Defining totality in the enumeration degrees

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joint work with Cai, Ganchev, Lempp and Miller

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The total enumeration degrees

The structure of the enumeration degrees is an upper semi lattice with jump operation which extends the structure of the Turing degrees. It arises naturally from enumeration reducibility, a notion introduced by Friedberg and Rogers in 1959.

The total enumeration degrees are the image of the Turing degrees under their natural, structure preserving embedding into the enumeration degrees.

Question (Rogers 67)

Is the set of total enumeration degrees first order definable in the structure of the enumeration degrees?

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Main ingredient: An analysis of the automorphism group of the enumeration degrees, based on Slaman and Woodin's framework.

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if A is c.e. then $A \leq_e B$ via $W = \{\langle x, \emptyset\rangle \mid x \in A\}$.

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- $\mathcal{D}_e = \langle D_e, \leq, \vee, ' \mathbf{0}_e\rangle$ is an upper semi-lattice with least element and jump operation.

What connects $\mathcal{D}_T$ and $\mathcal{D}_e$

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$$(\mathcal{D}_T, \leq_T, \vee, ', \mathbf{0}_T) \cong (\mathcal{TOT}, \leq_e, \vee, ', \mathbf{0}_e) \subseteq (\mathcal{D}_e, \leq_e, \vee, ', \mathbf{0}_e)$$

Semi-computable sets

Definition (Jockusch)

A set of natural numbers A is semi-computable if there is a total computable selector function s_A, such that $s_A(x, y) \in \{x, y\}$ and if $\{x, y\} \cap A \neq \emptyset$ then $s_A(x, y) \in A$.

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Theorem (Arslanov, Cooper, Kalimullin)

If A is a semi-computable set then for every X:

$$(d_e(X) \vee d_e(A)) \wedge (d_e(X) \vee d_e(\overline{A})) = d_e(X).$$

Kalimullin pairs

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- If $(m, n) \in W$ then $m \in A$ or $n \in B$.
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2. If A is a semi-computable set, then $\{A, \overline{A}\}$ is a $\mathcal{K}$-pair: $W = \{(m, n) \mid s_A(m, n) = m\}$.

Properties of $\mathcal{K}$-pairs

Proposition

Fix A. The set of all B, such that $\{A, B\}$ is a $\mathcal{K}$-pair is closed under least upper bound and downwards closed with respect to enumeration reducibility, i.e. it is an ideal.

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Definability of the enumeration jump

Theorem (Kalimullin)

A pair of sets A, B are a $\mathcal{K}$-pair if and only if their enumeration degrees $\mathbf{a}$ *and* $\mathbf{b}$ *satisfy:*

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$\mathbf{0}'_e$ *is the largest degree which can be represented as the least upper bound of a triple* $\mathbf{a}, \mathbf{b}, \mathbf{c}$*, such that* $\mathcal{K}(\mathbf{a}, \mathbf{b})$*,* $\mathcal{K}(\mathbf{b}, \mathbf{c})$ *and* $\mathcal{K}(\mathbf{c}, \mathbf{a})$*.*

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Corollary (Kalimullin)

The enumeration jump is first order definable in $\mathcal{D}_e$.

Definability in the local structure of the enumeration degrees

Theorem (Ganchev, S)

The class of $\mathcal{K}$-pairs below $\mathbf{0}'_e$ is first order definable in $\mathcal{D}_e(\leq \mathbf{0}'_e)$.

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The classes of the:

1. *Downwards properly Σ^0_2 enumeration degrees;*
2. *Upwards properly Σ^0_2 enumeration degrees;*
3. *Low enumeration degrees;*

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A semi-computable set and its complement form a maximal $\mathcal{K}$-pair.

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Theorem (Jockusch)

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In $\mathcal{D}_e(\leq \mathbf{0}'_e)$ a nonzero degree is total if and only if it is the least upper bound of a maximal $\mathcal{K}$-pair.

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A nonzero degree is total if and only if it is the least upper bound of a maximal $\mathcal{K}$-pair.

Main theorem

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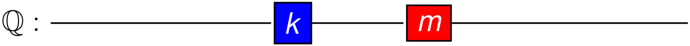
Proof:

- Let $\{A, B\}$ be a nontrivial $\mathcal{K}$-pair witnessed by W.
- C will be a left cut in the computable linear ordering $(\mathbb{Q}, \leq)$.
- We take two copies of the natural numbers: $\color{red}{\mathbb{N}}$ for A and $\color{blue}{\mathbb{N}}$ for B.
- Using W we dynamically label elements of $\mathbb{Q}$ with the elements of $\color{red}{\mathbb{N}} \cup \color{blue}{\mathbb{N}}$.
- A rational number can have at most one label, but many rationals will be given the same label.
- Ultimately A will be the set $\{m \mid \exists q \in C(q \text{ is labeled by } \color{red}{m})\}$ and B will be the set $\left\{k \mid \exists q \in \overline{C}(q \text{ is labeled by } \color{blue}{k})\right\}$.

The main rule for labeling

While $(m, k) \notin W$, keep blue labels for k to the left of red labels for m.



The main rule for labeling

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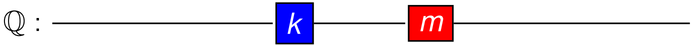

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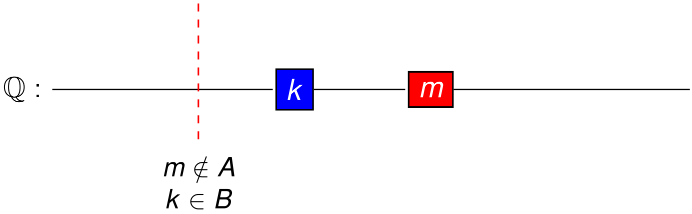

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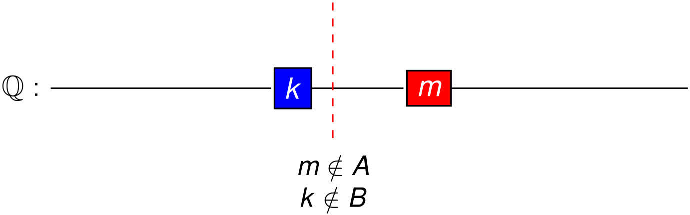

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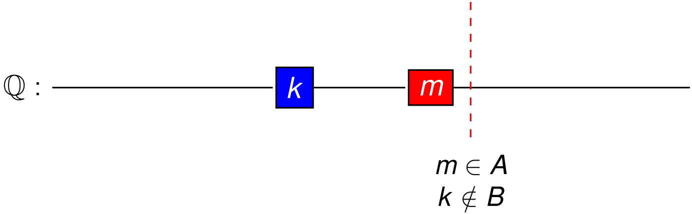

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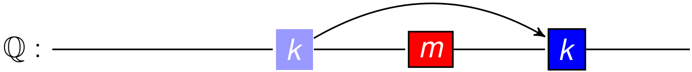

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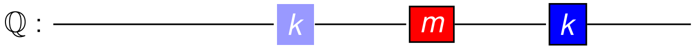

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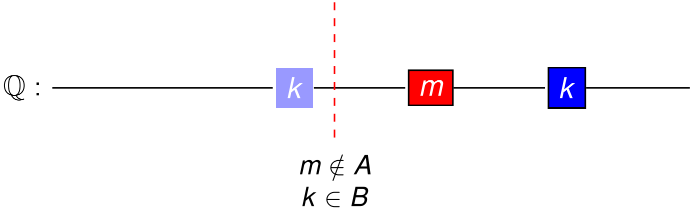

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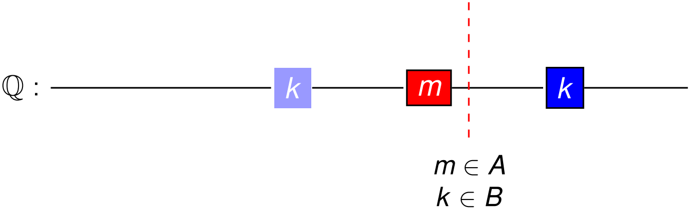

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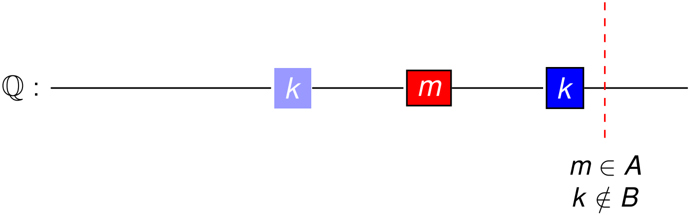
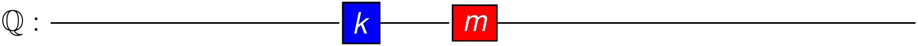


The checkerboard scenario

W	n	m
k		
j		

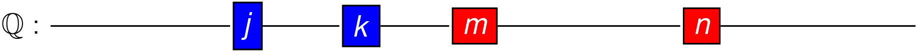
Initially no pair is in W.
First we label for the pair (m, k).



The checkerboard scenario

W	n	m
k		
j		

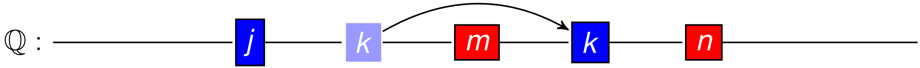
Initially no pair is in W.
Then we label for the pair (n, j).



The checkerboard scenario

W	n	m
k		✓
j		

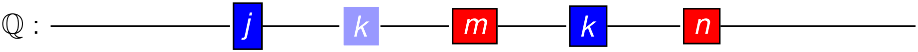
The pair (m, k) enters W.
We move the current label for k.



The checkerboard scenario

W	n	m
k		✓
j	✓	

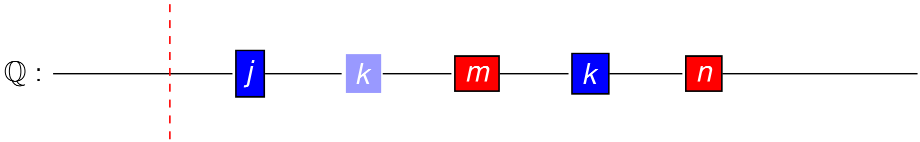
The pair (n, j) enters W.
There is no possible move.



The checkerboard scenario

W	n	m
k		$\checkmark$
j	$\checkmark$	

The pair (n, j) enters W.
There is no possible move.



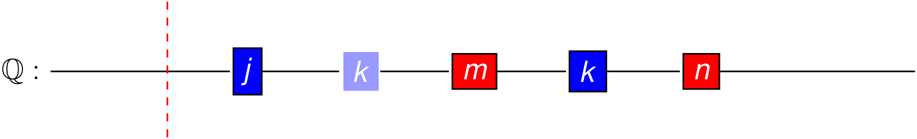
If this situation persists then we have the following:

$$j \in B$$

The checkerboard scenario

W	n	m
k		✓
j	✓	

The pair (n, j) enters W.
There is no possible move.



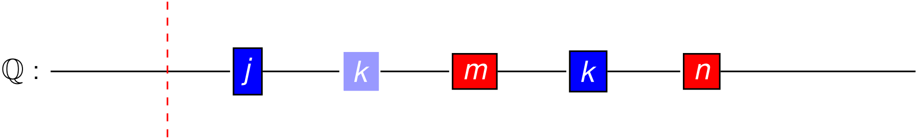
If this situation persists then we have the following:

$$j \in B \Rightarrow m \notin A$$

The checkerboard scenario

W	n	m
k		$\checkmark$
j	$\checkmark$	

The pair (n, j) enters W.
There is no possible move.



If this situation persists then we have the following:

$$j \in B \Rightarrow m \notin A \Rightarrow k \in B$$

The checkerboard scenario

W	n	m
k		✓
j	✓	

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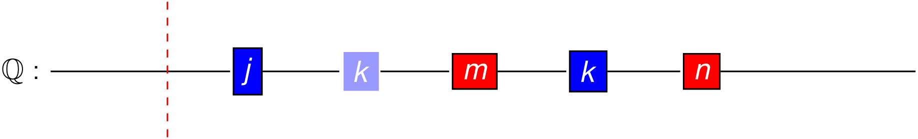


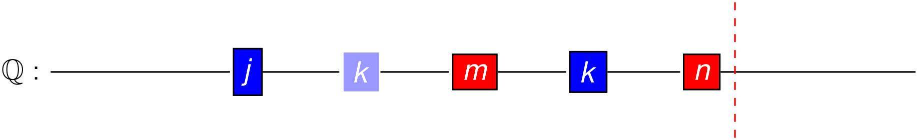
If this situation persists then we have the following:

$$j \in B \Rightarrow m \notin A \Rightarrow k \in B \Rightarrow n \notin A$$

The checkerboard scenario

W	n	m
k		$\checkmark$
j	$\checkmark$	

The pair (n, j) enters W.
There is no possible move.



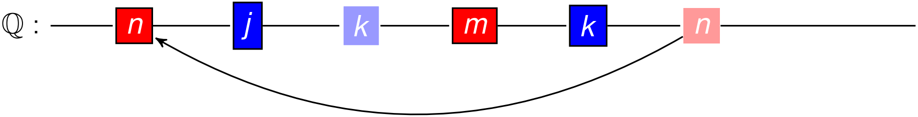
Or the following:

$$n \in A \Rightarrow k \notin B \Rightarrow m \in A \Rightarrow j \notin B$$

The checkerboard scenario

W	n	m
k	✓	✓
j	✓	

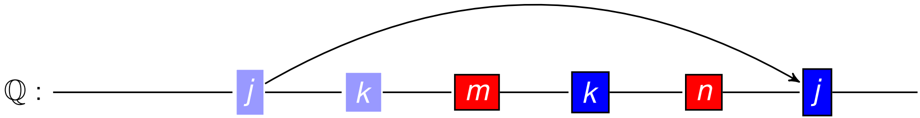
If the pair (n, k) enters W, the conflict is resolved.
We move the current label for n.



The checkerboard scenario

W	n	m
k		✓
j	✓	✓

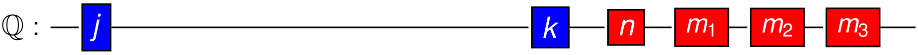
Or if the pair (m, j) enters W, the conflict is resolved.
We move the current label for j.



The infinite checkerboard scenario

W	n	m_1	m_2	$\ldots$
k				
j				

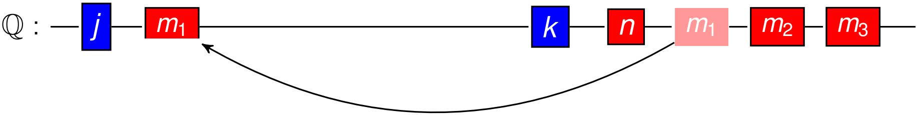
Initially no pair is in W.



The infinite checkerboard scenario

W	n	m_1	m_2	$\dots$
k		$\checkmark$		
j				

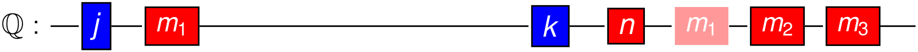
The pair (m_1, k) enters W.
We move the current label for m_1.



The infinite checkerboard scenario

W	n	m_1	m_2	$\dots$
k		✓		
j	✓			

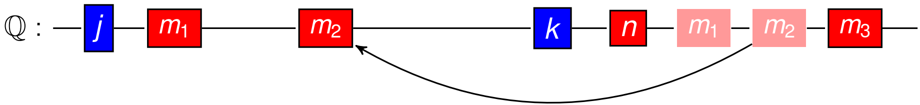
The pair (n, j) enters W.
No move is possible.



The infinite checkerboard scenario

W	n	m_1	m_2	$\dots$
k		✓	✓	
j	✓			

The pair (m_2, k) enters W.
We move the current label for m_2.

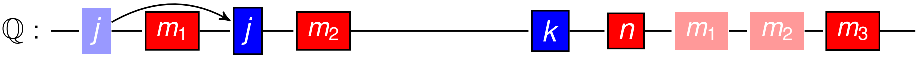


A second checkerboard scenario has appeared before the first one is resolved.

The infinite checkerboard scenario

W	n	m_1	m_2	$\dots$
k		✓	✓	
j	✓	✓		

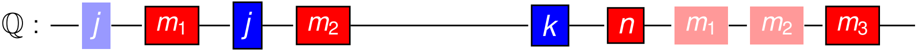
The pair (m_1, j) enters W.
We move the current label for j.



The first checkerboard turns out temporary.
The same thing repeats with $m_3, m_4, \dots$

The infinite checkerboard scenario

W	n	m_1	m_2	$\dots$
k		✓	✓	
j	✓	✓		

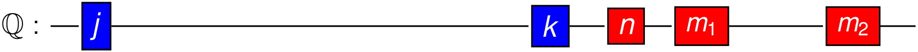


It is possible that $n \in A$ and $j \notin B$, but the dynamic of the enumeration of W prevents us from ever switching the order of these labels. Thus no correct cut is possible.

The infinite checkerboard scenario: resolved

W	n	m_1	m_2	$\ldots$
k				
j				

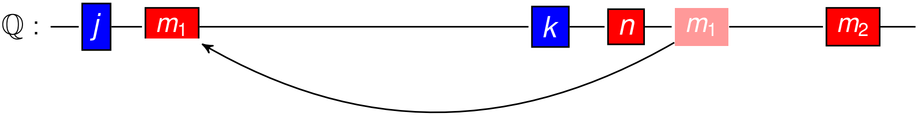
We introduce priority between pairs:
$(n, j) < (m_1, k) < (m_2, k)$.



The infinite checkerboard scenario: resolved

W	n	m_1	m_2	$\ldots$
k		✓		
j				

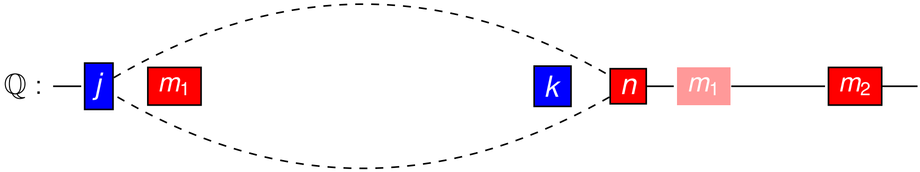
The pair (m_1, k) enters W.
We move the current label for m_1.



The infinite checkerboard scenario

W	n	m_1	m_2	$\dots$
k		$\checkmark$		
j	$\checkmark$			

The pair (n, j) enters W.
We announce the interval $[j, n]$ as a *dead zone*.

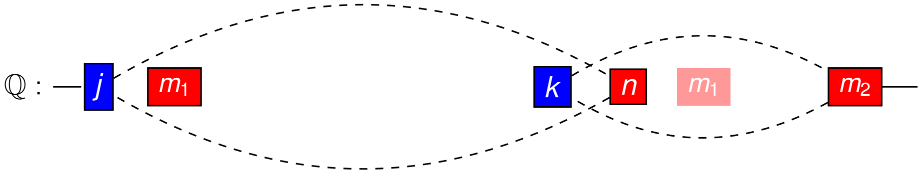


No pair of lower priority can label inside the dead zone.

The infinite checkerboard scenario: resolved

W	n	m_1	m_2	$\dots$
k		$\checkmark$	$\checkmark$	
j	$\checkmark$			

The pair (m_2, k) enters W.
The interval (k, m_2) becomes a dead zone.

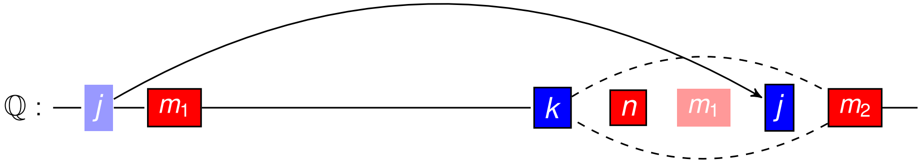


We cannot label m_2 inside the dead zone, as the pair (m_2, k) has lower priority than (n, j).

The infinite checkerboard scenario: resolved

W	n	m_1	m_2	$\dots$
k		✓	✓	
j	✓	✓		

The pair (m_1, j) enters W.
We move the current label for j.



We can label j inside the deadzone $[k, m_2]$, as (n, j) has higher priority.
The first checkerboard is resolved.
We now have a new checkerboard scenario of lower priority.

The Core of the argument

Say that two current labels are connected if the rationals labeled with them are in the same connected union of permanent dead zones.

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Lemma (Dead zone lemma)

- *If the current label m is connected to the current label k then* $m \in A \Leftrightarrow k \notin B$.
- *If the current label m is connected to the current label n then* $m \in A \Leftrightarrow n \in A$.
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C is the set of all rational q such that for some $k \in \overline{B}$:

- q is to the left of some k-labelled rational or
- q is in the same permanent deadzone with the current label k.

The relation *c.e. in*

Definition

A Turing degree $\mathbf{a}$ is *c.e. in* a Turing degree $\mathbf{x}$ if some $A \in \mathbf{a}$ is c.e. in some $X \in \mathbf{x}$.

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Theorem (Ganchev, S)

Let **a** *and* **x** *be Turing degrees such that* **a** *is not c.e. Then* **a** *is c.e. in* **x** *if and only if there is a nontrivial* $\mathcal{K}$*-pair* $\{C, \overline{C}\}$ *such that* $d_e(C) \leq_e \iota(\mathbf{x})$ *and* $\iota(\mathbf{a}) = d_e(C) \vee d_e(\overline{C})$.

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Thus for non c.e. Turing degrees $\mathbf{a}$, we have that $\mathbf{a}$ is c.e. in $\mathbf{x}$ if and only if there is a maximal $\mathcal{K}$-pair $\{\mathbf{c}, \mathbf{d}\}$ such that $\mathbf{c} \leq_e \iota(\mathbf{x})$ and $\iota(\mathbf{a}) = \mathbf{c} \vee \mathbf{d}$.

The images of c.e. degrees

If $\mathbf{a} \in \mathcal{D}_T$ is c.e. then for every $\mathbf{b}$ we have that $\mathbf{a} \vee \mathbf{b}$ is c.e. in $\mathbf{b}$.

The images of c.e. degrees

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Lemma (Cai and Shore)

If $\mathbf{a} \in \mathcal{D}_T$ is not c.e and $\mathbf{b}$ is 2-generic in $\mathbf{a}$ then $\mathbf{a} \vee \mathbf{b}$ is not c.e. in $\mathbf{b}$.

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Theorem (Cai, Ganchev, Lempp, Miller, S)

The set $\mathcal{CE} = \{\iota(\mathbf{a}) \mid \mathbf{a} \in \mathcal{D}_T \text{ is c.e.}\}$ is first order definable in $\mathcal{D}_e$.

The images of c.e. degrees

If $\mathbf{a} \in \mathcal{D}_T$ is c.e. then for every $\mathbf{b}$ we have that $\mathbf{a} \vee \mathbf{b}$ is c.e. in $\mathbf{b}$.

Lemma (Cai and Shore)

If $\mathbf{a} \in \mathcal{D}_T$ is not c.e and $\mathbf{b}$ is 2-generic in $\mathbf{a}$ then $\mathbf{a} \vee \mathbf{b}$ is not c.e. in $\mathbf{b}$.

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Corollary

The image of the relation c.e. in in the enumeration degrees is first order definable.

The automorphism analysis of degree structures

Theorem (Slaman and Woodin (95))

1. *Every member of* $Aut(\mathcal{D}_T)$ *is the identity on the cone above* $\mathbf{0}''$.
2. $Aut(\mathcal{D}_T)$ *is countable, every member has an arithmetic presentation.*
3. $\mathcal{D}_T$ *has a finite automorphism basis.*
4. *Every relation on* $\mathcal{D}_T$ *induced by a degree invariant relation definable in Second order arithmetic is definable in* $\mathcal{D}_T$ *from parameters.*
5. *Every relation on* $\mathcal{D}_T$ *induced by a degree invariant relation definable in Second order arithmetic and invariant under automorphisms is definable in* $\mathcal{D}_T$.

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Enumeration induced automorphisms

Theorem (Selman)

A is enumeration reducible to B if and only if
$\{\mathbf{x} \in \mathcal{D}_T \mid A \text{ is c.e. in } \mathbf{x}\} \supseteq \{\mathbf{x} \in \mathcal{D}_T \mid B \text{ is c.e. in } \mathbf{x}\}$.

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Definition

An automorphism π of $\mathcal{D}_T$ is *enumeration induced* if there is an automorphism of $\mathcal{D}_e$, π_e such that f $\pi(\mathbf{x}) = \iota^{-1}(\pi_e(\iota(\mathbf{x})))$.

Extendable automorphisms of $\mathcal{D}_T$

Definition

An automorphism π of $\mathcal{D}_T$ is *enumeration extendible* if there exists a function $P\colon 2^\omega \to 2^\omega$ such that for every set of natural numbers A and every Turing degree $\mathbf{x}$, we have that A is c.e. in $\mathbf{x}$ if and only if $P(A)$ is c.e. in $\pi(\mathbf{x})$.

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Question

Is every automorphism of the Turing degrees enumeration extendable?

Thank you

Gian-Carlo Rota

According to ...[the "one-shot"] ... view, mathematics would consist of a succession of targets, called problems, which mathematicians would be engaged in shooting down by well-aimed shots. But where do problems come from, and what are they for? If the problems of mathematics were not instrumental in revealing a broader truth, then they would be indistinguishable from chess problems or crossword puzzles. Mathematical problems are worked on because they are pieces of a larger puzzle.

Ivan Soskov

The good puzzles are the ones that will never be completely solved.