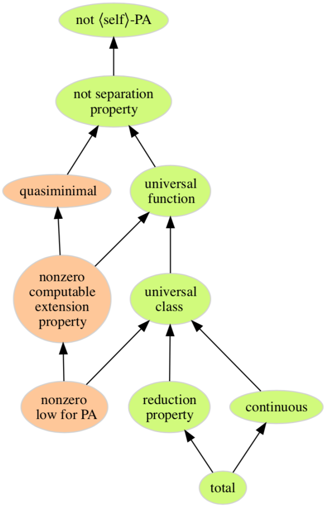
$3 \Rightarrow 2$ Given P define $U(\langle e, i \rangle, x) = y$ if

Think of $\Gamma_e(X)$ as the graph of a $\{0,1\}$-valued function φ and Φ_i^Y as a separator for $\{x \mid \varphi(x) = 1\}$ and $\{x \mid \varphi(x) = 0\}$ for every $Y \in P$.

1. $y \leqslant 1$ and $\langle x, y \rangle \in \Gamma_e(X)$,
2. there is a finite set $D \subseteq 2^{<\omega}$ such that $P \subseteq [D]$ and an n such if $\sigma \in 2^n \cap [D]$ then $\Phi_i^\sigma(x) \downarrow = y$.

□

A summary of the results by Goh, Kalimullin, Miller, and Soskova



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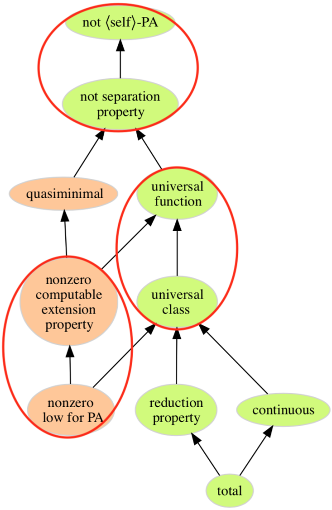


All of the arrows are strict! A forcing notion

Let $f(n) = 2^n$. We identify ω with $f^{<\omega}$—the set of sequences $\sigma \in \omega^{<\omega}$ such that $\sigma(n) < 2^n$ for all $n < |\sigma|$.

A forcing condition is a pair $\langle T, \varepsilon \rangle$:

- T is a finite subtree of $f^{<\omega}$ of height $|T|$;
- $\varepsilon \in (0, 1)$ is rational.

We associate the set $A_T = f^{\leqslant |T|} \smallsetminus T$ to the condition $\langle T, \varepsilon \rangle$.

$\langle S, \delta \rangle \leqslant \langle T, \varepsilon \rangle$ if and only if

- $T = S \upharpoonright |T|$,
- $\delta \leqslant \varepsilon$, and
- for every $\sigma \in S$ with $|T| \leqslant |\sigma| < |S|$, at least $\lceil (1 - \varepsilon) \cdot 2^{|\sigma|} \rceil$ of its immediate successors lie in S.

If $\mathcal{F}$ is a filter in this partial order then let $G = \bigcup_{\langle T, \varepsilon \rangle \in \mathcal{F}} T$ and $A_G = f^{<\omega} \smallsetminus G$.

Genericity ensures the computable extension property

Lemma. If G is sufficiently generic, then A_G has the computable extension property.

Proof: Fix $\langle T, \varepsilon \rangle$ and a pair of enumeration operators Γ_0 and Γ_1.

Suppose we cannot extend $\langle T, \varepsilon \rangle$ to $\langle S, \delta \rangle$ to make $\Gamma_0(A_S)$ and $\Gamma_1(A_S)$ intersect.

We want to extend $\langle T, \varepsilon \rangle$ to ensure that $\Gamma_0(A_G)$ and $\Gamma_1(A_G)$ are separated by disjoint c.e. sets.

We claim that $\langle T, \varepsilon/2 \rangle$ is such an extension: let C_i to be the set of all n for which there is some condition $\langle S, \delta \rangle$ extending $\langle T, \varepsilon/2 \rangle$ such that $n \in \Gamma_i(A_S)$.

- C_0 and C_1 are c.e.
- If we assume that they are not disjoint, say n is put in $\Gamma_0(A_{S_0})$ via $\langle S_0, \delta_0 \rangle$ and in $\Gamma_1(A_{S_1})$ via $\langle S_1, \delta_1 \rangle$, then $\langle S_0 \cap S_1, \varepsilon \rangle$ extends $\langle T, \varepsilon \rangle$ and has $n \in \Gamma_0(A_{S_0 \cap S_1}) \cap \Gamma_1(A_{S_0 \cap S_1})$ contradicting our assumption. □

The more difficult separations

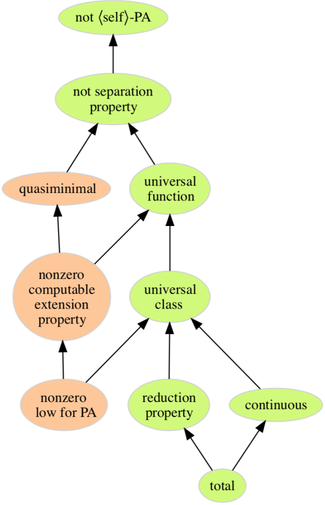
Lemma. If G is sufficiently generic, then A_G does not have a universal class.

Proof: A much more elaborate analysis of the forcing notion.

Lemma. There is a set A that has the separation property, but is not $\langle \text{self} \rangle$-PA.

Proof: A combination of the two two forcing notions that we discussed.

Thank You!



Open questions.

1. Is the extra uniformity that we added to the definition of universal class necessary?
2. If A has a universal class does A have a separating class that is universal?
3. Is the relation PA relative to an enumeration oracle definable?

Visit http://zoo.ludovicpatey.com/ to build your own pretty diagram!