Extensions of the Turing model for relative definability

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The real numbers within the complex numbers

- In his 1545 “Ars Magna”, Cardano provided the first complete expression for the solution of a general cubic equation. In this solution a certain class of cases - ones he referred to as “irreducible” - the equation always has three real solutions, but in order to derive his expression he was forced to take the square root of a negative number.
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The spectrum of relative definability

How can a set of natural numbers B be used to define a set of natural numbers A.

- There is an algorithm, which determines whether $x \in A$ using finite information about memberships in B: Turing reducibility.
- There is an algorithm, which enumerates instances of memberships in A from instances of memberships in B: enumeration reducibility.

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Preliminaries: Enumeration reducibility

$A \leq_T B$ iff there is a computable in B function φ^B, such that $\chi_A = \varphi^B$.

$A \leq_T B$ iff $A \oplus \overline{A}$ is c.e. in B.

$A \leq_T B$ iff there is a c.e. set W such that
$x \in A \oplus \overline{A}$ if and only if there are finite sets D_B and $D_{\overline{B}}$ such that
$\langle x, D_B \oplus D_{\overline{B}} \rangle \in W$ and $D_B \oplus D_{\overline{B}} \subseteq B \oplus \overline{B}$.

Definition

$A \leq_e B$ if and only if there is a set W, such that
$A = W(B) = \{x \mid \exists u(\langle x, u \rangle \in W \wedge D_u \subseteq B)\}$.

Note that $A \leq_T B$ if and only if $A \oplus \overline{A} \leq_e B \oplus \overline{B}$.

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The structure of the enumeration degrees

- $d_e(A) = \{B \mid A \leq_e B \ \& \ B \leq_e A\}$
- $d_e(A) \leq d_e(B)$ iff $A \leq_e B$.
- $\mathbf{0}_e = d_e(\emptyset) = \{W \mid W$ is c.e. $\}$.
- $d_e(A) \vee d_e(B) = d_e(A \oplus B)$.
- $d_e(A)' = d_e(A')$, where $A' = L_A \oplus \overline{L_A}$ and $L_A = \{x \mid x \in W_x(A)\}$.
- $\mathcal{D}_e = \langle D_e, \leq, \vee, ', \mathbf{0}_e \rangle$ is an upper semi-lattice with jump operation and least element.

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Proposition

The embedding $\iota : \mathcal{D}_T \to \mathcal{D}_e$, defined by $\iota(d_T(A)) = d_e(A \oplus \overline{A})$, preserves the order, the least upper bound and the jump operation.

The substructure of the total e-degrees is defined as $\mathcal{TOT} = \iota(D_T)$.

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More connections between $\mathcal{D}_T$ and $\mathcal{D}_e$

- B is c.e. in A if and only if $B \leq_e A \oplus \overline{A}$.
- Selman's Theorem: $A \leq_e B$ if and only if

$$\{X \mid B \text{ is c.e. in } X\} \subseteq \{X \mid A \text{ is c.e. in } X\}.$$

$$\left\{d_e(X \oplus \overline{X}) \mid B \leq_e X \oplus \overline{X}\right\} \subseteq \left\{d_e(X \oplus \overline{X}) \mid A \leq_e X \oplus \overline{X}\right\}$$

- Corollary: $\mathcal{TOT}$ is an automorphism base for $\mathcal{D}_e$.
- Soskov's Jump inversion theorem (JIT): For every $\mathbf{x} \in \mathcal{D}_e$ and every total $\mathbf{q} \geq \mathbf{x}'$ exists $\mathbf{a} \in \mathcal{TOT}$ such that $\mathbf{a} \geq \mathbf{x}$ and $\mathbf{a}' = \mathbf{q}$.

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$\mathcal{D}_T$: Coles, Downey and Slaman: "Every set has a least jump enumeration"

Given a set A let $\mathcal{C}(A) = \{X \mid A \text{ is c.e. in } X\}$.

Theorem (Richter)

There is a non-c.e. set A such that A is c.e. in two sets B and C which form a minimal pair. Hence there is a set A, such that $\mathcal{C}(A)$ does not have a member of least degree.

Theorem (Coles, Downey, Slaman)

For every sets A the set: $\mathcal{C}(A)' = \{X' \mid A \text{ is c.e. in } X\}$ has a member of least degree: $c'_\mu(A)$.

A set of the degree $c'_\mu(A)$ is obtained using forcing with finite conditions.

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Motivation: torsion-free abelian groups of rank 1

A torsion free abelian group of rank 1 G is (isomorphic to) a subgroup of $(\mathbb{Q}, +, =)$.

Definition

Let p be a prime number and $a \in G$.

$$h_p(a) = \begin{cases} \text{the largest } k, & \text{such that } p^k | a \text{ in } G; \\ \infty, & \text{if } \forall k (p^k | a \text{ in } G) . \end{cases}$$

Here $p^k | a$ in G if there exists $b \in G$ such that $p^k . b = a$.

Example: If $G = \mathbb{Q}$ then for all nonzero a and all p, $h_p(a) = \infty$, because for all k, $p^k . \frac{a}{p^k} = a$.
If $G = \mathbb{Z}$ then for all nonzero a and all but finitely many p, $h_p(a) = 0$.

In fact if $a, b \neq 0$ then for all but finitely many p, $h_p(a) = h_p(b)$.

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The type of G

Let $p_0 < p_1 < \cdots < p_n \ldots$ be an enumeration of all prime numbers.

Definition

The characteristic of an element $a \in G$ is the sequence:

$$\chi(a) = (h_{p_0}(a), h_{p_1}(a), \ldots h_{p_n}(a), \ldots)$$

So if $a, b \neq 0$ then $\chi(a) =^* \chi(b)$.
The type of G, denoted $\chi(G)$ is the equivalence class of $\chi(a)$ for any $a \neq 0$ in G.

Baer noticed that there is a TFA1 group of every possible type.

Theorem (Baer)

Two torsion-free abelian groups of rank 1 are isomorphic if and only if they have the same type.

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The standard type of G

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Let $S(G) = \{\langle i, j\rangle \mid j \leq$ the i-th element of $\chi(G)\}$.

Note that every S can be coded as (is m-equivalent to) $\{\langle i, j\rangle \mid j = 0 \vee i \in S \,\&\, j = 1\}$.

Theorem (Downey, Jockusch)

The degree spectrum of G: $\{d_T(H) \mid H \cong G\}$ is precisely $\{deg_T(Y) \mid S(G)$ is c.e. in $Y\}$.

$c'_\mu(A)$ exists for all A if and only if for every torsion-free abelian group G the jump spectrum of G, the set $\{d_T(H)' \mid H \cong G\}$ has a least element.

Corollary (Coles, Downey, Slaman)

Every every torsion-free abelian group of rank 1 G has a jump degree.

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The degree spectrum of G: $\{d_T(H) \mid H \cong G\}$ *is precisely* $\{deg_T(Y) \mid S(G)$ is c.e. in $Y\}$.

$c'_\mu(A)$ exists for all A if and only if for every torsion-free abelian group G the jump spectrum of G, the set $\{d_T(H)' \mid H \cong G\}$ has a least element.

Corollary (Coles, Downey, Slaman)

Every every torsion-free abelian group of rank 1 G has a jump degree.

The standard type of G

Definition

Let $S(G) = \{\langle i, j\rangle \mid j \leq$ the i-th element of $\chi(G)\}$.

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$\mathcal{D}_e$: Soskov "Degree spectra and co-spectra of structures"

Consider a structure $\mathcal{A} = (\mathbb{N}, R_1 \dots R_k)$.

Definition

An enumeration is any bijective mapping $f : \mathbb{N} \to \mathbb{N}$.

If R is an n-ary relation then $f^{-1}(R) = \{\langle x_1, \dots, x_n\rangle \mid (f(x_1), \dots f(x_n)) \in R\}$.

The pullback of $\mathcal{A}$ is the set $f^{-1}(\mathcal{A}) = f^{-1}(R_1) \oplus f^{-1}(R_2) \cdots \oplus f^{-1}(R_k)$.

- Consider the structure $\mathcal{A}^+ = (\mathbb{N}, R_1, \overline{R_1} \dots R_k, \overline{R_k})$.
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Enumeration degree spectrum

Fix $\mathcal{A} = (\mathbb{N}, R_1 \dots R_k)$.

Definition

The e-degree spectrum of $\mathcal{A}$ is
$DS_e(\mathcal{A}) = \{d_e(f^{-1}(\mathcal{A})) \mid f \text{ in an enumeration}\}$.

If $DS_e(\mathcal{A})$ has a least member, it is the (enumeration) degree of $\mathcal{A}$.

- In fact $DS_e(\mathcal{A}^+) = \{\iota(\mathbf{a}) \mid \mathbf{a} \in DS_T(\mathcal{A})\}$.
- $\mathcal{A}$ has T-degree $\mathbf{a}$ if and only if $\mathcal{A}^+$ has e-degree $\iota(\mathbf{a})$.

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Co-spectrum

Definition

The co-spectrum of $\mathcal{A}$ is the set $CS_e(\mathcal{A}) = \{\mathbf{b} \mid \forall \mathbf{a} \in DS_e(\mathcal{A})(\mathbf{b} \leq \mathbf{a})\}$.

If $CS_e(\mathcal{A})$ has a largest element, then it is called the co-degree of $\mathcal{A}$.

- If $\mathcal{A}$ has degree $\mathbf{a}$ then it has co-degree $\mathbf{a}$.
- There are examples of structures which have a co-degree but do not have a degree.

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The case of principal ideals

Let G be a torsion-free abelian group of rank 1.

Recall that the degree spectrum of G: $\{d_T(H) \mid H \cong G\}$ is precisely $\{deg_T(Y) \mid S(G) \text{ is c.e. in } Y\}$.

So the enumeration degree spectrum of G^+ is $DS_e(G^+) = \{\mathbf{a} \mid \mathbf{a} \in \mathcal{TOT} \ \& \ d_e(S(G)) \leq_e \mathbf{a}\}$.
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TFA groups in the e-degrees

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1. *The co-spectrum of G^+ is $CS(G^+) = \{\mathbf{b} \mid \mathbf{b} \leq \mathbf{s}_G\}$. Hence G^+ always has co-degree: $\mathbf{s}_G$.*
2. *Every principal ideal of e-degrees can be represented as the co-spectrum of a torsion-free abelian group of rank 1.*

1. Follows from Selman's theorem: if **a** and **b** are bounded by the same total degrees then $\mathbf{a} = \mathbf{b}$.
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$\mathcal{D}_T$: Shore and Slaman: "Defining the Turing jump"

Theorem (Shore, Slaman)

The Turing jump is first order definable in $\mathcal{D}_T$.

1. Slaman and Woodin: The double jump is first order definable in $\mathcal{D}_T$.
2. For every $\mathbf{a} \notin \Delta^0_2$ there is $\mathbf{g}$ such that $\mathbf{a} \vee \mathbf{g} = \mathbf{g}''$

Hence $\mathbf{0}'$ is the greatest degree which does not join any $\mathbf{g}$ to $\mathbf{g}''$.

Ingredient 1: Slaman and Woodin's analysis of the automorphisms of the Turing degrees and *"involves explicit translation of automorphism facts in definability facts via a coding of second order arithmetic".*

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$\mathcal{D}_e$: Kalimullin "Some notes on definability in the enumeration degrees"

Theorem (Kalimullin)

The enumeration jump is first order definable in $\mathcal{D}_e$.

Main ingredient: $\mathcal{K}$-pairs.

Definition

A pair of sets A, B are called a $\mathcal{K}$-pair if there is a c.e. set W, such that $A \times B \subseteq W$ and $\overline{A} \times \overline{B} \subseteq \overline{W}$.

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$\mathcal{K}$-pairs: An example

Definition (Jockusch)

A set of natural numbers A is semi-recursive if there is a computable function s_A such that for every pair of natural numbers (x, y):

1. $s_A(x, y) \in \{x, y\}$.
2. If $x \in A$ or $y \in A$ then $s_A(x, y) \in A$.

Example

Let A be a semi-recursive set. Then $(A, \overline{A})$ is a $\mathcal{K}$-pair.

$W = \{\langle x, y\rangle \mid s_A(x, y) = x\}$.

Theorem (Jockusch)

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Three steps

Theorem (Kalimullin)

1. *The property of being a $\mathcal{K}$-pair is degree theoretic and first order definable in $\mathcal{D}_e$ by:*

$$\mathcal{K}(\mathbf{a},\mathbf{b}) \leftrightharpoons (\forall \mathbf{x} \in \mathcal{D}_e)((\mathbf{a} \vee \mathbf{x}) \wedge (\mathbf{b} \vee \mathbf{x}) = \mathbf{x})$$

2. *There are $\mathbf{a}, \mathbf{b}, \mathbf{c}$, which form a $\mathcal{K}$-system, such that $\mathbf{a} \vee \mathbf{b} \vee \mathbf{c} = \mathbf{0}_e'$.*
3. *If A and B are a non-trivial $\mathcal{K}$-pair then $\overline{A} \oplus \mathbf{0}_e' \equiv_e B \oplus \mathbf{0}_e'$.*

   So $\mathbf{0}_e'$ joins every $\mathcal{K}$-partner of A to the same degree.

Hence $\mathbf{y} \geq_e \mathbf{0}_e'$ if an only if for all $\mathbf{a}, \mathbf{b}, \mathbf{c}$ if $\mathcal{K}(\mathbf{a},\mathbf{b})$ and $\mathcal{K}(\mathbf{a},\mathbf{c})$ then $\mathbf{y} \vee \mathbf{b} = \mathbf{y} \vee \mathbf{c}$.

All the steps can be proved for $\mathcal{K}$-pairs over any degree $\mathbf{u}$.

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   So $\mathbf{0}'_e$ joins every $\mathcal{K}$-partner of A to the same degree.

Hence $\mathbf{y} \geq_e \mathbf{0}'_e$ if an only if for all $\mathbf{a}, \mathbf{b}, \mathbf{c}$ if $\mathcal{K}(\mathbf{a}, \mathbf{b})$ and $\mathcal{K}(\mathbf{a}, \mathbf{c})$ then $\mathbf{y} \vee \mathbf{b} = \mathbf{y} \vee \mathbf{c}$.

All the steps can be proved for $\mathcal{K}$-pairs over any degree $\mathbf{u}$.

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Consequences

1. The set of total degrees **a** above $\mathbf{0}_e'$ is definable in $\mathcal{D}_e$.

   Follows for example from Soskov's Jump inversion theorem.

2. For every torsion-free abelian group of rank 1 G, the first jump degree spectrum of G is first order definable in $\mathcal{D}_e$ with parameter $\mathbf{s}_G$.

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Definability in the local structures

Consider the local structures $\mathcal{D}_T(\leq \mathbf{0}'_T)$ consisting of all Δ^0_2 Turing degrees and $\mathcal{D}_e(\leq \mathbf{0}'_e)$, consisting of all Σ^0_2 enumeration degrees.

Recall that $\iota : \mathcal{D}_T \to \mathcal{D}_e$ preserves the jump, hence $\mathcal{D}_T(\leq \mathbf{0}')$ embeds in $\mathcal{D}_e(\leq \mathbf{0}'_e)$.

Definition

1. For every $n \geq 1$ the class of low$_n$ degrees is $L_n = \{\mathbf{a} \leq \mathbf{0}' \mid \mathbf{a}^n = \mathbf{0}^n\}$.
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Theorem (Shore)

For every $n \geq 1$ the classes L_{n+1} and H_n are first order definable in $\mathcal{D}_T(\leq \mathbf{0}')$.

1. The theory of first order arithmetic can be interpreted in $\mathcal{D}_T(\leq \mathbf{0}')$.
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4. An additional proof that H_1 is first order definable, using the fact that $\mathbf{x} \in H_1$ if and only if $\forall \mathbf{w} \exists \mathbf{y} b \leq \mathbf{x}(\mathbf{w}'' = \mathbf{y}'')$.

Note that the definability of L_1 in $\mathcal{D}_T(\leq \mathbf{0}'_T)$ remains open.

Similar ideas are used by Nies Shore and Slaman in the proof of the corresponding result for the c.e. degrees.

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Initial motivation: Prove that the theory of first order arithmetic is interpretable in $\mathcal{D}_e(\leq \mathbf{0}_e')$.

Theorem (Slaman and Woodin)

A uniformly low antichain can be coded by parameters in $\mathcal{D}_e(\leq \mathbf{0}_e')$.

1. Non-trivial Σ^0_2 $\mathcal{K}$-pairs are low.
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$\mathcal{K}(\mathbf{a}, \mathbf{b}) \leftrightharpoons (\forall \mathbf{x})((\mathbf{a} \vee \mathbf{x}) \wedge (\mathbf{b} \vee \mathbf{x}) = \mathbf{x})$

Is it enough to require that this formula is satisfied by all Σ^0_2 e-degrees?

Theorem (Ganchev, S)

There is a first order formula $\mathcal{LK}(x, y)$, which defines $\mathcal{K}$-pairs in $\mathcal{D}_e(\leq \mathbf{0}'_e)$.

Proof flavor: We add a cupping property to the formula $\mathcal{K}(x, y)$, proved with a priority construction, similar to Harrington's non-splitting.

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$Th(\mathcal{D}_e(\leq \mathbf{0}'_e)) \equiv_1 Th(\mathbb{N})$.

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An easy consequence

Theorem

The class of downwards properly Σ^0_2 enumeration degrees is first order definable in $\mathcal{D}_e(\leq \mathbf{0}'_e)$.

a is downwards properly Σ^0_2 if and only if it bounds no $\mathcal{K}$-pairs.

A more important consequence

Jockusch:

For every incomputable set B there is a semi-recursive set $A \equiv_T B$ such that both A and $\overline{A}$ are not c.e.

It is not hard to see that a $\mathcal{K}$-pair of the form $\{A, \overline{A}\}$ is maximal, i.e. it cannot be extended to a $\mathcal{K}$-pair (B, C), with $A <_e B$ or $\overline{A} <_e C$

Corollary

Every nonzero total enumeration degree can be represented as the least upper bound of a maximal nontrivial $\mathcal{K}$-pair.

Theorem (Ganchev, S)

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Theorem (Giorgi, Sorbi, Yang)

If $\mathbf{a}$ *is a total degree then* $\mathbf{a}$ *is low iff* $\mathbf{a}$ *does not bound a downwards properly* Σ^0_2 *enumeration degree.*

By Soskov's Jump Inversion Theorem every low degree is bounded by a total low degree.

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One final consequence

Theorem (Giorgi, Sorbi, Yang)

If **a** *is a total degree then* **a** *is low iff* **a** *does not bound a downwards properly* Σ^0_2 *enumeration degree.*

By Soskov's Jump Inversion Theorem every low degree is bounded by a total low degree.

Corollary

The class of low e-degrees is first order definable in $\mathcal{D}_e(\leq \mathbf{0}'_e)$.

Open question

We know that:

- $\mathcal{TOT} \cap \mathcal{D}_e(\geq \mathbf{0}'_e)$ is first order definable.
- $\mathcal{TOT} \cap \mathcal{D}_e(\leq \mathbf{0}'_e)$ is first order definable.

Question

Is $\mathcal{TOT}$ first order definable in $\mathcal{D}_e$?

Recall that the total degrees are an automorphism base for $\mathcal{D}_e$.

A positive answer would connect the problems of the existence of a non-trivial automorphism in both structures.

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The end

Thank you!