Logic, degrees, and definability

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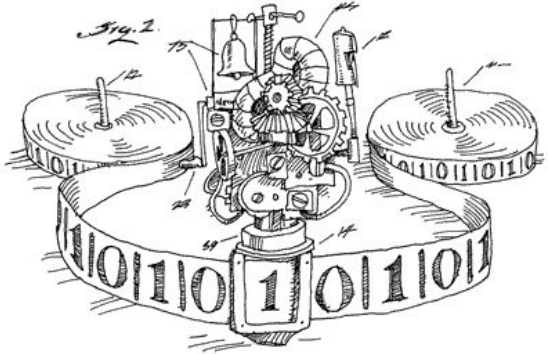
Computable sets and functions



Definition (Turing, Church 1936)

A function $f : \mathbb{N} \to \mathbb{N}$ is *computable* if there is a **computer program** which takes as input a natural number n and outputs $f(n)$.

A set $A \subseteq \mathbb{N}$ is *computable* if its characteristic function is computable.

Computably enumerable sets

Definition

A set $A \subseteq \mathbb{N}$ is *computably enumerable* (c.e.) if it can be enumerated by a computer program. Equivalently, if it is the domain of a computable function.

Example (Davis, Matyasevich, Putnam, Robinson 1970)

A set $S \subseteq \mathbb{N}$ is Diophantine if $S = \{n \mid \exists \bar{m}(P(n, \bar{m}) = 0)\}$.

The Diophantine sets are exactly the c.e. sets.

Example (Novikov, Boone 1955)

The word problem for a finitely presented group is c.e.

Every c.e. set can be coded as the word problem for a finitely presented group.

An incomputable c.e. set

A set A is computable if and only if both A and $\overline{A}$ are c.e.

We can code programs by natural numbers: P_e is the program with code e.

Definition

The *halting set* is $K = \{e \mid P_e \text{ halts on input } e\}$.

Theorem (Turing 1936)

The halting set K is c.e., but not computable.

Proof.

We prove, in fact, that $\overline{K}$ is not c.e.

Assume that it is and let e be such that P_e halts on n if and only if $n \in \overline{K}$.

P_e halts on $e \Leftrightarrow e \in \overline{K} \Leftrightarrow P_e$ does not halt on e. □

Comparing the information content of sets

Consider a program that has access to an external database, an *oracle*. During its computation the program can ask the oracle membership questions: does n belong to you or not?

Definition (Post 1944)

$A \leq_T B$ if and only if there is a program that computes the elements of A using B as an oracle.

Example

Consider a Diophantine set $S = \{n \mid (\exists \bar{m}) P(n, \bar{m}) = 0\}$. Let $P_{e(n)}$ be the program that ignores its input and for a listing $\{\bar{m}_i\}_{i \in \mathbb{N}}$ of all tuples $\bar{m}_i$ calculates $P(n, \bar{m}_0), P(n, \bar{m}_1), \ldots$ until it sees that the result is 0 and then halts. Then $S \leq_T K$ by the program that on input n computes $e(n)$ and asks the oracle whether $e(n) \in K$.

Comparing the information content of sets

Definition (Friedberg and Rogers 1959)

$A \leq_e B$ if there is a program that transforms an enumeration of B to an enumeration of A.

The program is a c.e. table of axioms of the sort:

$$\textit{If } \{x_1, x_2, \ldots, x_k\} \subseteq B \textit{ then } x \in A.$$

Example

Let $S = \{n \mid (\exists \bar{m}) P(n, \bar{m}) = 0\}$ again. $\overline{S} \leq_e \overline{K}$ by the program that consists of the axioms:

$$\textit{If } \{e(n)\} \subseteq \overline{K} \textit{ then } n \in \overline{S}.$$

Degree structures

Definition

1. $A \equiv B$ if and only if $A \leq B$ and $B \leq A$.
2. $\mathbf{d}(A) = \{B \mid A \equiv B\}$.
3. $\mathbf{d}(A) \leq \mathbf{d}(B)$ if an only if $A \leq B$.
4. Let $A \oplus B = \{2n \mid n \in A\} \cup \{2n+1 \mid n \in B\}$. Then $\mathbf{d}(A \oplus B) = \mathbf{d}(A) \vee \mathbf{d}(B)$.

And so we have two partial orders with least upper bound (upper semi-lattices):

1. The Turing degrees $\mathcal{D}_T$ with least element $\mathbf{0}_T$ consisting of all computable sets.
2. The enumeration degrees $\mathcal{D}_e$ with least element $\mathbf{0}_e$ consisting of all c.e. sets.

The jump operation

The halting set with respect to A is the set
$K_A = \{e \mid P_e \text{ using oracle } A \text{ halts on input } e\}$.

$A <_T K_A$.

Definition

The jump of a Turing degree is $\mathbf{d}_T(A)' = \mathbf{d}_T(K_A)$.

We can apply a similar construction to enumeration reducibility to obtain the enumeration jump.

We always have $\mathbf{a} < \mathbf{a}'$.

What connects $\mathcal{D}_T$ and $\mathcal{D}_e$

Proposition

$A \leq_T B \Leftrightarrow A \oplus \overline{A} \leq_e B \oplus \overline{B}$.

The embedding $\iota : \mathcal{D}_T \to \mathcal{D}_e$, defined by $\iota(\mathbf{d}_T(A)) = \mathbf{d}_e(A \oplus \overline{A})$, preserves the order, the least upper bound, and the jump operator.

$\mathcal{T} = \iota(\mathcal{D}_T)$ is the set of *total* enumeration degrees.

$$(\mathcal{D}_T, \leq_T, \mathbf{0}_T) \cong (\mathcal{T}, \leq_e, \mathbf{0}_e) \subseteq (\mathcal{D}_e, \leq_e, \mathbf{0}_e)$$

Medvedev (1955) proved that there are nontotal degrees.

Properties of the degree structures

Similarities

1. Both $\mathcal{D}_T$ and $\mathcal{D}_e$ are uncountable structures with least element and no greatest element.
2. They have uncountable chains and antichains.
3. They are not lattices: there are pairs of degrees with no greatest lower bound.

Differences

1. (Spector 1956) In $\mathcal{D}_T$ there are *minimal degrees*, nonzero degrees $\mathbf{m}$ such that the interval $(\mathbf{0}_T, \mathbf{m})$ is empty.
2. (Gutteridge 1971) $\mathcal{D}_e$ is downwards dense.

Three aspects of degree structures

We will consider questions about these degree structures from three interrelated aspects:

I. The theory of the degree structure and its fragments: what statements (in the language of partial orders) are true in the degree structure.
II. First order definability: what relations on the degree structure can be captured by a structural property;
III. Automorphisms: are there degrees that cannot be structurally distinguished?

The existential theory

To understand: *"What existential sentences in the language of partial order are true in a degree structure $\mathcal{D}$?"*

we ask *"What finite partial orders can be embedded in $\mathcal{D}$?"*.

The answer for both $\mathcal{D}_T$ and $\mathcal{D}_e$ is "all".

Theorem

There is an algorithm that decides whether an $\exists$-sentence is true in $\mathcal{D}_T$ or $\mathcal{D}_e$.

The two quantifier theory

The problem of deciding the 2-quantifier theory is equivalent to the following:

Problem

We are given a finite lattice P and partial orders $Q_1, \ldots Q_n \supseteq P$. Does every embedding of P extend to an embedding of one of the Q_i?

When $n = 1$, we call this the *Extension of embeddings problem*.

The two-quantifier theory of $\mathcal{D}_T$ is decidable

Theorem (Lerman 1971)

Every finite lattice can be embedded into $\mathcal{D}_T$ as an initial interval.

This presents a major obstacle to extension of embeddings: if P is embedded as an initial interval, then there is no extension to a partial order that puts a new element below any element of P.

Theorem (Shore 1978; Lerman 1983)

The two quantifier theory of the Turing degrees is decidable.

The two quantifier theory of $\mathcal{D}_e$

The downward density of $\mathcal{D}_e$ makes the approach used in $\mathcal{D}_T$ not applicable.

Theorem (Slaman, Calhoun 1996, Kent, Lewis-Pye, Sorbi 2012)

There are e-degrees $\mathbf{a} < \mathbf{b}$ such that the interval $(\mathbf{a}, \mathbf{b})$ is empty and for every degree $\mathbf{x} < \mathbf{b}$ we have $\mathbf{x} \leq \mathbf{a}$.

Theorem (Lempp, Slaman, S 2020)

Every finite distributive lattice can be embedded as an interval $[\mathbf{a}, \mathbf{b}]$ so that if $\mathbf{x} \leq \mathbf{b}$ then $\mathbf{x} \in [\mathbf{a}, \mathbf{b}]$ or $\mathbf{x} \leq \mathbf{a}$.

Theorem (Lempp, Slaman, S 2020)

The extension of embeddings problem in $\mathcal{D}_e$ is decidable.

Arithmetic vs Degrees

Second order arithmetic $\mathcal{Z}_2$ is the standard model of arithmetic $\mathcal{N} = (\mathbb{N}, 0, 1, +, *, <)$ with an additional sort for sets of natural numbers and a membership relation.

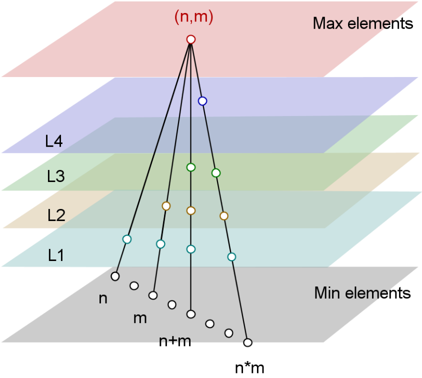
Classical results due to Kleene and Post show that $\leq_T$, $\leq_e$ are definable in second order arithmetic.

This means that any sentence φ in the language of partial orders can be effectively translated to a sentence ψ_T and to a sentence ψ_e in the language of second order arithmetic so that

1. φ is true in $\mathcal{D}_T$ if and only if ψ_T is true in $\mathcal{Z}_2$;
2. φ is true in $\mathcal{D}_e$ if and only if ψ_e is true in $\mathcal{Z}_2$;

The theory of second order arithmetic is (highly) undecidable.

Interpreting arithmetic in $\mathcal{D}_T$ and $\mathcal{D}_e$



There is a way to represent a model of arithmetic in $\mathcal{D}_T$.

Start by translating arithmetic into a partial order.

Prove that this partial order can be embedded into $\mathcal{D}_T$.

Theorem (Slaman, Woodin 1986, 1997)

Every countable relation $\mathcal{D}_T$ or $\mathcal{D}_e$ is uniformly definable in the respective structure using finitely many parameters.

The biinterpretability conjecture

Slaman and Woodin show that it is a definable property of finitely many parameters $\vec{\mathbf{p}}$ that they code a model of $(\mathbb{N}, +, \times, <, C)$ where C is a unary predicate on $\mathbb{N}$.

Theorem (Slaman, Woodin 1986, 1997)

The theories of $\mathcal{D}_T$, $\mathcal{D}_e$ and $\mathcal{Z}_2$ have the same complexity.

Conjecture (The Biinterpretability conjecture)

The relation Bi, where $Bi(\vec{\mathbf{p}}, \mathbf{c})$ holds when $\vec{\mathbf{p}}$ codes a model of $(\mathbb{N}, +, \times, <, C)$ and $\mathbf{d}_e(C) = \mathbf{c}$, is first order definable in $\mathcal{D}_e$.

Rogers' questions from 1969

Question

Is the jump operator definable in $\mathcal{D}_T$?

If a structure $\mathcal{A}$ has an automorphism that maps an element in a set X to an element outside X then X is not definable.

Question

Are there any nontrivial automorphisms of $\mathcal{D}_T$ or $\mathcal{D}_e$?

Question

Is the copy of $\mathcal{D}_T$ in $\mathcal{D}_e$ (i.e. the total enumeration degrees) definable in $\mathcal{D}_e$?

The three aspects: theory, automorphisms and definability

If $\mathcal{R}$ is definable in $\mathcal{D}_T$ then the set $S = \{X \mid \mathbf{d}_T(X) \in \mathcal{R}\}$ is:

1. definable in second order arithmetic;
2. invariant with respect to $\equiv_T$.

Lets call relations $\mathcal{R}$ on $\mathcal{D}_T$ that correspond to relations S in second order arithmetic of this sort *nice*.

Theorem (Slaman, Woodin 1986)

The following are equivalent:

1. The Biinterpretability conjecture for $\mathcal{D}_T$ is true.
2. $\mathcal{D}_T$ has no nontrivial automorphisms.
3. Every nice relation is definable in $\mathcal{D}_T$ (without any parameters).

The automorphism analysis for the Turing degrees

Cohen in 1963 invented the method of forcing to prove that the continuum hypothesis is independent from ZFC.

The method gives a way to extend a model V of ZFC to a model $V[G]$ in which a generic object with predetermined properties has been added.

Slaman and Woodin attempted to use forcing to build a generic model $V[G]$ which adds a nontrivial automorphism for $\mathcal{D}_T$. They found that if $V[G]$ has such an automorphism, then so does V.

Theorem (Slaman, Woodin 1986)

There are at most countably many automorphisms of $\mathcal{D}_T$.

The biinterpretability conjecture for $\mathcal{D}_T$ is true if we allow the use of one parameter.

Every nice relation is definable in $\mathcal{D}_T$ if we allow the use of a parameter.

Definaility in the Turing degrees

A further consequences of the automorphism analysis is that every automorphism of $\mathcal{D}_T$ fixes the degrees above $\mathbf{0}''_T$.

Theorem (Slaman, Woodin 1986)

The double jump is definable in $\mathcal{D}_T$.

Theorem (Slaman, Shore 1999)

The jump is definable in $\mathcal{D}_T$.

Definability of the enumeration jump

Definition (Kalimullin 2003)

A pair of degrees $\mathbf{a}, \mathbf{b}$ is called a $\mathcal{K}$-pair if and only if satisfy:

$$\mathcal{K}(\mathbf{a}, \mathbf{b}) \leftrightharpoons (\forall \mathbf{x})((\mathbf{x} \vee \mathbf{a}) \wedge (\mathbf{x} \vee \mathbf{b}) = \mathbf{x}).$$

A pair of degrees $\mathbf{a}, \mathbf{b}$ is called a $\mathcal{K}$-pair relative to $\mathbf{z}$ if and only if satisfy:

$$\mathcal{K}_{\mathbf{z}}(\mathbf{a}, \mathbf{b}) \leftrightharpoons (\forall \mathbf{x} \geq \mathbf{z})((\mathbf{x} \vee \mathbf{a} \vee \mathbf{z}) \wedge (\mathbf{x} \vee \mathbf{b} \vee \mathbf{z}) = \mathbf{x}).$$

Theorem (Kalimullin)

The enumeration jump is first order definable: $\mathbf{z}'$ is the largest degree which can be represented as the least upper bound of a triple $\mathbf{a}, \mathbf{b}, \mathbf{c}$, such that $\mathcal{K}_{\mathbf{z}}(\mathbf{a}, \mathbf{b})$, $\mathcal{K}_{\mathbf{z}}(\mathbf{b}, \mathbf{c})$ and $\mathcal{K}_{\mathbf{z}}(\mathbf{c}, \mathbf{a})$.

Maximal $\mathcal{K}$-pairs

Definition (Ganchev, S)

A $\mathcal{K}$-pair $\{\mathbf{a}, \mathbf{b}\}$ is maximal if for every $\mathcal{K}$-pair $\{\mathbf{c}, \mathbf{d}\}$ with $\mathbf{a} \leq \mathbf{c}$ and $\mathbf{b} \leq \mathbf{d}$, we have that $\mathbf{a} = \mathbf{c}$ and $\mathbf{b} = \mathbf{d}$.

Conjecture (Ganchev and S 2010)

The joins of maximal $\mathcal{K}$-pairs are exactly the nonzero total degrees.

Partial confirmation: we proved that the conjecture is true for e-degrees bounded by $\mathbf{0}'_e$.

The automorphisms of $\mathcal{D}_e$

Theorem (S 2012)

There are at most countably many automorphisms of $\mathcal{D}_e$.

The biinterpretability conjecture for $\mathcal{D}_e$ is true if we allow the use of one parameter.

Every nice relation is definable in $\mathcal{D}_e$ if we allow the use of a parameter.

Example

The set of total degrees is a nice relation, and hence it is definable in $\mathcal{D}_e$ with one parameter.

Defining totallity in $\mathcal{D}_e$

Theorem (Cai, Ganchev, Lempp, Miller, S 2016)

The set of total enumeration degrees is first order definable in $\mathcal{D}_e$. The nonzero total degrees are the joins of maximal $\mathcal{K}$-pairs.

Theorem (Selman 1971)

$\mathbf{a} \leq \mathbf{b}$ if and only if every total degree above $\mathbf{b}$ is also above $\mathbf{a}$.

Corollary

The total enumeration degrees form a definable automorphism base of the enumeration degrees. If $\mathcal{D}_e$ has a non-trivial automorphism then so does $\mathcal{D}_T$.

The continuous degrees

Definition (Lacombe 1957)

A *computable metric space* is a metric space $\mathcal{M}$ together with a countable dense sequence $Q^{\mathcal{M}} = \{q_n^{\mathcal{M}}\}_{n\in\mathbb{N}}$ on which the metric is computable.

For example $\mathbb{R}$, $\mathcal{C}[0,1]$, and *Hilbert cube* $[0,1]^{\mathbb{N}}$ can be thought of as computable.

Definition

A *name* for a point $x \in \mathcal{M}$ is a function that takes as input a rational precision ε and outputs the index i so that $d_{\mathcal{M}}(x, q_i^{\mathcal{M}}) < \varepsilon$.

Definition (Miller 2004)

If x and y are members of (possibly different) computable metric spaces, then $x \leq_r y$ if there is a uniform way to compute a name for x from a name for y.

The continuous degrees

The reducibility $\leq_r$ induces the *continuous degrees*.

Theorem (Miller 2004)

Every continuous degree contains a point from $[0,1]^{\mathbb{N}}$ and a point from $C[0,1]$.

For $\alpha \in [0,1]^{\mathbb{N}}$, let

$$C_\alpha = \bigoplus_{i \in \mathbb{N}} \{q \in \mathbb{Q} \mid q < \alpha(i)\} \oplus \{q \in \mathbb{Q} \mid q > \alpha(i)\}.$$

Enumerating C_α is exactly as hard as computing a name for α. So $\alpha \mapsto C_\alpha$ induces an embedding of the continuous degrees into the enumeration degrees.

Topology realized as a structural property

Elements of $\mathbb{R}$ are mapped to the *total* degree of their least Turing degree name, the degree of their binary representation.

Theorem (Miller 2004)

There is a nontotal continuous degree.

Every known proof of this result uses nontrivial topological facts: Brouwer's fixed point theorem for multivalued functions on an infinite dimensional space, or Sperner's lemma, or results from topological dimension theory: $[0,1]^{\mathbb{N}}$ is strongly infinite dimensional.

Theorem (Andrews, Igusa, Miller, S.)

An enumeration degree $\mathbf{a}$ is continuous if and only if it is *almost total*: if $\mathbf{x} \not\leq \mathbf{a}$ and $\mathbf{x}$ is total then $\mathbf{a} \vee \mathbf{x}$ is total.

The continuous degrees are definable in $\mathcal{D}_e$.

Topological classification of classes of e-degrees

Definition (Kihara, Pauly 2018)

Consider a second countable topological space X and listing of an open basis $B^X = \{B_i\}_{i<\mathbb{N}}$.

A name for a point $x \in X$ is an enumeration of the set $N_x = \{i \mid x \in B_i\}$.

Say that $x \leq y$ if every name for y (uniformly) computes a name for x.

Thus a represented space X gives rise to a class of e-degrees $\mathcal{D}_X \subset \mathcal{D}_e$.

Example

- $\mathcal{D}_{\mathbb{R}}$ is the total enumeration degrees.
- $\mathcal{D}_{[0,1]^{\mathbb{N}}}$ is the continuous degrees.
- $\mathcal{D}_{S^\infty} = \mathcal{D}_e$, where S is the Sierpinski topology $\{\emptyset, \{1\}, \{0,1\}\}$.

Kihara, Ng, and Pauly 2019 investigate $\mathcal{D}_X$, where X is the ∞-power of the: cofinite topology on $\mathbb{N}$, telophase space, double origin space, quasi-Polish Roy space, irregular lattice space.

The End

Thank you!