The e-verse

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Outline

Enumeration reducibility gives a way to compare the algorithmic complexity between sets of natural numbers.

The structure of the enumeration degrees can be viewed as an extension of the better studied partial order of the Turing degrees.

In certain cases, the enumeration degrees capture the algorithmic content of mathematical objects, while the Turing degrees fail.

Certain open problems in degree theory seem more approachable in the extended context of the enumeration degrees.

We have been working to develop a richer “e-verse”: a system of classes of enumeration degrees with interesting properties and relationships, in order to better understand the enumeration degrees.

Computable sets and functions

Definition (Turing, Church 1936)

A function $f : \mathbb{N} \to \mathbb{N}$ is *computable* if there is a **computer program** which takes as input a natural number n and outputs $f(n)$.

A set $A \subseteq \mathbb{N}$ is *computable* if its characteristic function is computable.

Computably enumerable sets

Definition

A set $A \subseteq \mathbb{N}$ is *computably enumerable* (c.e.) if it can be enumerated by a computer program.

Example (Davis, Matyasevich, Putnam, Robinson 1970)

A set $S \subseteq \mathbb{N}$ is Diophantine if $S = \{n \mid \exists \bar{m}(P(n, \bar{m}) = 0)\}$ for some polynomial $P(x, \bar{y})$.

The Diophantine sets are exactly the c.e. sets.

Example (Novikov, Boone 1955)

The word problem for a finitely presented group is c.e.

Every c.e. set can be coded as the word problem for a finitely presented group.

An incomputable c.e. set

A set A is computable if and only if both A and $\overline{A}$ are c.e.

We can code programs by natural numbers: P_e is the program with code e.

Definition

The *halting set* is $K = \{e \mid P_e \text{ halts on input } e\}$.

Theorem (Turing 1936)

The halting set K is c.e., but not computable.

Comparing the information content of sets

Consider a program that has access to an external database, an *oracle*. During its computation the program can ask the oracle membership questions: does n belong to you or not?

Definition (Post 1944)

A is *Turing reducible to* B ($A \leq_T B$) if and only if there is a program that computes the elements of A using B as an oracle.

Example

For every A, we have $\overline{A} \leq_T A$: on input n check whether n is in the oracle and output the opposite.

Comparing the information content of sets

Definition (Friedberg and Rogers 1959)

A is *enumeration reducible to* B ($A \leq_e B$) if there is a program that transforms an enumeration of B to an enumeration of A.

The program can always be taken to be a c.e. table of axioms of the sort:

$$\textit{If } \{x_1, x_2, \ldots, x_k\} \subseteq B \textit{ then } x \in A.$$

Example

$\overline{K} \nleq_e K$ because $\overline{K}$ is not c.e.

Degree structures

Let $\leq_* \in \{\leq_T, \leq_e\}$.

Definition

1. $A \equiv_* B$ if and only if $A \leq_* B$ and $B \leq_* A$.
2. The $*$-degree of A is $\mathbf{d}_*(A) = \{B \mid A \equiv_* B\}$.
3. $\mathbf{d}_*(A) \leq \mathbf{d}_*(B)$ if and only if $A \leq_* B$.
4. $\mathbf{d}_*(A \oplus B) = \mathbf{d}_*(A) \vee \mathbf{d}_*(B)$, where $A \oplus B = \{2n \mid n \in A\} \cup \{2n+1 \mid n \in B\}$.

And so we have two partial orders with least upper bound:

1. The Turing degrees $\mathcal{D}_T$ with least element $\mathbf{0}_T$ consisting of all computable sets.
2. The enumeration degrees $\mathcal{D}_e$ with least element $\mathbf{0}_e$ consisting of all c.e. sets.

The jump operation

The halting set with respect to A is the set
$K^A = \{e \mid P_e \text{ using oracle } A \text{ halts on input } e\}$.

$A <_T K^A$.

Definition

The *jump* of a Turing degree is $\mathbf{d}_T(A)' = \mathbf{d}_T(K^A)$.

We can apply a similar construction to enumeration reducibility to obtain the enumeration jump.

What connects $\mathcal{D}_T$ and $\mathcal{D}_e$

Proposition

$A \leq_T B \Leftrightarrow A \oplus \overline{A} \leq_e B \oplus \overline{B}$.

So mapping $\mathbf{d}_T(A)$ to $\mathbf{d}_e(A \oplus \overline{A})$ induces an embedding. It preserves the order, the least upper bound, and the jump operator.

The image of the Turing degrees under this embedding is the set $\mathcal{T}$ of *total* enumeration degrees.

$$(\mathcal{D}_T, \leq_T, \mathbf{0}_T) \cong (\mathcal{T}, \leq_e, \mathbf{0}_e) \subseteq (\mathcal{D}_e, \leq_e, \mathbf{0}_e)$$

Medvedev (1955) proved that there are nontotal degrees.

$\mathcal{D}_T$ and $\mathcal{D}_e$ exhibit structural differences: $\mathcal{D}_e$ is downwards dense, while $\mathcal{D}_T$ is not.

Measuring the computability of other mathematical objects

Question

What should the degree of a real number r be?

Fix some effective listing of the rational numbers: $\{q_i\}_{i\in\mathbb{N}}$. We can then describe r via a *name*: a function f on the natural numbers such that

$$|r - q_{f(n)}| < \frac{1}{n}.$$

Every real has infinitely many names, of infinitely many Turing degrees, but...

Proposition

For every real r there is a least Turing degree $\mathbf{d}_r$, such that r has a name of degree $\mathbf{d}_r$ (the degree of the binary representation of r).

Measuring the computability of other mathematical objects

Question

What should the degree of a continuous function F be?

We can approximate continuous functions via piece-wise linear function L_i with rational endpoints. We can fix some effective listing of these function: $\{L_i\}_{i \in \mathbb{N}}$. We can then describe F via a *name*: a function f on the natural numbers such that

$$\|F - L_{f(n)}\|_\infty < \frac{1}{n}.$$

Theorem (Miller 04)

Not every continuous function has a name of least Turing degree. However, there is a meaningful way to assign an enumeration degree to every continuous function.

Three aspects of degree structures

We will consider questions about these degree structures from three interrelated aspects:

I. The theory of the degree structure: what statements (in the language of partial orders) are true in the degree structure.
II. Automorphisms: are there degrees that cannot be structurally distinguished?
III. First order definability: what relations on the degree structure can be captured by a structural property;

Arithmetic vs Degrees

Second order arithmetic $\mathcal{Z}_2$ is the theory of $\mathcal{N} = (\mathbb{N}, +, \cdot)$ where we can quantify both over natural numbers and sets of natural numbers.

Classical results due to Kleene and Post show that $\leq_T$, $\leq_e$ are definable in second order arithmetic.

This means that any sentence φ in the language of partial orders can be effectively translated to a sentence ψ_T and to a sentence ψ_e in the language of second order arithmetic so that

1. φ is true in $\mathcal{D}_T$ if and only if ψ_T is true in $\mathcal{Z}_2$;
2. φ is true in $\mathcal{D}_e$ if and only if ψ_e is true in $\mathcal{Z}_2$;

The theory of second order arithmetic is (highly) undecidable.

Interpreting arithmetic in $\mathcal{D}_T$ and $\mathcal{D}_e$

There is a way to represent a model of second order arithmetic in $\mathcal{D} \in \{\mathcal{D}_T, \mathcal{D}_e\}$:

- Any countable set $M \subseteq \mathcal{D}^n$ can be defined using a finite sequence of parameters $\vec{\mathbf{p}}$ from $\mathcal{D}$.
- You can use a countable subset of $\mathcal{D}$ and two countable subsets of $\mathcal{D}^3$ to represent a copy of $(\mathbb{N}, +, \cdot)$.
- It is a definable property of $\vec{\mathbf{p}}$ whether it codes a copy of $(\mathbb{N}, +, \cdot, C)$, where $C \subseteq \mathbb{N}$.

Theorem (Simpson 1977; Slaman, Woodin 1997)

The theories of $\mathcal{D}_T$ and $\mathcal{D}_e$ have the same complexity as $\mathcal{Z}_2$.

Conjecture (The Biinterpretability conjecture for $\mathcal{D}$)

The relation Bi, where $Bi(\vec{\mathbf{p}}, \mathbf{c})$ holds when $\vec{\mathbf{p}}$ codes a model of $(\mathbb{N}, +, \cdot, C)$ and $\mathbf{d}_e(C) = \mathbf{c}$, is first order definable in $\mathcal{D}$.

The three aspects: theory, automorphisms and definability

If $\mathcal{R}$ is definable in $\mathcal{D} \in \{\mathcal{D}_e, \mathcal{D}_T\}$ then the set $S = \{X \mid \mathbf{d}_*(X) \in \mathcal{R}\}$ is:

1. definable in second order arithmetic;
2. invariant with respect to the corresponding equivalence relation.

Let's call relations $\mathcal{R}$ with this property *nice*.

Theorem (Slaman, Woodin 1986, S 2016)

The following are equivalent for $\mathcal{D} \in \{\mathcal{D}_T, \mathcal{D}_e\}$:

1. The Biinterpretability conjecture for $\mathcal{D}$ is true.
2. $\mathcal{D}$ has no nontrivial automorphisms.
3. Every nice relation is definable in $\mathcal{D}$ without any parameters.

The automorphism analysis

Theorem (Slaman, Woodin 1986, S 2016)

Let $\mathcal{D} \in \{\mathcal{D}_T, \mathcal{D}_e\}$:

There are at most countably many automorphisms of $\mathcal{D}$.

The biinterpretability conjecture for $\mathcal{D}$ is true if we allow the use of one parameter.

Every nice relation is definable in $\mathcal{D}$ if we allow the use of a parameter.

Definability in the Turing degrees

Theorem (Slaman, Shore 1999)

The jump is definable in $\mathcal{D}_T$.

The proof is far from "natural" and the definition passes through the coding machinery of the automorphism analysis.

Definability of the enumeration jump

Definition (Kalimullin 2003)

A pair of enumeration degrees $\mathbf{a}, \mathbf{b}$ is called a $\mathcal{K}$-pair if and only if satisfy:

$$\mathcal{K}(\mathbf{a}, \mathbf{b}) \leftrightharpoons (\forall \mathbf{x})((\mathbf{x} \vee \mathbf{a}) \wedge (\mathbf{x} \vee \mathbf{b}) = \mathbf{x}).$$

Theorem (Kalimullin 2003)

The enumeration jump is first order definable: $\mathbf{z}'$ is the largest degree which can be written as $\mathbf{a} \vee \mathbf{b}$, where $\mathcal{K}(\mathbf{a}, \mathbf{b})$, and $\mathbf{a} \leq \mathbf{z}$.

Theorem (Cai, Ganchev, Lempp, Miller, S 2016)

The set of total enumeration degrees is first order definable in $\mathcal{D}_e$: The nonzero total degrees are the joins of maximal $\mathcal{K}$-pairs.

Consequences for the automorphism problem

Theorem (Selman 1971)

For enumeration degrees $\mathbf{a}$ and $\mathbf{b}$ we have $\mathbf{a} \leq \mathbf{b}$ if and only if every total degree above $\mathbf{b}$ is also above $\mathbf{a}$.

Corollary

The total enumeration degrees form a definable automorphism base of the enumeration degrees. If $\mathcal{D}_e$ has a non-trivial automorphism then so does $\mathcal{D}_T$.

Insight

We need to identify more subclasses of e-degrees, understand how they interact with each other, to help us understand what's going on. We turn to effective mathematics, focusing on cases where enumeration degrees provide a better tool for capturing algorithmic content.

The continuous degrees

Definition (Lacombe 1957)

A *computable metric space* is a metric space $\mathcal{M}$ together with a countable dense sequence $Q^{\mathcal{M}} = \{q_n^{\mathcal{M}}\}_{n\in\mathbb{N}}$ on which the metric is computable.

For example $\mathbb{R}$, $\mathcal{C}[0,1]$, and the Hilbert cube $[0,1]^{\mathbb{N}}$ are computable.

Definition

A *name* for a point $x \in \mathcal{M}$ is a function f on $\mathbb{N}$ such that $d_{\mathcal{M}}(x, q_{f(n)}^{\mathcal{M}}) < \frac{1}{n}$.

Definition (Miller 2004)

If x and y are members of (possibly different) computable metric spaces, then $x \leq_r y$ if there is a way to compute a name for x from a name for y.

The continuous degrees

The reducibility $\leq_r$ induces the *continuous degrees*.

Theorem (Miller 2004)

Every continuous degree contains a point from $[0,1]^{\mathbb{N}}$ and a point from $C[0,1]$.

For $\alpha \in [0,1]^{\mathbb{N}}$, let

$$C_\alpha = \bigoplus_{i\in\mathbb{N}} \{q \in \mathbb{Q} \mid q < \alpha(i)\} \oplus \{q \in \mathbb{Q} \mid q > \alpha(i)\}.$$

Enumerating C_α is exactly as hard as computing a name for α. So $\alpha \mapsto C_\alpha$ induces an embedding of the continuous degrees into the enumeration degrees.

Topology realized as a structural property

Elements of $\mathbb{R}$ are mapped to the *total* degree of their least Turing degree name.

Theorem (Miller 2004)

There is a nontotal continuous degree.

Every known proof of this result uses topological facts: Brouwer's fixed point theorem for multivalued functions on an infinite dimensional space, or Sperner's lemma, or the results that $[0,1]^{\mathbb{N}}$ is strongly infinite dimensional.

Theorem (Andrews, Igusa, Miller, S. 2019)

An enumeration degree $\mathbf{a}$ is continuous if and only if it is *almost total*: if for all $\mathbf{x} \not\leq \mathbf{a}$ such that $\mathbf{x}$ is total we have $\mathbf{a} \vee \mathbf{x}$ is also total.

Therefore the continuous degrees are definable in $\mathcal{D}_e$.

Topological classification of classes of e-degrees

Definition (Kihara, Pauly 2018)

A *represented space* is a countably based topological space X together with listing of an open basis $B^X = \{B_i\}_{i<\mathbb{N}}$.

A name for a point $x \in X$ is an enumeration of the set $N_x = \{i \mid x \in B_i\}$. The enumeration degree of $x \in X$ is $\mathbf{d}_e(N_x)$.

Thus a represented space (X, B^X) gives rise to a class $\mathcal{D}_X \subseteq \mathcal{D}_e$.

Example

- $\mathcal{D}_\mathbb{R}$ is the total enumeration degrees.
- $\mathcal{D}_{[0,1]^\mathbb{N}}$ is the continuous degrees.
- $\mathcal{D}_{\mathbb{R}_<}$, where $\mathbb{R}_<$ is the reals with topology generated by the basis $\{(q,\infty) | \, q \in \mathbb{Q}\}$ is the degrees of members of maximal $\mathcal{K}$-pairs.

Kihara, Ng, and Pauly 2019 investigate $\mathcal{D}_X$, for many other spaces X.

The cotatal enumeration degrees

Recall that total degrees contain sets of the form $A \oplus \overline{A}$ or equivalently sets A such that $\overline{A} \leq_e A$.

Definition

A set A is *cototal* if $A \leq_e \overline{A}$. A degree is *cototal* if it contains a cototal set.

The cototal degrees contain the continuous degrees. Not every e-degree is cototal.

The cototal enumeration degrees are characterized as:

1. The degrees of languages of minimal subshifts by Jeandel and McCarthy 2018.
2. The degrees of complements of maximal independent sets in computable graphs by AGKLMSS 2019.
3. The degrees of points in countably based effectively G_δ topological spaces by Kihara, Ng, and Pauly.

The graph cototal enumeration degrees

An e-degree is total if and only if it contains the graph of a total function.

Definition

An enumeration degree is *graph cototal* if it contains the complement of the graph of a total function.

It is easy to see that graph cototal degrees are cototal, but are they the same class?

Theorem (AGKLMSS 2019)

There is a cototal degree that is not graph cototal.

Problem

Are all continuous degrees graph cototal?

Topological classification of the graph cototal degrees

Theorem (Kihara, Ng, Pauly)

$\mathcal{D}_{(\mathbb{N}_{cof})^{\mathbb{N}}}$ is the graph cototal degrees.

The problem "Is every continuous degree graph cototal?" can be restated as

Problem

Can you cover the Hilbert cube with countably many homeomorphic copies of subspaces of $(\mathbb{N}_{cof})^{\mathbb{N}}$?

The End: Thank you!