Enumeration Degrees

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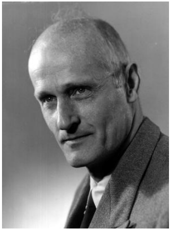
The Work of Alan Turing

- A function f is computable if there is Turing Machine which computes it.
- Using Oracle Turing machines we can compare the information content of different problems.

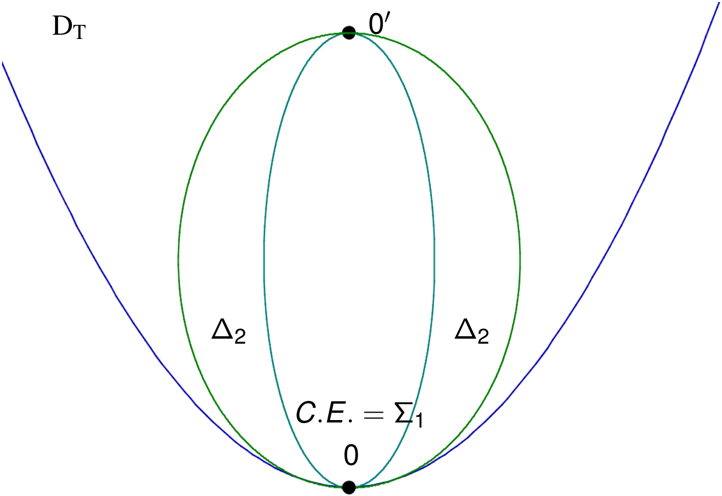
Kleene and Post



- Turing reducibility: A set A is Turing reducible to a set B if there is an oracle Turing machine which computes A when using oracle B.
- $A \equiv_T B$ if $A <_T B$ and $B <_T A$.
- The Turing degree of A is $d_T(A) = \{B | B \equiv_T A\}$.
- $\mathrm{D}_T = (D_T, \vee, 0)$ the semi-lattice of the Turing Degrees.

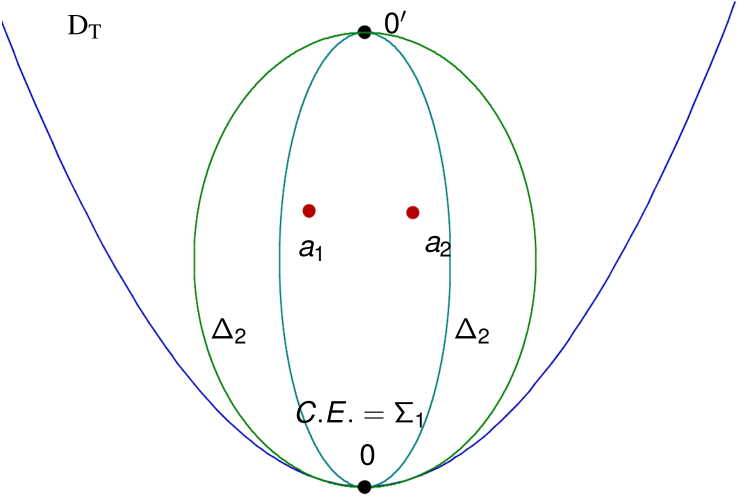
The c.e. degrees and Post's Program

- Problems arising outside of computability: computably enumerable.
- Construct a c.e. set that is neither computable nor complete by defining some structural property of this set.



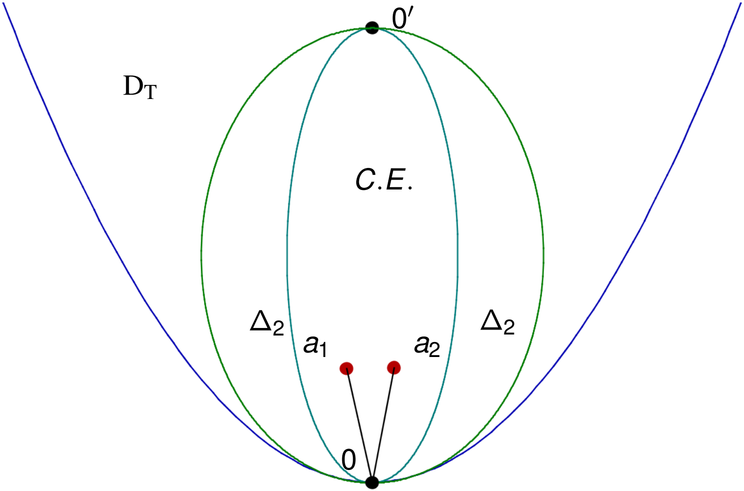
The solution to Post's Program and the Priority method

- Friedberg and Muchnik independently construct two incomparable c.e. sets.



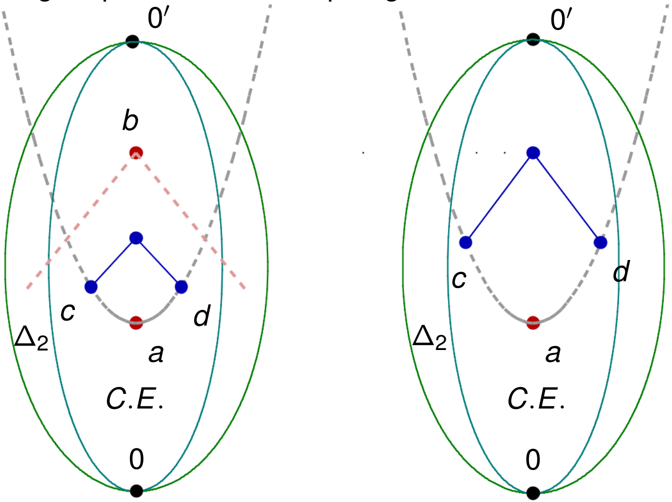
Infinite Injury Priority

- Sacks proved that the c.e. degrees are dense.
- Shoenfield's conjecture: The c.e. degrees form a countably infinite homogeneous semi-lattice.



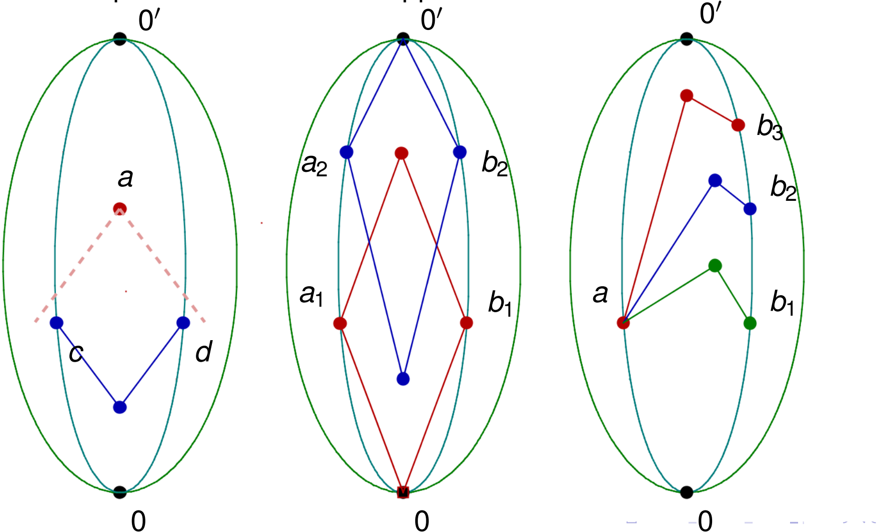
Lachlan's Monster theorem

There exist c.e. degrees $\mathbf{a} < \mathbf{b}$ such that $\mathbf{b}$ can not be split in the c.e. degrees above $\mathbf{a}$.
Harrington proved that the top degree can be taken to be $\mathbf{0}'$.



Further surprising results

- Lachlan's Non-bounding theorem.
- Lachlan's Non-diamond theorem.
- Cooper and Yates: Non-cuppable theorem.



The enumeration degrees

Definition

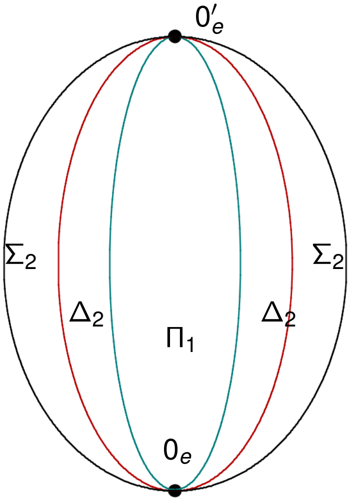
1. $A \leq_e B$ if there is a c.e. set Φ such that

$$n \in A \Leftrightarrow \exists D(\langle n, [D] \rangle \in \Phi \wedge D \subseteq B).$$

2. A is enumeration equivalent to B ($A \equiv_e B$) if $A \leq_e B$ and $B \leq_e A$.
3. Let $d_e(A) = \{B | A \equiv_e B\}$.
4. $(D_e, <, \cup, ', 0_e)$ is the semi-lattice of the enumeration degrees with the jump operator.

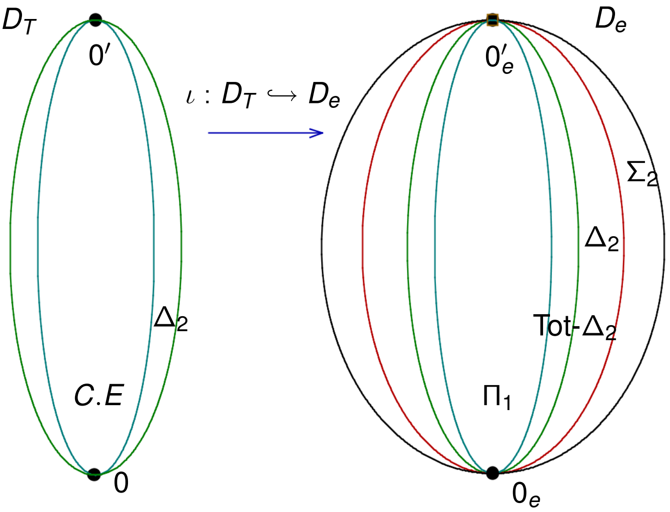
The Local structure

Below $0'_e$ are exactly the Σ_2 e-degrees.



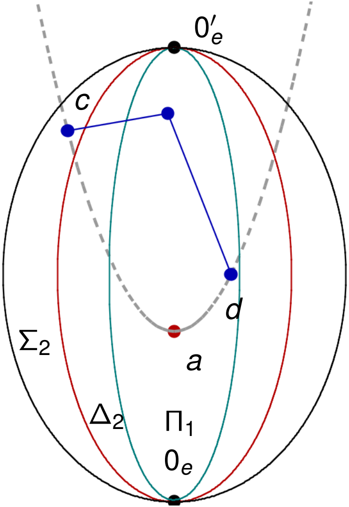
Natural Embedding

There is an order theoretic embedding of D_T into D_e.



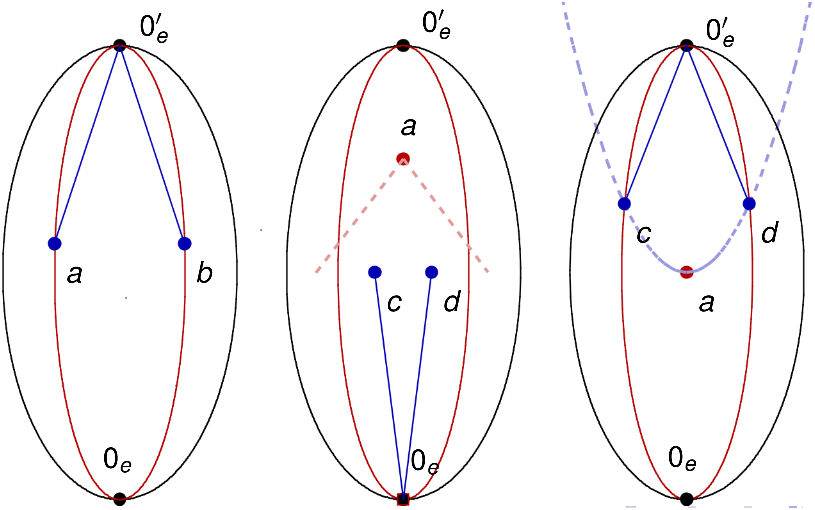
An extension of Harrington's Theorem

Cooper and M.S.: There exists a Π_1 e-degree $\mathbf{a} < \mathbf{0'_e}$ such that no pair of a Π_1 and a Σ_2 e-degree above $\mathbf{a}$ split $\mathbf{0'_e}$.



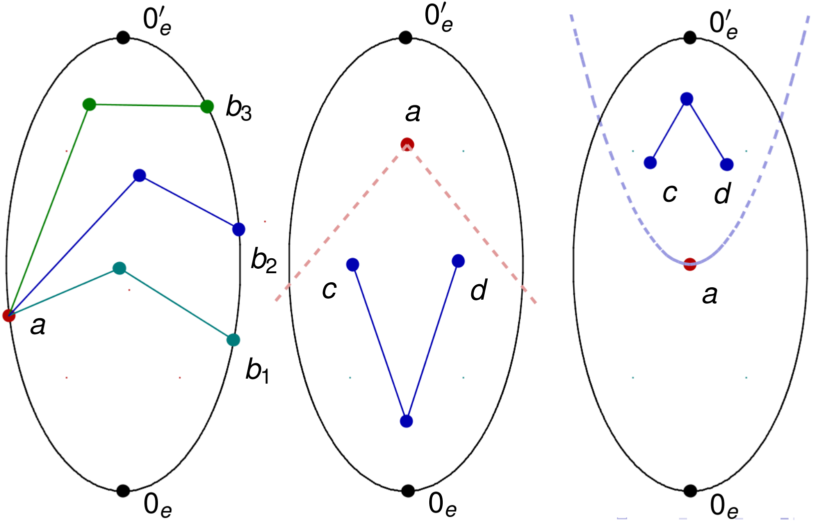
The Δ_2 - e-degrees

- Cooper, Sorbi, Yui: The Cupping theorem.
- Cooper, Li, Sorbi, Yang: The Bounding theorem.
- Arslanov, Sorbi: The Splitting theorem.



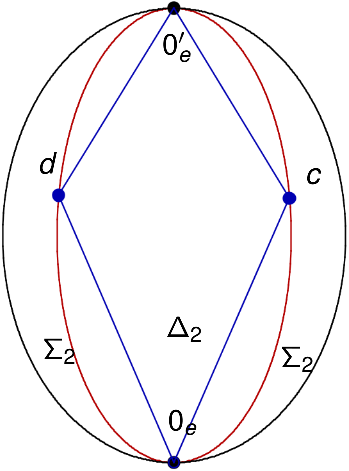
The Σ_2 e-degrees

- Cooper, Sorbi, Yui: The Non-cupping theorem.
- Cooper, Li, Sorbi, Yang: The Non-bounding theorem.
- M.S.: The Non-splitting theorem.



What is the connection?

- Cooper's conjecture: The Σ_2 e-degrees elementary equivalent to the c.e. degrees.
- Ahmad refutes the conjecture.
- Is there a mathematical reason for the connection?



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