Logic and degrees

Mariya I. Soskova[1]

Sofia University

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Computable sets and functions

Definition

A function $f : \mathbb{N}^k \to \mathbb{N}$ is computable if there is an algorithm which takes as input natural numbers $n_1, n_2 \ldots n_k$ and outputs $f(n_1, n_2, \ldots, n_k)$.

Example

- Addition, multiplication and exponentiation on the natural numbers.
- The sequence of the digits of π in its decimal expansion.
- The decision function (characteristic function) for the set of prime numbers.

Question (Hilbert's Entscheidungsproblem 1928)

Is the set of all *logically valid* formulas in first order logic computable?

First order logic

Propositional logic deals with simple declarative statements.

Example

Ladybugs are purple or green. Ladybugs are not green. Therefore ladybugs are purple.

$$((P \vee Q) \ \& \ \neg Q) \to P$$

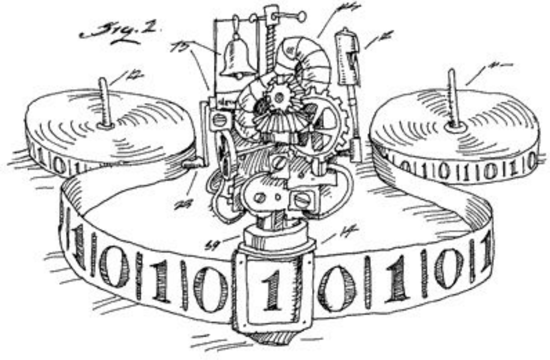
First order logic introduces the idea of prediacates, variables and quantifiers.

Example

All ladybugs are purple or green. Mary is a ladybug. Mary is not green. Therefore there are purple things.

$$(\forall X (L(X) \to (P(X) \vee Q(X))) \ \& \ L(Mary) \ \& \ \neg Q(Mary)) \to \exists X P(X).$$

Computable sets and functions



Definition (Turing, Church 1936)

A function $f : \mathbb{N}^k \to \mathbb{N}$ is computable if there is a **computer program** which takes as input natural numbers $n_1, n_2 \ldots n_k$ and outputs $f(n_1, n_2, \ldots, n_k)$.

Computably enumerable sets

Definition

A set $A \subseteq \mathbb{N}$ is *computably enumerable* (c.e.) if it can be enumerated by a computer program.

Example (Davis, Matyasevich, Putnam, Robinson 1970)

A set $S \subseteq \mathbb{N}$ is Diophantine if $S = \{n \mid \exists \bar{m}(P(n, \bar{m}) = 0)\}$, where $P(n, \bar{m})$ is a polynomial with integer coefficients.

E.g. $S = \left\{n \mid \exists m_1 \exists m_2 (n^4 - 55m_1^2 + 11m_2 - 16 = 0)\right\}$.The number 2 belongs to S, as witnessed by $m_1 = 1$ and $m_2 = 5$.

The Diophantine sets are exactly the c.e. sets.

Example (Novikov, Boone 1955)

The word problem for a finitely presented group is c.e.

Every c.e. set can be coded as the word problem for a finitely presented group.

Computably enumerable sets

Example (Hilbert's Entscheidungsproblem 1928)

The set of all logically valid formulas in first order logic is c.e.

Theorem (Gödel's completeness theorem)

The logically valid formulas in first order logic are exactly the ones that have formal proofs (deductions).

Question

Are c.e. sets and computable sets the same thing?

- If A is computable then A is c.e.
- If A and $\overline{A}$ are c.e. then A is computable.

An incomputable c.e. set

No general procedure for bug checks succeeds.
Now, I won't just assert that, I'll show where it leads:
I will prove that although you might work till you drop,
you cannot tell if a computation will stop.
…

From: "Scooping the loop snooper", G. Pullum

An incomputable c.e. set

- A program P can be coded by a natural number e.
- Not all programs P_e compute total functions, some are undefined (run forever) on certain inputs.

Definition

The *halting set* is the set K of all codes e for programs P such that P halts on input e.

- The set K is computably enumerable.
- Suppose K is computable by a program P.

Well, the truth is that P cannot possibly be,
because if you wrote it and gave it to me,
I could use it to set up a logical bind
that would shatter your reason and scramble your mind.

An incomputable c.e. set

- Suppose K, the set of all e such that P_e halts on e, is computable by a program P.
- We can define a program Q that on input e runs P to check if $e \in K$. If the answer is "yes" then Q goes into an infinite loop. If the answer is "no" then Q halts and outputs 42.
- Consider what happens when we run Q on Q's code $\hat{e}$:
  Q halts on $\hat{e}$ if and only if $e \notin K$ if and only if Q does not halt on $\hat{e}$.

Corollary (Church, Turing 1936)

The answer to Hilbert's Entsheidungs problem is 'NO'.

Comparing the information content of sets

Consider a program that has access to an external database, an *oracle*.
During its computation the program can ask the oracle membership questions: does n belong to you or not?

Definition (Post 1944)

$A \leq_T B$ if and only if there is a program that computes the elements of A using B as an oracle.

Example

Consider a Diophantine set $S = \{n \mid (\exists \bar{m}) P(n, \bar{m}) = 0\}$. Let $P_{e(n)}$ be the program that ignores its input and for a listing $\{\bar{m}_i\}_{i \in \mathbb{N}}$ of all tuples $\bar{m}_i$ calculates $P(n, \bar{m}_0), P(n, \bar{m}_1), \ldots$ until it sees that the result is 0 and then halts. Then $S \leq_T K$ by the program that on input n computes $e(n)$ and asks the oracle whether $e(n) \in K$.

Comparing the information content of sets

Definition (Friedberg and Rogers 1959)

$A \leq_e B$ if there is a program that transforms an enumeration of B to an enumeration of A.

The program is an infinite table of axioms of the sort:

$$\textit{If } \{x_1, x_2, \dots, x_k\} \subseteq B \textit{ then } x \in A.$$

Example

Let $S = \{n \mid (\exists \bar{m}) P(n, \bar{m}) = 0\}$ again. $\overline{S} \leq_e \overline{K}$ by the program that consists of the axioms:

$$\textit{If } \{e(n)\} \subseteq \overline{K} \textit{ then } n \in \overline{S}.$$

Degree structures

Definition

1. $A \equiv B$ if and only if $A \leq B$ and $B \leq A$.
2. $\mathbf{d}(A) = \{B \mid A \equiv B\}$.
3. $\mathbf{d}(A) \leq \mathbf{d}(B)$ if an only if $A \leq B$.
4. Let $A \oplus B = \{2n \mid n \in A\} \cup \{2n+1 \mid n \in B\}$. Then $\mathbf{d}(A \oplus B) = \mathbf{d}(A) \vee \mathbf{d}(B)$.

And so we have two orders:

1. The Turing degrees $\mathcal{D}_T$ with least element $\mathbf{0}_T$ consisting of all computable sets.
2. The enumeration degrees $\mathcal{D}_e$ with least element $\mathbf{0}_e$ consisting of al c.e. sets.

The jump operation

The halting set K is an example of an incomputable set.
Let K_A be the set of all e such that the oracle program P_e halts on e when using A as an oracle.
Then $A <_T K_A$.

Definition

The jump of a Turing degree is $\mathbf{d}_T(A)' = \mathbf{d}_T(K_A)$.

We can apply a similar construction within the enumeration degrees to obtain the enumeration jump of an enumeration degree.

We always have $\mathbf{a} < \mathbf{a}'$.

What connects $\mathcal{D}_T$ and $\mathcal{D}_e$

Proposition

$A \leq_T B \Leftrightarrow A \oplus \overline{A} \leq_e B \oplus \overline{B}$.

The embedding $\iota : \mathcal{D}_T \to \mathcal{D}_e$, defined by $\iota(\mathbf{d}_T(A)) = \mathbf{d}_e(A \oplus \overline{A})$, preserves the order, the least upper bound, and the jump operator.

$\mathcal{TOT} = \iota(\mathcal{D}_T)$ is the set of total enumeration degrees.

Properties of the degree structures

Similarities

1. Both $\mathcal{D}_T$ and $\mathcal{D}_e$ are uncountable structures with least element and no greatest element.
2. They are partial orders with uncountable chains and antichains.
3. Every pair of degrees has a least upper bound, but not always a greatest lower bound.

Differences

1. (Spector 1956) In $\mathcal{D}_T$ there are *minimal degrees*, nonzero degrees $\mathbf{m}$ such that the interval $(\mathbf{0}_T, \mathbf{m})$ is empty.
2. (Gutteridge 1971) $\mathcal{D}_e$ is downwards dense.

Definability

Definition

A set B is definable in a structure $\mathcal{A}$ if there is a formula $\varphi(x)$ (in the language of the structure) such that $n \in B$ if and only if $\varphi(n)$ is true in $\mathcal{A}$.

This concept naturally extends to definable relations and functions.

Example

In the field of real numbers $\mathcal{R} = (\mathbb{R}, 0, 1, +, \times)$ the set positive numbers is definable by the formula $\varphi_{\mathbb{R}^+}(x) : (\exists y)(x = y \times y) \ \&\ x \neq 0$.

In first order arithmetic $\mathcal{N} = (\mathbb{N}, 0, 1, +, \times)$ the set of prime numbers is definable by the formula
$\varphi_{prime}(x) : x \neq 1 \wedge (\forall y)((\exists z)(y \times z = x) \rightarrow (y = 1 \vee y = x))$.

In the partial order of the Turing degrees $(\mathcal{D}_T, \leq_T, \mathbf{0}_T)$ the set of minimal degrees is definable by the formula $\varphi_{min}(x)$:
$x \neq \mathbf{0}_T \wedge (\forall y)(y \leq_T x \rightarrow (y = x \vee y = \mathbf{0}_T))$.

Arithmetic vs Degrees

Second order arithmetic $\mathcal{Z}_2 = (\mathbb{N}, \mathcal{S}, \in, 0, 1, +, \times)$ is $\mathcal{N}$ with an additional sort for sets of natural numbers and a membership relation.

Example

In $\mathcal{Z}_2$ the pairs of sets X and Y such that X is the set of prime numbers that belong to Y is definable by $\psi(X, Y) : (\forall n)(n \in X \Leftrightarrow \varphi_{prime}(n) \;\&\; n \in Y)$.

Classical results due to Kleene and Post show that the relations and functions $\leq_T$, $\leq_e$, $\oplus$, K_A are definable in second order arithmetic.

Do they translate to definable relations and functions in our degree structures?

Example

The function that maps A and B to $A \oplus B$ is definable in $\mathcal{Z}_2$ by $\varphi(X, Y, Z)$: $(\forall n)(n \in X \leftrightarrow n + n \in Z) \wedge (n \in Y \leftrightarrow n + n + 1 \in Z)$.

The function that maps $\mathbf{d}(A)$ and $\mathbf{d}(B)$ to $\mathbf{d}(A \oplus B)$ is definable by $\psi(x, y, z)$: $x \leq z \;\wedge\; y \leq z \wedge (\forall u)(x \leq u \;\wedge\; y \leq u \to z \leq u)$.

Rogers' questions from 1969

Question 1: Is the jump operator definable in $\mathcal{D}_T$?

Definition

An automorphism of a structure $\mathcal{A}$ is a bijection on $\mathcal{A}$ that preserves all of the structure.

Example

Consider $\mathbb{Z} = (\mathbb{Z}, \leq)$. The function $f(z) = z + 1$ is an automorphism because it is a bijection and $z_1 \leq z_2$ if and only if $f(z_1) \leq f(z_2)$.

If a structure $\mathcal{A}$ has an automorphism that maps an element in a set X to an element outside X then X is not definable.

Question 2: Are there any nontrivial automorphisms of $\mathcal{D}_T$ or $\mathcal{D}_e$?

Question 3: Is the copy of $\mathcal{D}_T$ in $\mathcal{D}_e$ (i.e. the total enumeration degrees) definable in $\mathcal{D}_e$?

Definability in the Turing degrees

Theorem (Slaman, Woodin 1986)

There are at most countably many automorphisms of $\mathcal{D}_T$.

Using parameters you can represent a model of arithmetic within $\mathcal{D}_T$.

Definable relations in second order arithmetic correspond to relations that can be defined in $\mathcal{D}_T$ using parameters.

Relations that are in addition invariant under automorphisms is definable in $\mathcal{D}_T$.

The double jump is definable in $\mathcal{D}_T$.

Theorem (Slaman, Shore 1999)

The jump is definable in $\mathcal{D}_T$.

Definability of the enumeration jump

Definition (Kalimullin 2003)

A pair of sets A, B are called a $\mathcal{K}$-pair if there is a c.e. set W, such that $A \times B \subseteq W$ and $\overline{A} \times \overline{B} \subseteq \overline{W}$.

Theorem (Kalimullin)

A pair of sets A, B is a $\mathcal{K}$-pair if and only if their enumeration degrees $\mathbf{a}$ and $\mathbf{b}$ satisfy:

$$\mathcal{K}(\mathbf{a}, \mathbf{b}) \leftrightharpoons (\forall \mathbf{x})((\mathbf{a} \vee \mathbf{x}) \wedge (\mathbf{b} \vee \mathbf{x}) = \mathbf{x}).$$

Definability of the enumeration jump

Theorem (Kalimullin)

$\mathbf{0}'_e$ is the largest degree which can be represented as the least upper bound of a triple $\mathbf{a}, \mathbf{b}, \mathbf{c}$, such that $\mathcal{K}(\mathbf{a}, \mathbf{b})$, $\mathcal{K}(\mathbf{b}, \mathbf{c})$ and $\mathcal{K}(\mathbf{c}, \mathbf{a})$.

Corollary (Kalimullin 2003)

The enumeration jump is first order definable in $\mathcal{D}_e$.

Maximal $\mathcal{K}$-pairs

Definition (Ganchev, S)

A $\mathcal{K}$-pair $\{\mathbf{a}, \mathbf{b}\}$ is maximal if for every $\mathcal{K}$-pair $\{\mathbf{c}, \mathbf{d}\}$ with $\mathbf{a} \leq \mathbf{c}$ and $\mathbf{b} \leq \mathbf{d}$, we have that $\mathbf{a} = \mathbf{c}$ and $\mathbf{b} = \mathbf{d}$.

Theorem (Jockusch)

Every total enumeration degree is the least upper bound of the elements of a maximal $\mathcal{K}$-pair.

A partial solution

Definition

$\mathcal{D}_e(\leq \mathbf{0}'_e)$ is the substructure of the enumeration degrees that consist of all enumeration degrees bounded by $\mathbf{0}'_e$.

$\mathcal{D}_e(\leq \mathbf{0}'_e)$ is countable and Cooper (1984) showed it is dense.

The members of degrees in $\mathcal{D}_e(\leq \mathbf{0}'_e)$ are easier to construct and handle using finite approximations.

Theorem (Ganchev, S 2009)

$\mathcal{K}$-pairs are first order definable in $\mathcal{D}_e(\leq \mathbf{0}'_e)$.

Theorem (Ganchev, S 2010)

In $\mathcal{D}_e(\leq \mathbf{0}'_e)$ maximal $\mathcal{K}$-pairs have total least upper bounds.

The total degrees are definable in $\mathcal{D}_e(\leq \mathbf{0}'_e)$.

The automorphisms of $\mathcal{D}_e$

Theorem (S 2012)

There are at most countably many automorphisms of $\mathcal{D}_e$.

Using parameters you can represent a model of arithmetic within $\mathcal{D}_e$.

Definable relations in second order arithmetic correspond to relations that can be defined in $\mathcal{D}_e$ using parameters.

Relations that are in addition invariant under automorphisms is definable in $\mathcal{D}_e$.

Example

$\mathcal{TOT}$ corresponds to a definable relation in $\mathcal{Z}_2$.

So $\mathcal{TOT}$ is definable in $\mathcal{D}_e$ with parameters.

Defining totallity in $\mathcal{D}_e$

Theorem (Cai, Ganchev, Lempp, Miller, S)

Maximal $\mathcal{K}$-pairs have total least upper bounds.

If C is a left cut in a computable linear ordering then $\{C, \overline{C}\}$ is a maximal $\mathcal{K}$-pair.

Let W be a c.e. set witnessing that a pair of sets $\{A, B\}$ forms a $\mathcal{K}$-pair.

1. The countable component: we use W to construct the computable linear ordering.
2. The uncountable component: find an appropriate left cut in this ordering to define C.

Theorem (Cai, Ganchev, Lempp, Miller, S 2013)

The set of total enumeration degrees is first order definable in $\mathcal{D}_e$.

Rogers' questions from 1969

Question 1: Is the jump operator definable in $\mathcal{D}_T$? ✓

Question 2: Are there any nontrivial automorphisms of $\mathcal{D}_T$ or $\mathcal{D}_e$?

Question 3: Is the copy of $\mathcal{D}_T$ in $\mathcal{D}_e$ (i.e. the total enumeration degrees) definable in $\mathcal{D}_e$? ✓

Theorem (Cai, Ganchev, Lempp, Miller, S)

If $\mathcal{D}_e$ has a nontrivial automorphism then so does $\mathcal{D}_T$.

The end

Thank you!