Effective properties of Marker's Extensions

Ivan N. Soskov
presented by Mariya I. Soskova

Sofia University

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Marker's extensions in the literature

- 1989 Marker: $\aleph_0$-categorical non-Σ^0_n-axiomatizable almost strongly minimal theory.
- 2004 Goncharov, Khoussainov: An $\aleph_1$-categorical theory, Turing equivalent to $\mathbf{0}^{(n)}$ with a computable model.
- 2009 Soskov, Soskova: Jump inversion of spectra of structures.
- 2009 Stukachev: A jump inversion theorem for the semi-lattices of sigma-degrees.
- 2010 Fokina, Kalimullin, Miller: Degrees of categoricity of computable structures.

A parallel between classical computability theory and effective definability in abstract structures

1. A set X is *c.e. in* a set A if X can be enumerated by a computable in A function.
2. A set X is enumeration reducible to a set A if and only if there is an effective procedure to transform an enumeration of A to an enumeration of X

Denote by A^+ the set $A \oplus \overline{A}$.

Proposition

X is c.e. in A if and only if $X \leq_e A^+$.

Given a set A can we find a set M such that $X \leq_e A$ if and only if X is *c.e. in M*?
There are sets A which are not enumeration equivalent to any set of the form $M \oplus \overline{M}$, so the answer is "No".

Abstract structures

Let $\mathfrak{A} = (A; R_1, R_2, \dots R_n)$ be a countable structure.

- An enumeration of $\mathfrak{A}$ is a bijection $f : \mathbb{N} \to A$.
- $f^{-1}(\mathfrak{A}) = f^{-1}(R_1) \oplus f^{-1}(R_2) \oplus \cdots \oplus f^{-1}(R_n)$ computes the atomic diagram of an isomorphic copy of $\mathfrak{A}$.

Definition

A set $X \subseteq A$ is relatively intrinsically c.e. in $\mathfrak{A}$ (X c.e. in $\mathfrak{A}$) if for every enumeration f of $\mathfrak{A}$ we have that $f^{-1}(X)$ is c.e. in $f^{-1}(\mathfrak{A})$.

By Ash, Knight, Manasse, Slaman we have that X is c.e. in $\mathfrak{A}$ if and only if X is definable in $\mathfrak{A}$ by means of a computable infinitary Σ_1 formula with parameters.

Relatively intrinsically enumeration reducible

Definition

A set $X \subseteq A$ is (relatively intrinsically) enumeration reducible to $\mathfrak{A}$ ($X \leq_e \mathfrak{A}$) if for every enumeration f of $\mathfrak{A}$, $f^{-1}(X) \leq_e f^{-1}(\mathfrak{A})$.

$X \leq_e \mathfrak{A}$ if and only if X is definable in $\mathfrak{A}$ by means of a positive computable infinitary Σ_1 formula with parameters.
Given a structure $\mathfrak{A} = (A, R_1, \dots R_n)$ let $\mathfrak{A}^+ = (A, R_1, \overline{R_1}, \dots R_n, \overline{R_n})$.

Proposition

For every $X \subseteq A$, X c.e. in $\mathfrak{A}$ if and only if $X \leq_e \mathfrak{A}^+$.

Question

Given a structure $\mathfrak{A}$, does there exist a structure $\mathfrak{M}$, such that for every $X \subseteq A$, $X \leq_e \mathfrak{A}$ if and only if X c.e. in $\mathfrak{M}$?

The answer is "Yes"

Consider the simple case when $\mathfrak{A} = (A, R)$ and $R \subseteq A$. We can assume that R is infinite.

Marker's (0-th) extension of $\mathfrak{A}$

Let X be an infinite countable set disjoint from A.
Let $h : R \to X$ be a bijection. Let $M(a, x)$ be true if and only if $h(a) = x$.

$$\mathfrak{M} = (A \cup X; A, X, M).$$

R is Σ_1 definable in $\mathfrak{M}$: $R(a) \Leftrightarrow \exists x M(a, x)$.

1. If $Y \subseteq A$ and $Y \leq_e \mathfrak{A}$ then for every enumeration f of $\mathfrak{M}$, $f^{-1}(Y)$ is c.e. in $f^{-1}(\mathfrak{M})$.
2. If $Y \not\leq_e A$ then there is an enumeration f of $\mathfrak{M}$ such that $f^{-1}(Y)$ is not c.e. in $f^{-1}(\mathfrak{M})$.

Step 1

Lemma

If $Y \subseteq A$ and $Y \leq_e \mathfrak{A}$ then for every enumeration f of $\mathfrak{M}$, $f^{-1}(Y)$ is c.e. in $f^{-1}(\mathfrak{M})$.

Proof: $f^{-1}(A)$ is computable from $f^{-1}(\mathfrak{M})$.
Fix a computable in $f^{-1}(\mathfrak{M})$ bijection $\lambda : \mathbb{N} \to f^{-1}(A)$.
Let $g(n) = f(\lambda(n))$. Then for every $Y \subseteq A$:

$$y \in g^{-1}(Y) \iff g(y) \in Y \iff f(\lambda(y)) \in Y \iff \lambda(y) \in f^{-1}(Y)$$

In other words $g^{-1}(Y) \oplus f^{-1}(\mathfrak{M})^+ \equiv_e f^{-1}(Y) \oplus f^{-1}(\mathfrak{M})^+$.

- $g^{-1}(\mathfrak{A})$ is c.e. in $f^{-1}(\mathfrak{M})$:
  $g^{-1}(\mathfrak{A}) = g^{-1}(R) \leq_e f^{-1}(\mathfrak{M})^+ \oplus f^{-1}(R) \leq_e f^{-1}(\mathfrak{M})^+$.
- If $g^{-1}(Y) \leq_e g^{-1}(\mathfrak{A})$ then $f^{-1}(Y) \leq_e f^{-1}(\mathfrak{M})^+$.

Step 2

Proposition

If $Y \not\leq_e A$ then there is an enumeration f of $\mathfrak{M}$ such that $f^{-1}(Y)$ is not c.e. in $f^{-1}(\mathfrak{M})$.

Proof: Let g be an enumeration of $\mathfrak{A}$ such that $g^{-1}(Y) \not\leq_e g^{-1}(\mathfrak{A})$. We construct f so that $f(2n) = g(n)$.

1. $f^{-1}(A) = 2\mathbb{N}$ and $f^{-1}(X) = 2\mathbb{N} + 1$.
2. $f^{-1}(\mathfrak{M}) \equiv_e f^{-1}(M)$.
3. For every set $Y \subseteq A$, $g^{-1}(Y) \equiv_1 f^{-1}(Y)$.
4. $g^{-1}(\mathfrak{A})$ is c.e. in $f^{-1}(\mathfrak{M})$.

What remains to be done is construct a bijection $k : f^{-1}(R) \to 2\mathbb{N} + 1$. Recall that there is a bijection $h : R \to X$. Then we complete f by $f(2n+1) = h(f(k^{-1}(2n+1)))$.

Note that then $f^{-1}(\mathfrak{M}) \equiv_e f^{-1}(M) = G_k$.

Step 2: Continued

Proposition

If $Y \not\leq_e A$ then there is an enumeration f of $\mathfrak{M}$ such that $f^{-1}(Y)$ is not c.e. in $f^{-1}(\mathfrak{M})$.

Proof: We construct a bijection $k : f^{-1}(R) \to 2\mathbb{N} + 1$ using forcing so that statements of the form $x \in \Gamma_e(f^{-1}(M)^+)$ are decided at finite stages.

For $\sigma : f^{-1}(R) \to 2\mathbb{N} + 1$ we say that $\sigma \Vdash x \in \Gamma_e(M^+_G)$ if there exists v, such that $\langle x, v\rangle \in \Gamma_e$ and for every $u \in D_v$

1. $u = 2\langle a, x\rangle$ and $\sigma(a) = x$.
2. $u = 2\langle a, x\rangle + 1$ and $\sigma(b) = x$ for some $b \neq a$.

$\{x \mid \exists\sigma \supseteq \tau(\sigma \Vdash x \in \Gamma_e(M^+_G))\}$ is enumeration reducible to $g^{-1}(\mathfrak{A})$. We use this to ensuret $g^{-1}(Y) \neq \Gamma_e(f^{-1}(M)^+)$.

Finally note that $f^{-1}(\mathfrak{M})'$ can be computed from $g^{-1}(\mathfrak{A})'$. $\square$

From sets to sequences of sets

Theorem (Selman)

$X \leq_e A$ if an only if for every Z, if A is c.e. in Z then X is c.e. in Z.

1. A sequence of sets $\mathcal{X} = \{X_n\}_{n<\omega}$ is *c.e. in* a set A if for every n, X_n is c.e. in $A^{(n)}$ uniformly in n.
2. $\mathcal{X} = \{X_n\}_{n<\omega}$ is ω-enumeration reducible to a sequence $\mathcal{A} = \{A_n\}_{n<\omega}$ if and only for every set of natural numbers Z, if $\{A_n\}$ is c.e. in Z then $\{X_n\}$ is c.e. in Z.

The jump sequence $P(\mathcal{A})$ of a sequence $\mathcal{A}$ is defined by induction: $P_0(\mathcal{A}) = A_0$ and $P_{n+1}(\mathcal{A}) = P_n(\mathcal{A})' \oplus A_{n+1}$.

Theorem (Soskov)

$\mathcal{X} \leq_\omega \mathcal{A}$ if and only if for every n, $X_n \leq_e P_n(\mathcal{A})$ uniformly in n.

Sequences and structures

Let $\mathfrak{A} = (A; R_1, R_2, \dots R_m)$ be a countable structure.

Definition

A sequence $\{Y_n\}$ of subsets of A is (relatively intrinsically) c.e. in $\mathfrak{A}$ if for every enumeration f of $\mathfrak{A}$ the sequence $\{f^{-1}(Y_n)\}$ is c.e in $f^{-1}(\mathfrak{A})$.

By the methods of Ash, Knight, Manasse, Slaman one can show that $\{Y_n\}$ is c.e. in $\mathfrak{A}$ if and only if there is a computable sequence $\{F_n\}$ such that F_n is a computable infinitary Σ_{n+1} formula and parameters $t_1, \dots t_m$, such that Y_n is definable in $\mathfrak{A}$ using F_n and the parameters $t_1, \dots t_m$.

Sequences of structures

Now consider a sequence of structures $\vec{\mathfrak{A}} = \{\mathfrak{A}_n\}_{n<\omega}$, where $\mathfrak{A}_n = (A_n; R_1^n, R_2^n, \ldots R_{m_n}^n)$. Let $A = \bigcup_n A_n$.
An enumeration f of $\vec{\mathfrak{A}}$ is a bijection from $\mathbb{N} \to A$.
$f^{-1}(\vec{\mathfrak{A}})$ is the sequence $\{f^{-1}(A_n) \oplus f^{-1}(R_1^n) \cdots \oplus f^{-1}(R_{m_n}^n)\}_{n<\omega}$

Definition

A sequence $\{Y_n\}$ of subsets of A is (relatively intrinsically) ω-enumeration reducible to $\vec{\mathfrak{A}}$ if for every enumeration f of $\vec{\mathfrak{A}}$, $\{f^{-1}(Y_n)\} \leq_\omega f^{-1}(\vec{\mathfrak{A}})$.

Soskov and Baleva show that this is equivalent to Y_n is uniformly in n definable by a Σ^+_{n+1} formula: a positive computable infinitary formula with predicates only from the first n structures, such that the predicates for the n-th appear for the first time at level $n+1$ positively.

The bigger question

Question

Given a sequence of structures $\vec{\mathfrak{A}}$, *does there exist a structure* $\mathfrak{M}$, *such that for every sequence* $\mathcal{X}$ *of subsets of* $A = \bigcup_n A_n$, $\mathcal{X} \leq_\omega \vec{\mathfrak{A}}$ *if and only if* $\mathcal{X}$ *c.e. in* $\mathfrak{M}$*?*

Marker's extensions

Let $\vec{\mathfrak{A}} = \{\mathfrak{A}_n\}_{n<\omega}$, and $A = \bigcup_n A_n$. Let $R \subseteq A^m$.

The n-th Marker's extension $\mathfrak{M}_n(R)$ of R

Let $X_0, X_1, \dots X_n$ be infinite disjoint countable - companions to $\mathfrak{M}_n(R)$.
Fix bijections: $h_0 : R \to X_0$
$h_1 : (A^m \times X_0) \setminus G_{h_0} \to X_1 \dots$
$h_n : (A^m \times X_0 \times X_1 \dots \times X_{n-1}) \setminus G_{h_{n-1}} \to X_n$

Let $M_n = G_{h_n}$ and $\mathfrak{M}_n(R) = (A \cup X_0 \cup \dots \cup X_n; X_0, X_1, \dots X_n, M_n)$.

If n is even then $a \in R \iff \exists x_0 \in X_0[(a, x_0) \in G_{h_0}] \iff$

$\exists x_0 \in X_0 \forall x_1 \in X_1[(a, x_0, x_1) \notin G_{h_1}] \iff$

$\exists x_0 \in X_0 \forall x_1 \in X_1 \exists x_2 \in X_2[(a, x_0, x_1, 2) \in G_{h_2}] \iff \dots$

$\exists x_0 \in X_0 \forall x_1 \in X_1 \dots \exists x_n \in X_n[M_n(a, x_0, \dots x_n)]$.

Marker's extensions

Given two structures $\mathfrak{A} = (A; R_1, R_2, \dots R_m)$ and $\mathfrak{B} = (B; P_1, P_2, \dots P_k)$ let $\mathfrak{A} \cup \mathfrak{B} = (A \cup B; R_1, R_2, \dots R_m, P_1, P_2, \dots P_k)$.

Let $\vec{\mathfrak{A}} = \{\mathfrak{A}_n\}_{n<\omega}$, and $A = \bigcup_n A_n$.

1. For every n construct the n-th Markers's extensions of A_n, R_1^n, $\dots R_{m_n}^n$ with disjoint companions.
2. For every n let $\mathfrak{M}_n(\mathfrak{A}_n) = \mathfrak{M}_n(A_n) \cup \mathfrak{M}_n(R_1^n) \cup \dots \cup \mathfrak{M}_n(R_{m_n}^n)$.
3. Set $\mathfrak{M}(\vec{\mathfrak{A}})$ to be $\bigcup_n \mathfrak{M}_n(\mathfrak{A}_n)$ with one additional predicate for A.

Theorem

A sequence $\mathcal{Y}$ of subsets of A is (r.i.) ω-enumeration reducible to $\vec{\mathfrak{A}}$ if and only if $\mathcal{Y}$ is (r.i) c.e. in $\mathfrak{M}(\vec{\mathfrak{A}})$.

Two steps

Lemma

For every enumeration f of $\mathfrak{M}(\vec{\mathfrak{A}})$ *there is an enumeration g of* $\vec{\mathfrak{A}}$*:*

1. *There is a computable in* $f^{-1}(\mathfrak{M}(\vec{\mathfrak{A}}))$ *injective function* λ *such that for every* $Y \subseteq A$, $x \in g^{-1}(Y)$ *iff* $\lambda(x) \in f^{-1}(Y)$.
2. $g^{-1}(\vec{\mathfrak{A}})$ *is c.e. in* $f^{-1}(\mathfrak{M}(\vec{\mathfrak{A}}))$.

Theorem

Let g be an enumeration of $\vec{\mathfrak{A}}$ *and* $\mathcal{Y} \not\leq_\omega g^{-1}(\vec{\mathfrak{A}})$. *There is an enumeration f of* $\mathfrak{M}(\vec{\mathfrak{A}})$*:*

1. *There is a computable injective function* λ *such that for every* $Y \subseteq A$, $x \in g^{-1}(Y)$ *iff* $\lambda(x) \in f^{-1}(Y)$.
2. $\bigoplus_n P_n(g^{-1}(\vec{\mathfrak{A}})) \equiv_e (f^{-1}(\mathfrak{M}(\vec{\mathfrak{A}})))^{(\omega)}$.
3. $\mathcal{Y}$ *is not c.e. in* $f^{-1}(\mathfrak{M}(\vec{\mathfrak{A}}))$.

Proof flavour

We consider the case when $\mathfrak{A}_n$ has no predicates, i.e.
$g^{-1}(\vec{\mathfrak{A}}) = \{g^{-1}(A_n)\}$ and $\mathfrak{M} = (|\mathfrak{M}|, A, \{M_n, X_0^n, \ldots, X_n^n\}_{n<\omega})$.

Let $\{Z_0^n, \ldots, Z_n^n\}_{n<\omega}$ be a partition of $2\mathbb{N}+1$ in uniformly computable sets.

We define f so that $f(2n) = g(n)$ and $f(Z_i^n) = X_i^n$ for every i, n.

What remains to be constructed is a system of bijections
$\{k_0^n, \ldots, k_n^n\}_{n<\omega}$ (an acceptable system) such that:
$k_0^n : f^{-1}(A_n) \to Z_0^n$.
$k_1^n : (2\mathbb{N} \times Z_0^n) \setminus G_{k_0^n} \to Z_1^n \ldots$
$k_n^n : (2\mathbb{N} \times Z_0^n \times Z_{n-1}^n) \setminus G_{k_{n-1}^n} \to Z_n^n$

We do so by forcing, so that statements of the form
$x \in \Gamma_e((f^{-1}(\mathfrak{M})^+)^{(n)})$ are decided at finite stages using only information from $P_{n+1}(g^{-1}(\vec{\mathfrak{A}}))$.

Degree structures

- The enumeration degree of set A is $d_e(A) = \{B \mid A \equiv_e B\}$.

  The structure of the enumeration degrees $\mathcal{D}_e$ is an upper semi-lattice with jump operation.

  The Turing degrees are embedded in to the enumeration degrees by: $\iota(d_T(A)) = d_e(A^+)$.

- The ω-enumeration degree of a sequence $\mathcal{A}$ is $d_\omega(\mathcal{A}) = \{\mathcal{B} = \{B_n\}_{n<\omega} \mid \forall n(P_n(\mathcal{A}) \equiv_e P_n(\mathcal{B})$ uniformly in $n)\}$

  The structure of the ω-enumeration degrees $\mathcal{D}_\omega$ is an upper semi-lattice with jump operation.

  The enumeration degrees are embedded in to the ω enumeration degrees by: $\kappa(d_e(A)) = d_\omega(\{A^{(n)}\}_{n<\omega})$.

$\mathcal{D}_T \subset \mathcal{D}_e \subset \mathcal{D}_\omega$

- There are sets A which are not enumeration equivalent to any set of the form M^+.
- There are sequences $\mathcal{R} = \{R_n\}_{n<\omega}$ such that:
  - $P_n(\mathcal{R}) \equiv_e \emptyset^{(n)}$ for every n.
  - $\{\emptyset^{(n)}\}_{n<\omega} <_\omega \mathcal{R}$.

To make $\mathcal{R} \not\leq_\omega \{\emptyset^{(n)}\}_{n<\omega}$ it is sufficient to ensure $\mathcal{R} \neq \{\Gamma_e^{[n]}(\emptyset^{(n)})\}_{n<\omega}$, where $\Gamma_e^{[n]}$ is the n-th column of Γ_e.

$$R_n = \begin{cases} \{1\}, & \text{if } 0 \in \Gamma_n^{[n]}(\emptyset^{(n)}); \\ \{0\}, & \text{otherwise.} \end{cases}$$

Sequences with this property are called *almost zero*.
Note that R_n is uniformly reducible to $\emptyset^{(n+1)}$

Spectra and Co-spectra

Let $\mathfrak{M}$ be a structure.

Definition

The spectrum of $\mathfrak{M}$ is the set
$Sp(\mathfrak{M}) = \{\mathbf{a} \in \mathcal{D}_T \mid \exists f(d_T(f^{-1}(\mathfrak{M}))) \leq \mathbf{a}\}$.
The n-th jump spectrum of $\mathfrak{M}$ is the set $Sp_n(\mathfrak{M}) = \{\mathbf{a}^n \mid \mathbf{a} \in Sp(\mathfrak{M})\}$.

We can view $Sp_n(\mathfrak{M})$ as a subset of $\mathcal{D}_e$ or as a subset of $\mathcal{D}_\omega$.

Definition

The n-th co-spectrum of $\mathfrak{M}$ is the set
$CoSp_n(\mathfrak{M}) = \{\mathbf{a} \in \mathcal{D}_e \mid \forall \mathbf{x} \in Sp_n(\mathfrak{M})(\mathbf{a} \leq_e \mathbf{x})\}$.
The ω-enumeration co-pectrum of $\mathfrak{M}$ is the set
$Ocsp(\mathfrak{M}) = \{\mathbf{a} \in \mathcal{D}_\omega \mid \forall \mathbf{x} \in Sp(\mathfrak{M})(\mathbf{a} \leq_\omega \mathbf{x})\}$.

Co-spectra of Marker's extensions

Theorem

Fix $\vec{\mathfrak{A}} = \{\mathfrak{A}_n\}_{n<\omega}$ *and let* $\mathfrak{M} = \mathfrak{M}(\vec{\mathfrak{A}})$*.*

1. $CoSp_n(\mathfrak{M}) = \left\{ d_e(Y) \mid (\forall g)(Y \leq_e P_n(g^{-1}(\vec{\mathfrak{A}}))) \right\}$.
2. $Ocsp(\mathfrak{M}) = \left\{ d_\omega(\mathcal{Y}) \mid (\forall g)(\mathcal{Y} \leq_\omega g^{-1}(\vec{\mathfrak{A}})) \right\}$.

Example: Let $\mathcal{R} = \{R_n\}$ be a sequence of sets. Consider the sequence $\vec{\mathfrak{A}}$, where $\mathfrak{A}_0 = (\mathbb{N} : G_s, R_0)$ and for all $n \geq 1$, $\mathfrak{A}_n = (\mathbb{N}; R_n)$, For every enumeration g of $\vec{\mathfrak{A}}$, $\mathcal{R} \leq_\omega g^{-1}(\vec{\mathfrak{A}})$:

- g is computable from $g^{-1}(G_s)$.
- $x \in R_n$ if and only if $g^{-1}(x) \in g^{-1}(R_n)$.

Considering that $g = id$ is an enumeration, we obtain that

1. $CoSp_n(\mathfrak{M}) = \{d_e(Y) \mid Y \leq_e P_n(\mathcal{R})\}$.
2. $Ocsp(\mathfrak{M}) = \{d_\omega(\mathcal{Y}) \mid \mathcal{Y} \leq_\omega \mathcal{R}\}$.

Example: continued

Definition

The least element of $Sp_n(\mathfrak{M})$ if it exists is the n-th jump degree of $\mathfrak{M}$. The greatest element of $CoSp_n(\mathfrak{M})$ if it exists is the n-th co-degree of $\mathfrak{M}$.

Richter, Knight: linear orderings have co-degree $\mathbf{0}_e$ and first co-degree $\mathbf{0}'_e$ but not always a degree or a jump degree.

- $CoSp_n(\mathfrak{M}) = \{d_e(Y) \mid Y \leq_e P_n(\mathcal{R})\}$.

Consider the *almost zero* sequence $\mathcal{R}$:

1. $P_n(\mathcal{R}) \equiv_e \emptyset^{(n)}$ for every n. Hence the n-th co-degree of $\mathfrak{M}$ is $\mathbf{0}_e^{(n)}$
2. $\{\emptyset^{(n)}\}_{n<\omega} <_\omega \mathcal{R}$. Hence $\mathfrak{M}$ has no n-th jump degree for any n.

Spectra of sequences of structures

Let $\vec{\mathfrak{A}} = \{\mathfrak{A}_n\}_{n<\omega}$ be given and let $A = \bigcup_n A_n$.

Definition

- The relative degree spectrum of $\vec{\mathfrak{A}}$ is
  $Rsp(\vec{\mathfrak{A}}) = \left\{ g^{-1}(\vec{\mathfrak{A}}) \mid g \text{ is an enumeration of } A \right\}$.
- The joint spectrum of $\vec{\mathfrak{A}}$ is
  $Jsp(\vec{\mathfrak{A}}) = \left\{ \{g_n^{-1}(\mathfrak{A}_n)\}_{n<\omega} \mid g_n \text{ is an enumeration of } A_n \right\}$.

If $\vec{\mathfrak{A}}$ and $\vec{\mathfrak{A}}^*$ are such that for every n $\mathfrak{A}_n \cong \mathfrak{A}_n^*$ then $Jsp(\vec{\mathfrak{A}}) = Jsp(\vec{\mathfrak{A}}^*)$.

Consider the structure $\vec{\mathfrak{A}}$ obtained from the almost zero sequence $\mathcal{R}$:
$\mathfrak{A}_0 = (\mathbb{N} : G_s, R_0)$ and for all $n \geq 1$, $\mathfrak{A}_n = (\mathbb{N}; R_n)$.
As $\mathcal{R} \not\leq_\omega \{\emptyset^{(n)}\}_{n<\omega}$, $\{\emptyset^{(n)}\}_{n<\omega} \notin Rsp(\vec{\mathfrak{A}})$.
Now consider the $\vec{\mathfrak{A}}^*$ obtained from the sequence $(R_0, R_0, R_0, \dots)$.

Main theorem

Theorem

Let $\vec{\mathfrak{A}} = \{\mathfrak{A}_n\}_{n<\omega}$ be a sequence of structures.

1. *There exists a structure $\mathfrak{M}$ such that* $Sp(\mathfrak{M}) = \left\{ d_T(B) \mid (\exists \mathcal{Y} \in Rsp(\vec{\mathfrak{A}}))(\mathcal{Y} \text{ is c.e. in } B) \right\}$.
2. *There exists a structure $\mathfrak{M}$ such that* $Sp(\mathfrak{M}) = \left\{ d_T(B) \mid (\exists \mathcal{Y} \in Jsp(\vec{\mathfrak{A}}))(\mathcal{Y} \text{ is c.e. in } B) \right\}$.

Note: (2) follows from (1): take an isomorphic copy $\vec{\mathfrak{A}}^*$of $\vec{\mathfrak{A}}$ with $\{A_n^*\}_{n<\omega}$ disjoint and use the structure $\mathfrak{M}^*$ for $\vec{\mathfrak{A}}^*$.

Proof of 1:
Consider $\mathfrak{M} = \mathfrak{M}(\vec{\mathfrak{A}})$. One direction we know: if B computes $f^{-1}(\mathfrak{M})$ we can find an enumeration g of $\vec{\mathfrak{A}}$ such that $g^{-1}(\vec{\mathfrak{A}})$ is c.e. in $f^{-1}(\mathfrak{M})$ and hence in B.
For the other direction we need a new construction.

The other direction

Theorem

Let $\vec{\mathfrak{A}} = \{\mathfrak{A}_n\}_{n<\omega}$ be a sequence of structures.

1. *There exists a structure $\mathfrak{M}$ such that* $Sp(\mathfrak{M}) = \left\{ d_T(B) \mid (\exists \mathcal{Y} \in Rsp(\vec{\mathfrak{A}}))(\mathcal{Y} \text{ is c.e. in } B) \right\}$.

Let g be an enumeration of $\vec{\mathfrak{A}}$. And let $g^{-1}(\vec{\mathfrak{A}})$ be c.e. in B. Consider again the case when $\mathfrak{A}_n$ has no predicates, i.e. $g^{-1}(\vec{\mathfrak{A}}) = \{g^{-1}(A_n)\}_{n<\omega}$ and hence $g^{-1}(A_n)$ is Σ_{n+1} in B uniformly in n.

We want an enumeration f of $\mathfrak{M}$, such that $f^{-1}(\mathfrak{M})$ is computable in B. We set again $f(2n) = g(n)$ and partition $2\mathbb{N}+1$ into infinite uniformly computable sets Z_i^n. To complete f we need an acceptable system :
$k_0^n : f^{-1}(A_n) \to Z_0^n$.
$k_1^n : (2\mathbb{N} \times Z_0^n) \setminus G_{k_0^n} \to Z_1^n \dots$
$k_n^n : (2\mathbb{N} \times Z_0^n \times Z_{n-1}^n) \setminus G_{k_{n-1}^n} \to Z_n^n$ with G_{k_n} uniformly computable in B.

Generalized Goncharov and Khoussainov Lemma

Proposition

Let $n \geq 0$ and R be a $\Sigma^0_{n+1}(B)$ set with an infinite computable subset. Then there exists functions $k_0 \dots k_n$ such that the graph of k_n is computable in B, uniformly in an index for R and n and
$k_0 : R \to \mathbb{N}$.
$k_1 : \mathbb{N}^2 \setminus G_{k_0} \to \mathbb{N} \dots$
$k_n : \mathbb{N}^{n+1} \setminus G_{k_{n-1}} \to \mathbb{N}$.

As $f^{-1}(A_n) \equiv_1 g^{-1}(A_n)$, it is Σ_{n+1} in B uniformly in n.
To ensure $f^{-1}(A_n)$ has an infinite computable set this we change $\vec{\mathfrak{A}}$:
Let $\perp \notin A$ be a new symbol.

- For every $R \subset A$ we set $R^\perp = \{(a, t) \mid a \in R \vee t = \perp\}$.
- Set $\mathfrak{A}_n = (A \cup \{\perp\}; A_n^\perp, (P_0^n)^\perp, \dots (P_{k_m}^n)^\perp)$
- Set $\vec{\mathfrak{A}}^\perp = \{A_n^\perp\}_{n<\omega}$.

$\mathfrak{M} = \mathfrak{M}(\vec{\mathfrak{A}}^\perp)$ is the required structure.

The inductive step

Lemma

Let R be $\Sigma^0_2(X)$ and $S \subseteq R$ be infinite and computable. There exists a bijection $k : R \to \mathbb{N}$ such that $\mathbb{N}^2 \setminus G_k$ is $\Sigma^0_1(X)$ and has an infinite computable subset.

Proof:
Fix a computable in X approximation $\{R_s\}_{s<\omega}$ such that $x \in R$ if and only if for all but finitely many s, $x \in R_s$. Assume $R_s \triangle R_{s+1} \leq 1$.
We can map an element x of R to the least stage s_x, such that for all $t \geq s_x$ we have $x \in R_t$.
The set $B = \{s \mid s \neq s_x \text{ for any } x\} \cup \{s \mid s = s_x \to x \in S\}$ is $\Sigma^0_1(X)$.
We map elements x in $R \setminus S$ to s_x and computably in X map S to B.

Let $k(0) = a$. Then $\{(0, b) \mid b \neq a\}$ is a computable subset of $\mathbb{N}^2 \setminus G_k$.

Embedding the ω-enumeration degrees

Consider again the structure $\vec{\mathfrak{A}}$ obtained from a sequence of sets $\mathcal{R}$. $\mathfrak{A}_0 = (\mathbb{N} : G_s, R_0)$ and for all $n \geq 1$, $\mathfrak{A}_n = (\mathbb{N}; R_n)$.

- Recall that for every enumeration g of $\vec{\mathfrak{A}}$, $\mathcal{R} \leq_\omega g^{-1}(\vec{\mathfrak{A}})$.
- By Main Theorem there is a structure $\mathfrak{M}_\mathcal{R}$ such that $Sp(\mathfrak{M}_\mathcal{R}) = \left\{ d_T(B) \mid (\exists g)(g^{-1}(\vec{\mathfrak{A}}) \text{ is c.e. in } B) \right\}$.
- If $g^{-1}(\vec{\mathfrak{A}})$ is c.e. in B then $\mathcal{R}$ is c.e. in B.
- If $\mathcal{R}$ is c.e. in B then as $\mathcal{R} \equiv_\omega id^{-1}(\vec{\mathfrak{A}})$, $B \in Sp(\mathfrak{M}_\mathcal{R})$.
- $Sp(\mathfrak{M}_\mathcal{R}) = \{ d_T(B) \mid \mathcal{R} \text{ is c.e. in } B \}$.

This allows us to embed $\mathcal{D}_\omega$ into the Muchnik degrees generated by spectra of structures.

$\mathcal{R} \leq_\omega \mathcal{Q} \iff$
$\{ d_T(B) \mid \mathcal{R} \text{ is c.e. in } B \} \supseteq \{ d_T(B) \mid \mathcal{Q} \text{ is c.e. in } B \} \iff$
$Sp(\mathcal{M}_R) \supseteq Sp(\mathcal{M}_\mathcal{Q})$.

Enumeration spectra

Let $\mathfrak{A} = (A; P_1 \dots P_m)$ be a structure

Definition

A partial enumeration of $\mathfrak{A}$ is a partial injective map φ from $\mathbb{N}$ onto A.
$\varphi^{-1}(\mathfrak{A}) = dom(\varphi) \oplus \varphi^{-1}(P_1) \oplus \dots \oplus \varphi^{-1}(P_m)$.

The enumeration spectrum of $\mathfrak{A}$ is
$Esp(\mathfrak{A}) = \{d_e(\varphi^{-1}(\mathfrak{A})) \mid \varphi$ is a partial enumeration of $\mathfrak{A}\}$.
The enumeration co-spectrum of $\mathfrak{A}$ is
$CoEsp(\mathfrak{A}) = \left\{\mathbf{y} \mid (\forall \mathbf{x} \in Esp(\vec{\mathfrak{A}}))(\mathbf{y} \leq \mathbf{x})\right\}$.

- Kalimullin showed that there is a structure $\mathfrak{A}$ with $Esp(\mathfrak{A}) = \{\mathbf{x} \mid \mathbf{x} \neq \mathbf{0}_e\}$.
- Soskov showed that every countable ideal of enumeration degrees can be represented as the $CoEsp(\mathfrak{A})$ for some $\mathfrak{A}$.

Enumerations spectra and spectra of structures

Recall the standard embedding $\iota : \mathcal{D}_T \to \mathcal{D}_e$.

Theorem

For every structure $\mathfrak{A}$ there exists a structure $\mathfrak{M}$ such that
$Sp(\mathfrak{M}) = \{\mathbf{a} \in \mathcal{D}_T \mid (\exists \mathbf{x} \in Esp(\mathfrak{A}))(\iota(\mathbf{a}) \geq \mathbf{x})\}$

Proof: Let $C \supseteq A$ be such that $C \setminus A$ is infinite and
$\mathfrak{C} = (C; A, P_1, \dots P_m)$.
Any partial enumeration φ of $\mathfrak{A}$ can be extended to a total enumeration f of $\mathfrak{C}$ with $\varphi^{-1}(\mathfrak{A}) \equiv_e f^{-1}(\mathfrak{C})$ and vice versa.
Let $\mathfrak{M}$ be such that $Sp(\mathfrak{M}) = \left\{d_T(B) \mid \exists f(f^{-1}(\mathfrak{C}) \text{ is c.e. in } B)\right\}$.

- Slaman and Wehner's result follows from Kalimullin's: there is a structure $\mathfrak{M}$ with $Sp(\mathfrak{M}) = \{\mathbf{x} \mid \mathbf{x} \neq \mathbf{0}\}$.
- Every countable ideal of enumeration degrees can be represented as the $CoSp(\mathfrak{M})$ for some $\mathfrak{M}$.

The main theorem for Turing degrees

Recall that

1. $Jsp(\vec{\mathfrak{A}}) = \Big\{\{g_n^{-1}(\mathfrak{A}_n)\}_{n<\omega} \mid g_n \text{ is an enumeration of } A_n\Big\}$.
2. There exists a structure $\mathfrak{M}$ such that $Sp(\mathfrak{M}) = \Big\{d_T(B) \mid (\exists \mathcal{Y} \in Jsp(\vec{\mathfrak{A}}))(\mathcal{Y} \text{ is c.e. in } B)\Big\}$.

- Consider $\vec{\mathfrak{A}}^+$, where $\mathfrak{A}_n^+ = (A_n; P_1, \overline{P_1}, \dots, P_m, \overline{P_m})$.
- Then $\{g^{-1}(\mathfrak{A}_n)\} \in Jsp(\vec{\mathfrak{A}}) \iff \{g^{-1}(\mathfrak{A}_n) \oplus \overline{g^{-1}(\mathfrak{A}_n)}\} \in Jsp(\vec{\mathfrak{A}})^+$.
- $\{g^{-1}(\mathfrak{A}_n) \oplus \overline{g^{-1}(\mathfrak{A}_n)}\}$ is c.e. in $B \iff g^{-1}(\mathfrak{A}_n) \leq_T B^{(n)}$ uniformly in n.

Theorem

For every sequence $\vec{\mathfrak{A}} = \{\mathfrak{A}_n\}_{n<\omega}$ there exists a structure $\mathfrak{M}$ such that $Sp(\mathfrak{M}) = \Big\{d_T(B) \mid (\exists \{Y_n\}_{n<\omega} \in Jsp(\vec{\mathfrak{A}}))(Y_n \leq_T B^{(n)} \text{ uniformly in } n)\Big\}$.

Wehner's construction

Let $\mathcal{F}$ be a countable family of sets.

Definition

An enumeration of $\mathcal{F}$ is a set $U \subseteq \mathbb{N}^2$ such that:

1. For every a, $\{n \mid (a,n) \in U\} \in \mathcal{F}$.
2. For every $F \in \mathcal{F}$ there is an a such that $\{n \mid (a,n) \in U\} = F$.

Let $\mathfrak{A}_{\mathcal{F}} = (A; S, Z, I)$ where $A = \mathcal{F} \times \mathbb{N}^2$;
$Z = \{(F, x, 0) \mid F \in \mathcal{F}, x \in \mathbb{N}\}$,
$S = \{((F, x, n), (F, x, n+1)) \mid F \in \mathcal{F}, x, n \in \mathbb{N}\}$ and
$I = \{(F, x, n) \mid n \in F\}$

Proposition (Wehner)

1. *There is a uniform way to compute an enumeration of $\mathcal{F}$ in any isomorphic copy $\mathfrak{B}$ of $\mathfrak{A}_{\mathcal{F}}$.*
2. *There is a uniform way to compute an isomorphic copy $\mathfrak{B}$ of $\mathfrak{A}_{\mathcal{F}}$ in any enumeration of $\mathcal{F}$.*

A structure with a nice spectrum

Consider the family $\mathcal{F}^X = \{\{n\} \oplus F \mid F \text{ is finite and } F \neq W_n^X\}$.

1. No enumeration of $\mathcal{F}^X$ is c.e. in X.
2. If $B \not\leq_T X$ then one can compute uniformly in B and X an enumeration of $\mathcal{F}$.

Theorem

There is a structure $\mathfrak{M}$ with $Sp(\mathfrak{M}) = \{\mathbf{b} \mid \forall n(\mathbf{b}^{(n)} > \mathbf{0}^{(n)})\}$.

Proof: Consider the sequence $\vec{\mathfrak{A}}$ where $\mathfrak{A}_n = \mathfrak{A}_{\mathcal{F}^{\emptyset^{(n)}}}$. Let $\mathfrak{M}$ be such that

$Sp(\mathfrak{M}) = \left\{d_T(B) \mid (\exists \{Y_n\}_{n<\omega} \in Jsp(\vec{\mathfrak{A}}))(Y_n \leq_T B^{(n)} \text{ uniformly in } n)\right\}$.

If $d_T(\mathcal{B}) \in Sp(\mathfrak{M})$ then $B^{(n)}$ computes an enumeration of $\mathcal{F}^{\emptyset^{(n)}}$ and hence $B^{(n)} \not\leq_T \emptyset^{(n)}$. If $B^{(n)} \not\leq_T \emptyset^{(n)}$ for every n then as $\emptyset^{(n)} \leq_T B^{(n)}$ uniformly in n, it follows that $B^{(n)}$ computes an enumeration of $\mathcal{F}^{\emptyset^{(n)}}$.

The end

Gian-Carlo Rota

According to ...[the “one-shot”] ... view, mathematics would consist of a succession of targets, called problems, which mathematicians would be engaged in shooting down by well-aimed shots. But where do problems come from, and what are they for? If the problems of mathematics were not instrumental in revealing a broader truth, then they would be indistinguishable from chess problems or crossword puzzles. Mathematical problems are worked on because they are pieces of a larger puzzle.

Ivan Soskov

The good puzzles are the ones that will never be completely solved.