Randomness relative to an enumeration oracle

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Enumeration reducibility

Definition

$A \leq_e B$ if there is a c.e. set W, such that

$$A = W(B) = \{x \mid \exists D(\langle x, D\rangle \in W \; \& \; D \subseteq B)\}\,.$$

Theorem (Selman)

$A \leq_e B$ if and only if every enumeration of B computes an enumeration of A.

The degree structure induced by $\leq_e$ is $\mathcal{D}_e$ the structure of the enumeration degrees, an upper semi-lattice with least element.

The total enumeration degrees

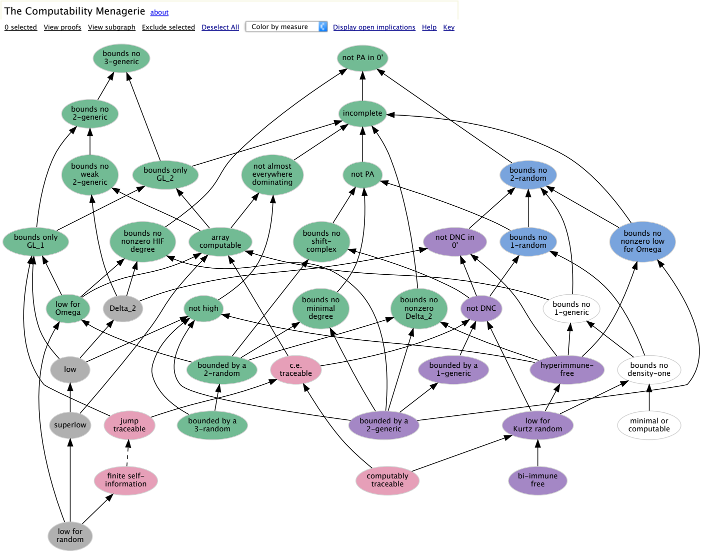
Proposition

1. A is c.e. in B if and only if $A \leq_e B \oplus \overline{B}$.
2. $A \leq_T B$ if and only if $A \oplus \overline{A} \leq_e B \oplus \overline{B}$.

The embedding $\iota : \mathcal{D}_T \to \mathcal{D}_e$, defined by $\iota(d_T(A)) = d_e(A \oplus \overline{A})$, defines an isomorphic copy of the Turing degrees in the enumeration degrees: the total enumeration degrees.

What is missing?

The Computability Menagerie   http://menagerie.math.wisc.edu/menagerie#coloring=meas



The enumeration menagerie

1. Total degrees.
2. Quasiminimal degrees, disjoint from the total degrees.
3. Generic degrees are all quasiminimal.
4. All total degrees are continuous, but no continuous degree is quasiminimal.
5. Halves of $\mathcal{K}$-pairs are quasiminimal and at most 1-generic.
6. Semi-computable sets that are not co-c.e. are quasiminimal, but not 1-generic.

Algorithmic randomness

Definition

1. A test is a uniform sequence of Σ^0_1 classes $\{V_n\}_{n<\omega}$ such that $\mu V_n \leq 2^{-n}$ for every n.
2. A set Z passes the test V if $Z \notin \bigcap_{n<\omega} V_n$.
3. The set Z is random if it passes all tests.

Definition

1. A Solovay test is a set c.e. set W (subset of $2^{<\omega}$) with finite weight $\mathrm{wt}(W) = \sum_{\sigma \in W} 2^{-|\sigma|}$.
2. A set Z passes the test W if only finitely many initial segments of Z are in W.
3. The set Z is Solovay random if it passes all Solovay tests.

Randomness relative to a Turing oracle

Definition

1. An A-test is a uniform sequence of $\Sigma^0_1(A)$ classes $\{V_n\}_{n<\omega}$ such that $\mu V_n \leq 2^{-n}$ for every n.
2. A sequence $Z \in 2^\omega$ passes the test V if $Z \notin \bigcap_{n<\omega} V_n$.
3. The sequence Z is A-random if it passes all A-tests.

Definition

1. A Solovay A-test is a c.e. in A set W with finite weight $\mathrm{wt}(W) = \sum_{\sigma \in W} 2^{-|\sigma|}$.
2. A sequence Z passes the test W if only finitely many initial segments of Z are in W.
3. The sequence Z is Solovay A-random if it passes all Solovay A-tests.

Randomness relative to an enumeration oracle $\langle A \rangle$

We have a well defined notion of randomness relative to a total oracle: Z is $\langle A \oplus \overline{A} \rangle$-random if and only if Z is A-random.

Approach I: Structural

- A sequence Z is upwards $\langle A \rangle$-random if it is random relative to some total set enumeration above A.
- A sequence Z is downwards $\langle A \rangle$-random if it is random relative to every total set enumeration below A.

Randomness relative to an enumeration oracle $\langle A \rangle$

Approach II: Using the fact that 'c.e. in A' is the same as '$\leq_e A \oplus \overline{A}$'.

Definition

1. V is a $\Sigma^0_1\langle A \rangle$ class if there is a set $U \leq_e A$ such that $V = [U]^{\preceq}$.
2. An $\langle A \rangle$-test is a uniform sequence of $\Sigma^0_1\langle A \rangle$ classes $\{V_n\}_{n<\omega}$ such that $\mu V_n \leq 2^{-n}$ for every n.
3. A sequence $Z \in 2^\omega$ passes the test V if $Z \notin \bigcap_{n<\omega} V_n$.
4. The sequence Z is A-random if it passes all $\langle A \rangle$-tests.

Definition

1. A Solovay $\langle A \rangle$-test is a set $W \leq_e A$ with finite weight $\mathrm{wt}(W) = \sum_{\sigma \in W} 2^{-|\sigma|}$.
2. A sequence Z passes the test W if only finitely many initial segments of Z are in W.
3. The sequence Z is Solovay $\langle A \rangle$-random if it passes all Solovay A-tests.

Relationships

Theorem

For any set A and sequence Z, consider the following "relative randomness" notions:

1. Z is X-random for some X such that $A \leq_e X \oplus \overline{X}$,
2. Z is $\langle A \rangle$-random,
3. Z is Solovay $\langle A \rangle$-random,
4. Z is X-random for every X such that $X \oplus \overline{X} \leq_e A$.

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4).
Furthermore, each of these implications can be strict.

(2) $\Rightarrow$ (3): If W is a Solovay $\langle A \rangle$-test then $\{V_n\}_{n<\omega}$ is and $\langle A \rangle$-test:

$$V_n = [\{\sigma \mid \sigma \in W \text{ and there are at least } 2^n - 1 \text{ proper prefixes of } \sigma \text{ in } W\}]^{\preceq}.$$

What fails?

Suppose that we wanted to show $(3) \Rightarrow (2)$. The usual reasoning is:

> *"Suppose that Z is not $\langle A \rangle$-random. Let $\{V_n\}_{n<\omega}$ be an $\langle A \rangle$-test. Since we may assume that each V_n is given by a a prefix-free set $U_n \leq_e A$ and $\mathrm{wt}\, U_n = \mu V_n \leq 2^{-n}$, the set $W = \bigcup_n U_n$ of strings has weight less than* 1*."*

To transform U_n into a prefix free set P_n we enumerate U_n one by one. If σ appears, but we have already enumerated an extension ofσ in P_n we do not add σ, but instead pick the longest extension τ of σ in P_n and add all extensions of σ of length $|\tau|$.

The prefix free version of U_n depends on the order in which we enumerate U_n: if σ appears before any of its extensions then $\sigma \in P_n$ and otherwise $\sigma \notin P_n$.

Separating $\langle A\rangle$-randomness from Solovay $\langle A\rangle$-randomness

Step 1: An alternative characterization of $\langle A\rangle$-randomness

Definition

A *Kučera* $\langle A\rangle$*-test* is a $\Sigma^0_1\langle A\rangle$ class U with $\mu U < 1$.
A set X *passes* the test U if not every tail of X is in U.

Theorem

A set X is $\langle A\rangle$-random if and only if X passes every Kučera $\langle A\rangle$-test.

Separating $\langle A \rangle$-randomness from Solovay $\langle A \rangle$-randomness

Step 2: A sufficient condition

Lemma

For any set A, consider the following properties:

1. There is a $\Sigma^0_1\langle A \rangle$-class U with $\mu(U) < 1$ such that if $W \leq_e A$ is a set of strings with $\mathrm{wt}(W) < 1$, then $U \smallsetminus [W]^{\prec}$ has positive measure.
2. There if a Solovay $\langle A \rangle$-random sequence that is not $\langle A \rangle$-random.
3. There is a $\Sigma^0_1\langle A \rangle$-class U with $\mu(U) < 1$ such that if $W \leq_e A$ is a set of strings with $\mathrm{wt}(W) < 1$, then $U \not\subseteq [W]^{\prec}$.

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3).

Separating $\langle A\rangle$-randomness from Solovay $\langle A\rangle$-randomness

Step 3: Building a class that is hard to cover.

Lemma

There is a set A and a $\Sigma^0_1\langle A\rangle$-class U with $\mu(U) < 1$ such that if $W \leq_e A$ is a set of strings with $\mathrm{wt}(W) < 1$, then $U \setminus [W]^{\prec}$ has positive measure.

We build $A \subseteq 2^{<\omega}$ and set $U = [A]^{\preceq}$ via forcing.

Conditions are (S, q), where $S \subseteq 2^{<\omega}$, $q \leq 1$ and $\mu[S]^{\preceq} < q$.

A condition (T, p) extends (S, q) if $T \supseteq S$ and $p \leq q$.

We show that there is a dense set of conditions that force:

$$R_\Phi : \text{either } \mathrm{wt}(\Phi(A)) \geq 1 \text{ or } U \setminus [\Phi(A)]^{\prec} \text{ has positive measure.}$$

Separating $\langle A \rangle$-randomness from Solovay $\langle A \rangle$-randomness

Step 3: Building a class that is hard to cover.

Let σ be a string of large enough length n.

If σ enters A then the opponent responds by adding $g(\sigma)$ to $\Phi(A)$.

If instead $\tau \succeq \sigma$ enters A then the opponent:

1. Responds efficiently by covering τ differently and spending less weight.
2. Responds inefficiently by covering σ.

Consider $G_\sigma = \bigcup_{\tau \succeq \sigma} g(\sigma)$.

Case 1: G_σ has infinite weight for some σ of length n then we can add all extensions of σ to A.

Case 2: The opponent covers too many strings inefficiently:
$I = \{\rho \mid (\exists^\infty \tau)[g(\tau) \ni \rho]\}$ has weight ≥ 1.
Then we can add I to $\Phi(A)$ by spending very little measure.

Lowness for randomness

Definition

A set A is *low for randomness* if every 1-random is $\langle A \rangle$-random;

A is *low for Solovay randomness* if every 1-random is Solovay $\langle A \rangle$-random.

Proposition

The following two conditions are equivalent:

1. A is low for randomness.
2. Every $\Sigma^0_1\langle A \rangle$ class of measure < 1 is covered by a Σ^0_1 class of measure < 1.

Uncountably many low for randomness sets

Proposition

Every 1-generic set is low for randomness.

Proof.

For every enumeration operator Φ and natural number n, consider the c.e. set

$$W(\Phi, n) = \left\{\sigma \mid \mu[\Phi(\{x \mid \sigma(x) = 1\})]^{\preceq} > 1 - 2^{-n}\right\}.$$

If G meets $W(\Phi, n)$ then $\mu[\Phi(G)]^{\preceq} > 1 - 2^{-n}$.

If $[\Phi(G)]^{\preceq}$ is a $\Sigma^0_1\langle G\rangle$ class with measure < 1 then there must be some n such that G avoids $W(\Phi, n)$.

There is a $\sigma \preceq G$ such that

$$\mu[\Phi(\{x \mid \sigma(x) = 1\} \cup \mathbb{N}^{\geq|\sigma|})]^{\preceq} < 1.$$

□

Separating $\langle A \rangle$-random from upwards $\langle A \rangle$-random

Proposition

If A is weakly 1-generic relative to Z and $A \leq_e X \oplus \overline{X}$ then Z is not $\langle X \oplus \overline{X} \rangle$-random.

Fix a weakly 2-generic set A.

- A is weakly 1-generic relative to Chaitin's Ω.
- A is 1-generic and hence low for randomness.

Ω is $\langle A \rangle$-random, but not $\langle X \oplus \overline{X} \rangle$-random for any $X \oplus \overline{X} \geq_e A$.

Semi-computable sets

Definition (Jockusch)

A set A is semi-computable if it is a left cut is some computable linear ordering on the natural numbers.

Proposition (Jockusch)

Every total enumeration degree $\mathbf{x}$ contains $A \oplus \overline{A}$ for some semicomputable A. If $\mathbf{x}$ is nonzero then A and $\overline{A}$ are not c.e.

Theorem (Rozinas)

Every enumeration degree is the meet of two total enumeration degrees.

Low for randomness sets generate the e-degrees

Proposition

If A is semicomputable and not co-c.e. then A is low for randomness.

Proof.

Let A be a left cut in the computable ordering $L = (\mathbb{N}, \leq_L)$.

Suppose that $\mu[\Phi(A)]^{\preceq} < 1 - 2^{-n}$ and consider

$$W = \left\{x \mid \mu[\Phi(\{y \mid y \leq_L x\})]^{\preceq} \geq 1 - 2^{-n}\right\}.$$

$W \cap A = \emptyset$. As A is not co-c.e. it follows that $W \subset \overline{A}$.

Let $z \in \overline{A} \setminus W$, then $[\Phi(\{y \mid y \leq_L z\})]^{\preceq}$ is a Σ^0_1 class with measure < 1 that covers $[\Phi(A)]^{\preceq}$.

□

Not all quasiminimal are low for randomness

Proposition

For any sequence Z, there is a set A of quasi-minimal enumeration degree such that Z is not Solovay $\langle A \rangle$-random.

If Z is random then Z is random relative to all total sets reducible to A, but not Solovay $\langle A \rangle$-random.

Relativizing PA

Definition

X is PA above Y if and only if X computes an element in every nonempty $\Pi^0_1(Y)$ class.

A $\Pi^0_1\langle A\rangle$ class is the complement of some $\Sigma^0_1\langle A\rangle$ class.

Definition

A set A is $\langle PA\rangle$ *above* B if every nonempty $\Pi^0_1\langle B\rangle$ class contains an element Z whose characteristic function is enumeration reducible to A (i.e., $Z \oplus \overline{Z} \leq_e A$).

Self $\langle PA \rangle$ sets

Theorem

There is a set A such that A is $\langle PA \rangle$ above A.

Proof.

We construct A in stages as $\bigcup_e A_e$.
The set A_e has total columns $A^{[i]}$ for $i < e$ and finite columns $A^{[j]}$ for $j \geq e$.

1. First we try to force $[\Phi_e(A)]^{\preceq}$ to be 2^ω:

   If $[\Phi_e(A_e \cup \mathbb{N}^{\geq e})]^{\preceq} = 2^\omega$ then by compactness there is a finite set F such that $[\Phi_e(A_e \cup F)]^{\preceq} = 2^\omega$.

   Otherwise $2^\omega \setminus [\Phi_e(A_e \cup \mathbb{N}^{\geq e})]^{\preceq}$ is a nonempty $\Pi^0_1(\bigoplus_{i<e} A^{[i]})$-class contained in $2^\omega \setminus [\Phi_e(A)]^{\preceq}$.

2. Then we code in $A^{[e]}$ a set $X \oplus \overline{X}$ that is PA above $\bigoplus_{i<e} A_e^{[i]}$.

□

Randomness properties of self-$\langle PA \rangle$ sets

Proposition

If A is self-$\langle PA \rangle$ then there is neither a universal $\langle A \rangle$-test nor a universal Solovay $\langle A \rangle$-test.

Proposition

Let A be self-$\langle PA \rangle$ enumeration degree and U be a $\Sigma^0_1\langle A \rangle$ class U of measure < 1. There exists a set Y such that $Y \oplus \overline{Y} \leq_e A$ and a $\Sigma^0_1(Y)$ class V of measure < 1, such that $U \subseteq V$.

Corollary

If A is self-$\langle PA \rangle$ then for every X the following are equivalent:

1. X is $\langle A \rangle$-random;
2. X is Solovay $\langle A \rangle$-random;
3. X is Y-random for every Y, such that $Y \oplus \overline{Y} \leq_e A$.

Other proporties of self-$\langle PA \rangle$ sets

Proposition

If A is self-$\langle PA \rangle$ then the set of total degrees below A is a Scott set.

Proposition

Let $\mathcal{S}$ be a countable Scott set of total enumeration degrees. There exists a self-$\langle PA \rangle$ set A such that $\mathcal{S}$ is the set of total enumeration degrees below A.

Proposition

If X is PA above Y if and only if then there is a self-$\langle PA \rangle$ A such that $Y \oplus \overline{Y} <_e A <_e X \oplus \overline{X}$.

Continuous degrees

"Does every continuous function on the unit interval have a name of least Turing degree?"

Definition (Miller)

There is a way to assign to every continuous function f on the unit interval an enumeration degree $c(f)$, so that the total degrees above $c(f)$ correspond to Turing degrees of names for f.
The degree $c(f)$ is called a *continuous enumeration degree*.

Theorem (Miller)

There are non-total continuous degrees.

Non-total continuous degrees

Theorem (Miller)

1. The total degrees below a non-total continuous degree form a Scott ideal.
2. Every Scot ideal can be realized as the set of total enumeration degrees below some non-total continuous degree.
3. For $\mathbf{a}$, $\mathbf{b}$-total, "$\mathbf{b}$ is PA above $\mathbf{a}$" if and only if there is a non-total continuous degree $\mathbf{x}$ such that $\mathbf{a} < \mathbf{x} < \mathbf{b}$.

Proposition

If A has continuous degree, then there is a universal (Martin-Löf) $\langle A \rangle$-test.

If A has continuous degree, then Z is $\langle A \rangle$-random iff Z is X-random for some X such that $A \leq_e X \oplus \overline{X}$.

The end

Thank you!