Logic, degrees, and definability

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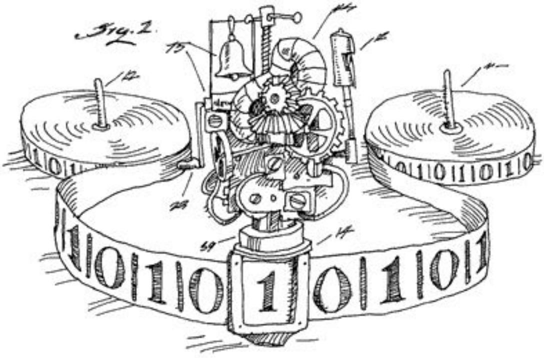
Computable sets and functions



Definition (Turing, Church 1936)

A function $f : \mathbb{N} \to \mathbb{N}$ is *computable* if there is a **computer program** which takes as input a natural number n and outputs $f(n)$.

A set $A \subseteq \mathbb{N}$ is *computable* if its characteristic function is computable.

Computably enumerable sets

Definition

A set $A \subseteq \mathbb{N}$ is *computably enumerable* (c.e.) if it is the range (domain) of a computable function.

Example (Davis, Matyasevich, Putnam, Robinson 1970)

A set $S \subseteq \mathbb{N}$ is Diophantine if $S = \{n \mid \exists \bar{m}(P(n, \bar{m}) = 0)\}$.

The Diophantine sets are exactly the c.e. sets.

Example (Novikov, Boone 1955)

The word problem for a finitely presented group is c.e.

Every c.e. set can be coded as the word problem for a finitely presented group.

An incomputable c.e. set

A set A is computable if and only if both A and $\overline{A}$ are c.e.

We can code programs by natural numbers: P_e is the program with code e.

Definition

The *halting set* is $K = \{e \mid P_e \text{ halts on input } e\}$.

Theorem (Turing 1936)

The halting set K is c.e., but not computable.

Proof.

We prove, in fact, that $\overline{K}$ is not c.e.

Assume that it is and let e be such that P_e halts on n if and only if $n \in \overline{K}$.

P_e halts on $e \Leftrightarrow e \in \overline{K} \Leftrightarrow P_e$ does not halt on e. $\square$

Comparing the information content of sets

Consider a program that has access to an external database, an *oracle*. During its computation the program can ask the oracle membership questions: does n belong to you or not?

Definition (Post 1944)

$A \leq_T B$ if and only if there is a program that computes the elements of A using B as an oracle.

Example

If A is computable then $A \leq_T B$ for every B.

Comparing the information content of sets

Definition (Uspensky 195?, Friedberg and Rogers 1959)

$A \leq_e B$ if there is a program that transforms an enumeration of B to an enumeration of A.

The program is a c.e. table of axioms of the sort:

$$\textit{If } \{x_1, x_2, \dots, x_k\} \subseteq B \textit{ then } x \in A.$$

Example

If A is c.e. then $A \leq_e B$ for every B.

Degree structures

Definition

1. $A \equiv B$ if and only if $A \leq B$ and $B \leq A$.
2. $\mathbf{d}(A) = \{B \mid A \equiv B\}$.
3. $\mathbf{d}(A) \leq \mathbf{d}(B)$ if an only if $A \leq B$.
4. Let $A \oplus B = \{2n \mid n \in A\} \cup \{2n+1 \mid n \in B\}$. Then $\mathbf{d}(A \oplus B) = \mathbf{d}(A) \vee \mathbf{d}(B)$.

And so we have two partial orders with least upper bound (upper semi-lattices):

1. The Turing degrees $\mathcal{D}_T$ with least element $\mathbf{0}_T$ consisting of all computable sets.
2. The enumeration degrees $\mathcal{D}_e$ with least element $\mathbf{0}_e$ consisting of all c.e. sets.

The jump operation

The halting set with respect to A is the set
$K_A = \{e \mid P_e$ using oracle A halts on input $e\}$.

$A <_T K_A$.

Definition

The jump of a Turing degree is $\mathbf{d}_T(A)' = \mathbf{d}_T(K_A)$.

We can apply a similar construction to enumeration reducibility to obtain the enumeration jump.

We always have $\mathbf{a} < \mathbf{a}'$.

Local structures

The Turing degrees of the c.e. sets form a countable substructure of $\mathcal{D}_T$.

Definition

We denote by $\mathcal{R}$ the collection of c.e. Turing degrees.

$\mathcal{R}$ lives inside the interval $[\mathbf{0}_T, \mathbf{0}'_T]$.

The analog $\mathcal{R}$ in the enumeration degrees is the structure $\mathcal{E} = [\mathbf{0}_e, \mathbf{0}'_e]$. We call it the *local structure of the e-degrees*.

What connects $\mathcal{D}_T$ and $\mathcal{D}_e$

Proposition

$A \leq_T B \Leftrightarrow A \oplus \overline{A} \leq_e B \oplus \overline{B}$.

The embedding $\iota : \mathcal{D}_T \to \mathcal{D}_e$, defined by $\iota(\mathbf{d}_T(A)) = \mathbf{d}_e(A \oplus \overline{A})$, preserves the order, the least upper bound, and the jump operator.

$\mathcal{T} = \iota(\mathcal{D}_T)$ is the set of *total* enumeration degrees.

$$(\mathcal{D}_T, \leq_T, \mathbf{0}_T) \cong (\mathcal{T}, \leq_e, \mathbf{0}_e) \subset (\mathcal{D}_e, \leq_e, \mathbf{0}_e)$$

$\mathcal{P} = \iota(\mathcal{R})$ is the set of all Π^0_1 enumeration degrees.

$$(\mathcal{R}, \leq_T, \mathbf{0}_T) \cong (\mathcal{P}, \leq_e, \mathbf{0}_e) \subset (\mathcal{E}, \leq_e, \mathbf{0}_e)$$

Properties of the degree structures

Similarities

1. Both $\mathcal{D}_T$ and $\mathcal{D}_e$ are uncountable structures with least element and no greatest element.
2. They have uncountable chains and antichains.
3. They are not lattices: there are pairs of degrees with no greatest lower bound.

Differences

1. (Spector 1956) In $\mathcal{D}_T$ there are *minimal degrees*, nonzero degrees $\mathbf{m}$ such that the interval $(\mathbf{0}_T, \mathbf{m})$ is empty.
2. (Gutteridge 1971) $\mathcal{D}_e$ is downwards dense.

Properties of the local degree structures

Similarities

1. Both $\mathcal{R}$ and $\mathcal{E}$ are dense countable structures with least element and greatest element.
2. They have countable chains and antichains.
3. They are not lattices: there are pairs of degrees with no greatest lower bound.

Differences

1. (Sacks 1963) In $\mathcal{R}$ every nonzero $\mathbf{a}$ degree can be split into two lesser ones: $\mathbf{c} \vee \mathbf{d} = \mathbf{a}$.
2. (Ahmad 1998) There are non-splittable degrees in $\mathcal{E}$.

Three aspects of degree structures

We will consider questions about these degree structures from three interrelated aspects:

I. The theory of the degree structure and its fragments: what statements (in the language of partial orders) are true in the degree structure.
II. First order definability: what relations on the degree structure can be captured by a structural property;
III. Automorphisms: are there degrees that cannot be structurally distinguished?

Arithmetic vs Degrees

Second order arithmetic $\mathcal{Z}_2$ is the standard model of arithmetic $\mathcal{N} = (\mathbb{N}, 0, 1, +, *, <)$ with an additional sort for sets of natural numbers and a membership relation.

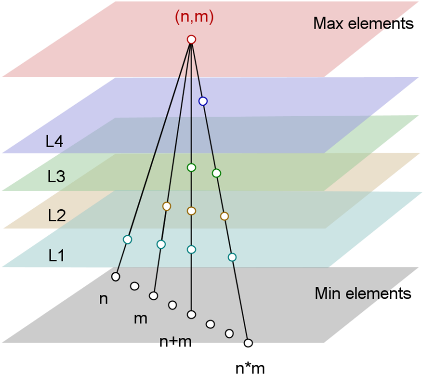
Classical results due to Kleene and Post show that $\leq_T$, $\leq_e$ are definable in second order arithmetic.

This means that any sentence φ in the language of partial orders can be effectively translated to a sentence ψ_T and to a sentence ψ_e in the language of second order arithmetic so that

1. φ is true in $\mathcal{D}_T$ if and only if ψ_T is true in $\mathcal{Z}_2$;
2. φ is true in $\mathcal{D}_e$ if and only if ψ_e is true in $\mathcal{Z}_2$;

The theory of second order arithmetic is (highly) undecidable.

Interpreting arithmetic in $\mathcal{D}_T$ and $\mathcal{D}_e$



There is a way to represent a model of arithmetic in $\mathcal{D}_T$.

Start by translating arithmetic into a partial order.

Prove that this partial order can be embedded into $\mathcal{D}_T$.

The biinterpretability conjecture

Slaman and Woodin show that we can code a model of $(\mathbb{N}, +, \times, <, C)$ where C is a unary predicate on $\mathbb{N}$ using finitely many parameters $\vec{\mathbf{p}}$.

Theorem (Slaman, Woodin 1986, 1997)

The theories of $\mathcal{D}_T$, $\mathcal{D}_e$ and $\mathcal{Z}_2$ have the same complexity.

Theorem (Harrington and Slaman 1995; Ganchev and S. 2012)

The theories of $\mathcal{R}$, $\mathcal{E}$ and $\mathcal{N}$ have the same complexity.

Conjecture (The Biinterpretability conjecture for $\mathcal{D}$)

The relation Bi, where $Bi(\vec{\mathbf{p}}, \mathbf{c})$ holds when $\vec{\mathbf{p}}$ codes a model of $(\mathbb{N}, +, \times, <, C)$ and $\mathbf{d}_e(C) = \mathbf{c}$, is first order definable in $\mathcal{D}$.

Rogers' questions from 1969

Question

Is the jump operator definable in $\mathcal{D}_T$?

If a structure $\mathcal{A}$ has an automorphism that maps an element in a set X to an element outside X then X is not definable.

Question

Are there any nontrivial automorphisms of $\mathcal{D}_T$ or $\mathcal{D}_e$?

Question

Is the copy of $\mathcal{D}_T$ in $\mathcal{D}_e$ (i.e. the total enumeration degrees) definable in $\mathcal{D}_e$?

The three aspects: theory, automorphisms and definability

Let $\mathcal{D}$ denote $\mathcal{D}_T$ or $\mathcal{D}_e$.

If $\mathcal{R}$ is definable in $\mathcal{D}$ then the set $S = \{X \mid \mathbf{d}(X) \in \mathcal{R}\}$ is:

1. definable in second order arithmetic;
2. invariant with respect to $\equiv$.

Lets call such relations $\mathcal{R}$ on $\mathcal{D}$ *nice*.

Example

The graph of Turing jump and the total enumeration degrees are nice relations.

Theorem (Slaman, Woodin 1986, Soskova 2016)

Let $\mathcal{D}$ be $\mathcal{D}_T$ or $\mathcal{D}_e$. The following are equivalent:

1. The Biinterpretability conjecture for $\mathcal{D}$ is true.
2. $\mathcal{D}$ has no nontrivial automorphisms.
3. Every nice relation is definable in $\mathcal{D}$ (without any parameters).

The automorphism analysis

Theorem (Slaman, Woodin 1986, Soskova 2016)

Let $\mathcal{D}$ be $\mathcal{D}_T$ or $\mathcal{D}_e$.

There are at most countably many automorphisms of $\mathcal{D}$.

The biinterpretability conjecture for $\mathcal{D}$ is true if we allow the use of one parameter.

Every nice relation is definable in $\mathcal{D}$ if we allow the use of a parameter.

Definaility in the Turing degrees

A further consequences of the automorphism analysis is that every automorphism of $\mathcal{D}_T$ fixes the degrees above $\mathbf{0}''_T$.

Theorem (Slaman, Woodin 1986)

The double jump is definable in $\mathcal{D}_T$.

Theorem (Slaman, Shore 1999)

The jump is definable in $\mathcal{D}_T$.

Definability of the enumeration jump

Definition (Kalimullin 2003)

A pair of degrees $\mathbf{a}, \mathbf{b}$ is called a $\mathcal{K}$-pair if and only if satisfy:

$$\mathcal{K}(\mathbf{a}, \mathbf{b}) \leftrightharpoons (\forall \mathbf{x})((\mathbf{x} \vee \mathbf{a}) \wedge (\mathbf{x} \vee \mathbf{b}) = \mathbf{x}).$$

Theorem (Kalimullin 2003; Ganchev, S 2015)

The enumeration jump is first order definable: $\mathbf{z}'$ is the largest degree which can be represented as the least upper bound of a $\mathcal{K}$-pair $\mathbf{a}, \mathbf{b}$, such that $\mathbf{a} \leq \mathbf{z}$.

Maximal $\mathcal{K}$-pairs

Definition (Ganchev, S)

A $\mathcal{K}$-pair $\{\mathbf{a}, \mathbf{b}\}$ is maximal if for every $\mathcal{K}$-pair $\{\mathbf{c}, \mathbf{d}\}$ with $\mathbf{a} \leq \mathbf{c}$ and $\mathbf{b} \leq \mathbf{d}$, we have that $\mathbf{a} = \mathbf{c}$ and $\mathbf{b} = \mathbf{d}$.

Conjecture (Ganchev and S 2010)

The joins of maximal $\mathcal{K}$-pairs are exactly the nonzero total degrees.

Partial confirmation: we proved that the conjecture is true in $\mathcal{E}$.

Defining totallity in $\mathcal{D}_e$

Theorem (Cai, Ganchev, Lempp, Miller, S 2016)

The set of total enumeration degrees is first order definable in $\mathcal{D}_e$. The nonzero total degrees are the joins of maximal $\mathcal{K}$-pairs.

Theorem (Selman 1971)

$\mathbf{a} \leq \mathbf{b}$ if and only if every total degree above $\mathbf{b}$ is also above $\mathbf{a}$.

Corollary

Any automorphism of $\mathcal{D}_e$ induces an automorphism of $\mathcal{D}_T$.

If an automorphism of $\mathcal{D}_e$ does not move any total degree then it must be the identity.

If $\mathcal{D}_e$ has a non-trivial automorphism then so does $\mathcal{D}_T$.

Local and global structural interactions

Theorem (Slaman, S (2017))

There is a finite set B of a degrees in $\mathcal{E}$ such that B determines the behavior of every automorphism of $\mathcal{D}_e$: if f, g are two automorphisms such that for ever $\mathbf{x} \in B$ we have $f(\mathbf{x}) = g(\mathbf{x})$ then $f = g$.

Theorem (Slaman, S (2017))

If $\mathcal{D}_e$ has a nontrivial automorphism then so does $\mathcal{E}$ and even $\mathcal{R}$.

The End

Thank you!