Logic, degrees, and definability

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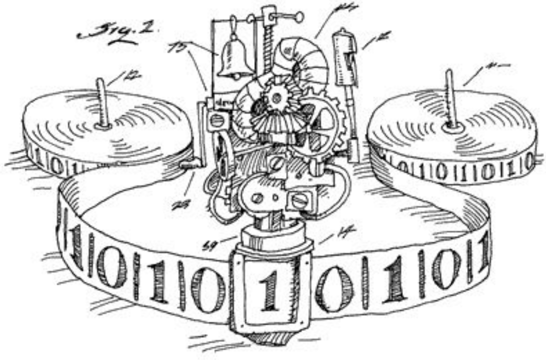
Computable sets and functions



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A function $f : \mathbb{N} \to \mathbb{N}$ is *computable* if there is a **computer program** which takes as input a natural number n and outputs $f(n)$.

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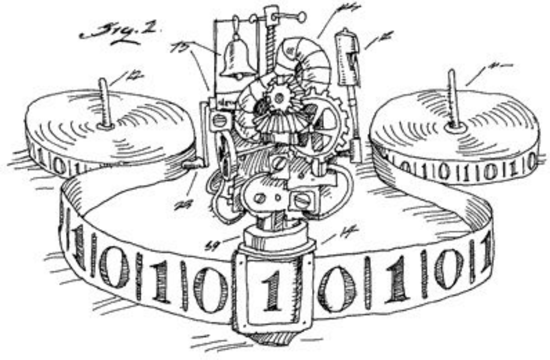

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A set $A \subseteq \mathbb{N}$ is *computable* if its characteristic function is computable: the function $A(n)$, where $A(n) = 1$ if $n \in A$ and $A(n) = 0$ if $n \notin A$.

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Example (Davis, Matyasevich, Putnam, Robinson 1970)

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The Diophantine sets are exactly the c.e. sets.

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Proof.

No general procedure for bug checks will do.
Now, I won't just assert that, I'll prove it to you.
I will prove that although you might work till you drop,
you cannot tell if a computation will stop.
For imagine you had a procedure called P, ... Geoffrey K. Pullum ☐

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P_e halts on $e \Leftrightarrow e \in \overline{K} \Leftrightarrow P_e$ does not halt on e. □

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Let $S = \{n \mid (\exists \bar{m}) P(n, \bar{m}) = 0\}$ again. $\overline{S} \leq_e \overline{K}$ by the program that consists of the axioms:

$$\textit{If } \{e(n)\} \subseteq \overline{K} \textit{ then } n \in \overline{S}.$$

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And so we have two partial orders with least upper bound (upper semi-lattices):

1. The Turing degrees $\mathcal{D}_T$ with least element $\mathbf{0}_T$ consisting of all computable sets.
2. The enumeration degrees $\mathcal{D}_e$ with least element $\mathbf{0}_e$ consisting of all c.e. sets.

The jump operation

The halting set with respect to A is the set
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We always have $\mathbf{a} < \mathbf{a}'$.

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We can think of the $\mathcal{D}_T$ as living inside $\mathcal{D}_e$, namely the Turing degrees are the enumeration degrees of sets of the form $A \oplus \overline{A}$.

Properties of the degree structures

Similarities

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2. (Gutteridge 1971) $\mathcal{D}_e$ has no minimal degrees.

Three aspects of degree structures

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I. The theory of the degree structure: what statements (in the language of partial orders) are true in the degree structure.
II. First order definability: what relations on the degree structure can be captured by a structural property;
III. Automorphisms: are there degrees that cannot be structurally distinguished?

Arithmetic vs Degrees

In second order arithmetic $\mathcal{Z}_2$ we have the natural numbers with the usual arithmetic operations and relations $\mathcal{N} = (\mathbb{N}, 0, 1, +, *, <)$, but we also talk about sets of natural numbers and their members.

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Classical results due to Kleene and Post show that $\leq_T$, $\leq_e$ are definable in second order arithmetic.

This means that any statement φ about degrees can be translated to a statement ψ_T and to a statement ψ_e about sets in second order arithmetic so that

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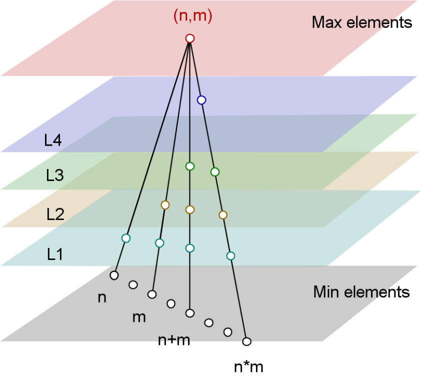
The theory of second order arithmetic is (highly) undecidable.

Interpreting arithmetic in $\mathcal{D}_T$ and $\mathcal{D}_e$



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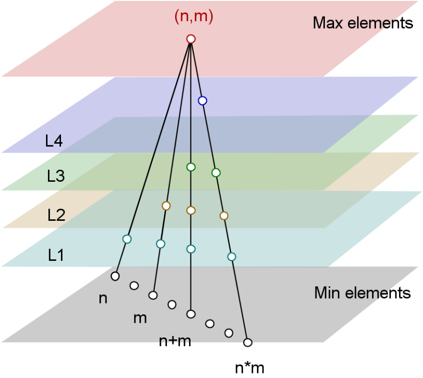

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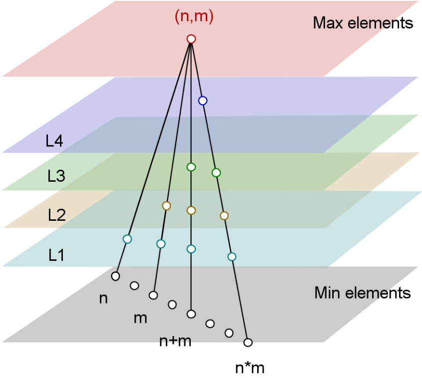

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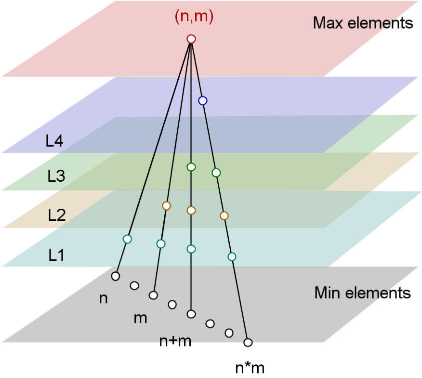

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Theorem (Slaman, Woodin 1986, 1997)

Every countable relation on $\mathcal{D}_T$ or $\mathcal{D}_e$ is uniformly definable in the respective structure using finitely many parameters.

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Conjecture (The Biinterpretability conjecture)

The relationship between $\mathcal{D}_T$ and $\mathcal{Z}_2$ and the relationship between $\mathcal{D}_e$ and $\mathcal{Z}_2$ is much stronger: once we code arithmetic in the degree structure we can structurally identify the relationship between a set in the coded model and its actual degree.

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Question

Is the copy of $\mathcal{D}_T$ in $\mathcal{D}_e$ definable in $\mathcal{D}_e$?

The three aspects: theory, automorphisms and definability

Theorem (Slaman, Woodin 1986)

The following are equivalent:

1. The Biinterpretability conjecture for $\mathcal{D}_T$ is true.
2. $\mathcal{D}_T$ has no nontrivial automorphisms.
3. There is a complete characterization of the relation that are definable in $\mathcal{D}_T$ in terms of relations definable in second order arithmetic (which includes the jump operator).

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Theorem (Slaman, Woodin 1986, Soskova 2016)

There are very few (at most countably many) automorphisms of $\mathcal{D}_T$ and of $\mathcal{D}_e$.

Theorem (Slaman, Shore 1999)

The jump is definable in $\mathcal{D}_T$.

Definability of the enumeration jump

Definition (Kalimullin 2003)

A pair of degrees $\mathbf{a}, \mathbf{b}$ is called a $\mathcal{K}$-pair if and only if satisfy:

$$\mathcal{K}(\mathbf{a}, \mathbf{b}) \leftrightharpoons (\forall \mathbf{x})((\mathbf{x} \vee \mathbf{a}) \wedge (\mathbf{x} \vee \mathbf{b}) = \mathbf{x}).$$

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Theorem (Kalimullin)

The enumeration jump is first order definable: $\mathbf{z}'$ is the largest degree which can be represented as the least upper bound of a pair $\mathbf{a}, \mathbf{b}$, such that $\mathcal{K}(\mathbf{a}, \mathbf{b})$ and $\mathbf{a} \leq \mathbf{z}$.

Maximal $\mathcal{K}$-pairs

Definition (Ganchev, S)

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Conjecture (Ganchev and S 2010)

The joins of maximal $\mathcal{K}$-pairs are exactly the nonzero total degrees.

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A $\mathcal{K}$-pair $\{\mathbf{a}, \mathbf{b}\}$ is maximal if for every $\mathcal{K}$-pair $\{\mathbf{c}, \mathbf{d}\}$ with $\mathbf{a} \leq \mathbf{c}$ and $\mathbf{b} \leq \mathbf{d}$, we have that $\mathbf{a} = \mathbf{c}$ and $\mathbf{b} = \mathbf{d}$.

Conjecture (Ganchev and S 2010)

The joins of maximal $\mathcal{K}$-pairs are exactly the nonzero total degrees.

Theorem (Cai, Ganchev, Lempp, Miller, S 2016)

The set of total enumeration degrees is first order definable in $\mathcal{D}_e$.

The open problem remains

Corollary

If $\mathcal{D}_e$ has a non-trivial automorphism then so does $\mathcal{D}_T$.

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Corollary

If $\mathcal{D}_e$ has a non-trivial automorphism then so does $\mathcal{D}_T$.

Question

Are there any nontrivial automorphisms of $\mathcal{D}_T$ or $\mathcal{D}_e$?

Thank you

From “Inside the mind of Alan Turing, the genius behind The Imitation Game” by Barry Cooper. Illustration by Jin Wicked.