The enumeration degrees: the known and the unknown

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Outline

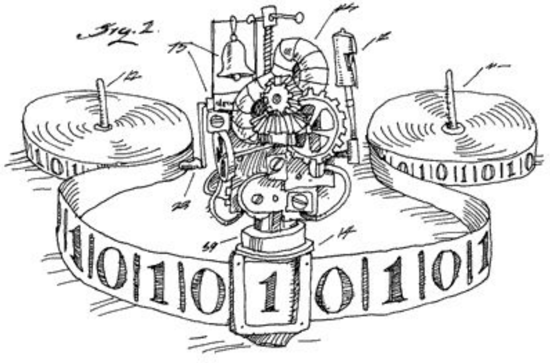
Enumeration reducibility captures a natural relationship between sets of natural numbers in which positive information about the first set is used to produce positive information about the second set.

By identifying sets that are reducible to each other we obtain an algebraic representation of this reducibility as a partial order: the structure of the enumeration degrees $\mathcal{D}_e$.

Motivation for the interest in this area comes from its nontrivial connections to the study of the Turing degrees.

I. The first order theory of $\mathcal{D}_e$ and its fragments;
II. First order definability;
III. Automorphisms and automorphism bases.

Computable sets and functions



Definition (Turing, Church 1936)

A function $f : \mathbb{N} \to \mathbb{N}$ is *computable* if there is a **computer program** which takes as input a natural number n and outputs $f(n)$.

A set $A \subseteq \mathbb{N}$ is *computable* if its characteristic function is computable.

Computably enumerable sets

Definition

A set $A \subseteq \mathbb{N}$ is *computably enumerable* (c.e.) if it can be enumerated by a computer program.

Example (Davis, Matyasevich, Putnam, Robinson 1970)

A set $S \subseteq \mathbb{N}$ is Diophantine if $S = \{n \mid \exists \bar{m}(P(n, \bar{m}) = 0)\}$.

The Diophantine sets are exactly the c.e. sets.

Example (Novikov, Boone 1955)

The word problem for a finitely presented group is c.e.

Every c.e. set can be coded as the word problem for a finitely presented group.

Example (Hilbert's Entscheidungsproblem 1928)

The set of provable formulas in first order logic is c.e.

An incomputable c.e. set

A set A is computable if and only if both A and $\overline{A}$ are c.e.

We can code programs by natural numbers: P_e is the program with code e.

Definition

The *halting set* is $K = \{e \mid P_e \text{ halts on input } e\}$.

Theorem (Turing 1936)

The halting set K is not computable.

Proof.

If K were computable then $\overline{K}$ would be computable.

Let e be such that P_e halts on n if and only if $n \in \overline{K}$.

P_e halts on $e \Leftrightarrow e \in \overline{K} \Leftrightarrow P_e$ does not halt on e. □

Note! This means that $\overline{K}$ is not c.e.

Comparing the information content of sets

Consider a program that has access to an external database, an *oracle*. During its computation the program can ask the oracle membership questions: does n belong to you or not?

Definition (Post 1944)

$A \leqslant_T B$ if and only if there is a program that computes the elements of A using B as an oracle.

Example

Consider a Diophantine set $S = \{n \mid (\exists \bar{m}) P(n, \bar{m}) = 0\}$. Let $P_{e(n)}$ be the program that ignores its input and for a listing $\{\bar{m}_i\}_{i \in \mathbb{N}}$ of all tuples $\bar{m}_i$ calculates $P(n, \bar{m}_0)$, $P(n, \bar{m}_1), \dots$ until it sees that the result is 0 and then halts. Then $S \leqslant_T K$ by the program that on input n computes $e(n)$ and asks the oracle whether $e(n) \in K$.

Comparing the information content of sets

Definition (Friedberg and Rogers 1959)

$A \leqslant_e B$ if there is a program that transforms an enumeration of B (i.e. a function on the natural numbers with range B) to an enumeration of A.

The program is a c.e. table of axioms of the sort:

$$\textit{If } \{x_1, x_2, \ldots, x_k\} \subseteq B \textit{ then } x \in A.$$

Example

Let $S = \{n \mid (\exists \bar{m}) P(n, \bar{m}) = 0\}$ again. $\overline{S} \leqslant_e \overline{K}$ by the program that consists of the axioms:

$$\textit{If } \{e(n)\} \subseteq \overline{K} \textit{ then } n \in \overline{S}.$$

Degree structures

Definition

1. $A \equiv B$ if and only if $A \leqslant B$ and $B \leqslant A$.
2. $\mathbf{d}(A) = \{B \mid A \equiv B\}$.
3. $\mathbf{d}(A) \leqslant \mathbf{d}(B)$ if an only if $A \leqslant B$.
4. Let $A \oplus B = \{2n \mid n \in A\} \cup \{2n+1 \mid n \in B\}$. Then $\mathbf{d}(A \oplus B) = \mathbf{d}(A) \vee \mathbf{d}(B)$.

And so we have two partial orders with least upper bound:

1. The Turing degrees $\mathcal{D}_T$ with least element $\mathbf{0}_T$ consisting of all computable sets.
2. The enumeration degrees $\mathcal{D}_e$ with least element $\mathbf{0}_e$ consisting of all c.e. sets.

Definability

Definition

A set B is definable in a structure $\mathcal{A}$ if there is a formula $\varphi(x)$ (in the language of the structure) such that $n \in B$ if and only if $\varphi(n)$ is true in $\mathcal{A}$.

This concept naturally extends to definable relations and functions.

Example

In the field of real numbers $\mathcal{R} = (\mathbb{R}, 0, 1, +, \times)$ the set of non-negative numbers is definable by the formula $\varphi(x) : (\exists y)(x = y \times y)$.

In arithmetic $\mathcal{N} = (\mathbb{N}, 0, 1, +, \times)$ the set of prime numbers is definable by the formula $\varphi(x) : x \neq 1 \wedge (\forall y)((\exists z)(y \times z = x) \rightarrow (y = 1 \vee y = x))$.

In the partial order of the Turing degrees $(\mathcal{D}_T, \leqslant_T, \mathbf{0}_T)$ the set of *minimal degrees* is definable by the formula $\varphi(x)$:
$x \neq \mathbf{0}_T \wedge (\forall y)(y \leqslant_T x \rightarrow (y = x \vee y = \mathbf{0}_T))$.

Arithmetic vs Degrees

Second order arithmetic Z_2 is $\mathcal{N}$ with an additional sort for sets of natural numbers and a membership relation.

Classical results due to Kleene and Post show that the relations and functions $\leqslant_T$, $\leqslant_e$, $\oplus$, are definable in second order arithmetic.

And so statements about the degree structure can be translated into statements of second order arithmetic:

Example

The statement that there is no greatest degree: $\forall \mathbf{x} \exists \mathbf{y} (\mathbf{x} < \mathbf{y})$ translates into: $(\forall X)(\exists Y)(\varphi_e(X, Y) \,\&\, \neg \varphi_e(Y, X))$, where φ_e defines $\leqslant_e$ in Z_2.

We say that the degree structures $\mathcal{D}_T$ and $\mathcal{D}_e$ can be *interpreted* in Z_2.

The main problem

The three parts of this talk address three aspects of the same problem:

Theorem (S 2016 (following Slaman, Woodin))

The following are equivalent:

1. $\mathcal{D}_e$ is *biinterpretable* with second order arithmetic.
2. The definable relations in $\mathcal{D}_e$ are exactly the ones induced by degree invariant definable relations in second order arithmetic.
3. $\mathcal{D}_e$ is a rigid structure.

Problem

Are these statements true or false?

Part I: The first order theory of $\mathcal{D}_e$ and its fragments

The existential theory

To understand: *"What existential sentences in the language of partial order are true in $\mathcal{D}_e$?"*

we ask *"What finite partial orders can be embedded in $\mathcal{D}_e$?"*.

The answer is "all". And so the $\exists$-Th($\mathcal{D}_e$) is decidable.

Theorem (Slaman, Sorbi 2014)

Every countable partial order can be embedded below any non-zero element of $\mathcal{D}_e$.

Note! This is a generalization of:

Theorem (Gutteridge 1971)

The enumeration degrees are downwards dense and so $\mathcal{D}_T \not\equiv \mathcal{D}_e$.

The two quantifier theory

Problem

Is the $\forall\exists$-theory of $\mathcal{D}_e$ decidable?

The problem of deciding the 2-quantifier theory is equivalent to the following:

Problem

We are given a finite lattice P and a partial orders $Q_0, \ldots Q_n \supseteq P$. Does every embedding of P extend to an embedding of one of the Q_i?

If $n = 0$ then we call this *the extension of embeddings problem.*

The algorithm for deciding $\exists\forall$-Th$(\mathcal{D}_T)$

In $\mathcal{D}_T$ the problem is solved through the following:

Theorem (Lerman 1971)

Every finite lattice can be embedded into $\mathcal{D}_T$ as an initial segment.

- Suppose that P is a finite lattice and $Q \supseteq P$ is a partial order extending P.
- The initial segment embedding of P can be extended to an embedding of Q only if no element in $Q \setminus P$ is below any element of P.
- Q also needs to respect least upper bounds if $x \in Q \setminus P$ and $u, v \in P$ and $x \geqslant u, v$ then $x \geqslant u \vee v$.

Theorem (Shore 1978; Lerman 1983)

That is the only obstacle.

The downward density of $\mathcal{D}_e$ makes this approach not applicable.

Towards a solution

Theorem (Kent, Lewis-Pye, Sorbi 2012 following Slaman, Calhoun 1996)

There are e-degrees $\mathbf{a} < \mathbf{b}$ such that $\mathbf{b}$ is a *strong minimal cover* of $\mathbf{a}$: if $\mathbf{x} < \mathbf{b}$ then $\mathbf{x} \leqslant \mathbf{a}$.

Theorem (Lempp, Slaman, S.)

Every finite distributive lattice can be embedded as an interval $[\mathbf{a}, \mathbf{b}]$ so that if $\mathbf{x} \leqslant \mathbf{b}$ then $\mathbf{x} \in [\mathbf{a}, \mathbf{b}]$ or $\mathbf{x} \leqslant \mathbf{a}$.

Corollary

The $\exists\forall\exists$-theory of $\mathcal{D}_e$ is undecidable.

Corollary

The extension of embeddings problem for $\mathcal{D}_e$ is decidable.

The complexity of the full theory of $\mathcal{D}_e$

Theorem (Slaman, Woodin 1997)

The theory of $\mathcal{D}_e$ is computably isomorphic to the theory of second order arithmetic Z_2: there are algorithms that translate a formula φ in the language of partial orders to a formula ψ in the language of second order arithmetic and vise versa so that:

$$\mathcal{D}_e \models \varphi \text{ if and only if } Z_2 \models \psi.$$

To translate ψ into φ they use their Coding Theorem and prove that it is a definable property of finitely many parameters $\vec{\mathbf{p}}$ that they code a model of $(\mathbb{N}, +, \times, <, C)$ where C is a unary predicate on $\mathbb{N}$.

The biinterpretability conjecture

Conjecture

The relation Bi, where $Bi(\vec{\mathbf{p}}, \mathbf{c})$ holds when $\vec{\mathbf{p}}$ codes a model of $(\mathbb{N}, +, \times, <, C)$ and $\deg_e(C) = \mathbf{c}$, is first order definable in $\mathcal{D}_e$.

Theorem (S 2016 (following Slaman and Woodin))

There is a parameter $\mathbf{g}$ such that relation Bi is first order definable in $\mathcal{D}_e$ with parameter $\mathbf{g}$.

Equivalently,

Corollary

If R is an n-ary relation invariant under $\equiv_e$ and definable in Z_2 then $\mathcal{R} = \{(\deg_e(A_1), \ldots \deg_e(A_n)) \mid Z_2 \models R(A_1, \ldots A_n)\}$ is definable in $\mathcal{D}_e$ with one parameter $\mathbf{g}$.

Part II: First order definability

The total enumeration degrees

Proposition. $A \leqslant_T B$ iff $A \oplus \overline{A}$ is B-c.e. iff $A \oplus \overline{A} \leqslant_e B \oplus \overline{B}$.

This suggests a natural embedding of the Turing degrees into the enumeration degrees.

Proposition. The function $\iota\colon \mathcal{D}_T \to \mathcal{D}_e$, where

$$\iota(d_T(A)) = d_e(A \oplus \overline{A}),$$

is an embedding of $\mathcal{D}_T$ into $\mathcal{D}_e$.

Definition

A set A is *total* if $A \geqslant_e \overline{A}$ (or equivalently if $A \equiv_e A \oplus \overline{A}$). An enumeration degree is *total* if it contains a total set.

The image of the Turing degrees under the embedding ι is exactly the set of total enumeration degrees.

Are the total degrees first order definable?

Semicomputable sets and natural definability

Jockusch introduced the *semicomputable sets* as left cuts in computable linear orderings on $\mathbb{N}$.

Theorem (Jockusch 1968)

Every nonzero Turing degree contains a semicomputable set that is not c.e. or co-c.e., so every nonzero total degree can be represented as $\deg_e(A) \vee \deg_e(\overline{A})$ for such a set A.

Theorem (Arslanov, Cooper, Kalimullin 2003)

If A is semicomputable and not c.e. or co-c.e. then the degrees $\mathbf{a} = \deg_e(A)$ and $\bar{\mathbf{a}} = \deg_e(\overline{A})$ are a *robust minimal pair*:

$$(\forall \mathbf{x} \in \mathcal{D}_e)[(\mathbf{a} \vee \mathbf{x}) \wedge (\bar{\mathbf{a}} \vee \mathbf{x}) = \mathbf{x}].$$

A definable copy of the Turing degrees

Theorem (Cai, Ganchev, Lempp, Miller, S 2016)

The pairs of degrees of a semicomputable set and its complement are first order definable in $\mathcal{D}_e$. They are *maximal* robust minimal pairs.

The total enumeration degrees are first order definable

Note! The characterization of the complexity of $\mathrm{Th}(\mathcal{D}_e)$ and Biinterpretability with parameters for $\mathcal{D}_e$ now follow from the corresponding theorems for $\mathcal{D}_T$.

Theorem (Cai, Ganchev, Lempp, Miller, S 2016)

The image of the relation "c.e. in" on Turing degrees is first order definable in $\mathcal{D}_e$.

The continuous degrees

Definition (Lacombe 1957)

A *computable metric space* is a metric space $\mathcal{M}$ together with a countable dense sequence $Q^{\mathcal{M}} = \{q_n^{\mathcal{M}}\}_{n\in\omega}$ on which the metric is computable, i.e. there is a computable function that maps a pairs of indices i, j and a precision $\varepsilon \in \mathbb{Q}^+$ to a rational that is within ε of $d_{\mathcal{M}}(q_i, q_j)$.

Examples:. 2^ω, ω^ω, $\mathcal{R}$, $\mathcal{C}[0,1]$, and *Hilbert cube* $[0,1]^\omega$.

Definition

$\lambda\colon \mathbb{Q}^+ \to \omega$ is a *name* of a point $x \in \mathcal{M}$ if for all rationals $\varepsilon > 0$ we have $d_{\mathcal{M}}(x, q_{\lambda(\varepsilon)}^{\mathcal{M}}) < \varepsilon$.

Definition (Miller 2004)

If x and y are members of (possibly different) computable metric spaces, then $x \leqslant_r y$ if there is a uniform way to compute a name for x from a name for y.

This reducibility induces the *continuous degrees*.

The continuous degrees

Theorem (Miller 2004)

Every continuous degree contains a point from $[0,1]^\omega$ and a point from $C[0,1]$.

For $\alpha \in [0,1]^\omega$, let

$$C_\alpha = \bigoplus_{i\in\omega} \{q \in \mathbb{Q} \mid q < \alpha(i)\} \oplus \{q \in \mathbb{Q} \mid q > \alpha(i)\}.$$

Observation. Enumerating C_α is exactly as hard as computing a name for α. So $\alpha \mapsto C_\alpha$ induces an embedding of the continuous degrees into the enumeration degrees.

Elements of 2^ω, ω^ω, and $\mathcal{R}$ are mapped onto the *total* degree of their least Turing degree name.

Theorem (Miller 2004)

There is a nontotal continuous degree.

Every known proof of this result uses nontrivial topological facts: Brouwer's fixed point theorem for multivalued functions on an infinite dimensional space, or Sperner's lemma, or results from topological dimension theory.

Topology realized as a structural property

Definition

A Turing degree $\mathbf{a}$ is *PA*

- if $\mathbf{a}$ computes a complete extension of Peano Arithmetic, or equivalently
- if $\mathbf{a}$ computes a path in every infinite computable tree.

An enumeration degree $\mathbf{a}$ is continuous if and only if it is *almost total*: if $\mathbf{x} \not\leqslant \mathbf{a}$ and $\mathbf{x}$ is total then $\mathbf{a} \vee \mathbf{x}$ is total.

The continuous degrees are definable in $\mathcal{D}_e$.

The degree $\mathbf{a}$ is *PA above* $\mathbf{b}$ if $\mathbf{a}$ computes a path in every infinite $\mathbf{b}$-computable tree.

Theorem (Miller 2004)

For total degrees $\mathbf{a}$ is *PA above* $\mathbf{b}$ if and only if there is a nontotal continuous degree $\mathbf{c}$ such that $\mathbf{b} < \mathbf{c} < \mathbf{a}$.

The image of the relation “PA above” is first order definable in $\mathcal{D}_e$.

The cototal enumeration degrees

Definition

A set A is *cototal* if $A \leqslant_e \overline{A}$. A degree is *cototal* if it contains a cototal set.

The cototal degrees contain the continuous degrees. Not every e-degree is cototal.

The cototal enumeration degrees are characterized as:

1. The degrees of complements of maximal independent sets in computable graphs by AGKLMSS 2019.
2. The degrees of complements of maximal antichains in $\omega^{<\omega}$ by McCarthy 2018.
3. The degrees of languages of minimal subshifts by McCarthy 2018.
4. The degrees of sets with good approximations by Miller and S 2018.
5. The degrees of points in computable G_δ topological spaces by Kihara, Ng, and Pauly 2019.

Problem

Are the cototal degrees first order definable in $\mathcal{D}_e$?

Topological classification of classes of e-degrees

Definition (Kihara, Pauly 2018)

A *represented space* is a pair of a second countable topological space X and listing of an open basis $B^X = \{B_i\}_{i<\omega}$.

A name for a point $x \in X$ is an enumeration of the set $N_x = \{i \mid x \in B_i\}$.

For $x, y \in X$, say that $x \leqslant y$ if every name for y (uniformly) computes a name for x.

Thus a represented space X gives rise to a class of e-degrees $\mathcal{D}_X \subset \mathcal{D}_e$.

Examples:.

- $\mathcal{D}_{2^\omega} = \mathcal{D}_{\mathbb{R}}$ is the total enumeration degrees.
- $\mathcal{D}_{[0,1]^\omega}$ is the continuous degrees.
- $\mathcal{D}_{S^\infty} = \mathcal{D}_e$, where S is the Sierpinski topology $\{\emptyset, \{1\}, \{0,1\}\}$.
- $\mathcal{D}_{\mathbb{R}^<}$, where $\mathbb{R}^<$ is the real line with topology generated by $\{(q,\infty)\}_{q\in\mathbb{Q}}$, is exactly the semicomputable degrees.

Kihara, Ng, and Pauly 2019 investigate $\mathcal{D}_X$, where X is the ω-power of the: cofinite topology on ω, telophase space, double origin space, quasi-Polish Roy space, irregular lattice space.

Part III: Automorphisms and automorphism bases

Slaman and Woodin's automorphism analysis

Theorem (Slaman, Woodin 1986)

The Turing degrees have at most countably many automorphisms.

There is a single degree $\mathbf{g} \leqslant \mathbf{0}^{(5)}$ that is an *automorphism base* for $\mathcal{D}_T$: if π is an automorphism such that $\pi(\mathbf{g}) = \mathbf{g}$ then $\pi = \mathrm{id}$.

Relations on $\mathcal{D}_T$ induced by definable relations in Z_2 are first order definable in $\mathcal{D}_T$ with such a parameter $\mathbf{g}$.

Relations on $\mathcal{D}_T$ induced by definable relations in Z_2 that are furthermore invariant under automorphism are first order definable in $\mathcal{D}_T$ (without parameters).

Theorem (Selman 1971)

$\mathbf{a} \leqslant \mathbf{b}$ if and only if every total degree above $\mathbf{b}$ is above $\mathbf{a}$.

Implications for the e-degrees

Corollary

The total enumeration degrees form a definable automorphism base for $\mathcal{D}_e$.

- Every nontrivial automorphism of $\mathcal{D}_e$ gives rise to a unique non-trivial automorphism of $\mathcal{D}_T$.
- This automorphism preserves the relations "c.e. in" and "PA above".
- $\mathcal{D}_e$ has at most countably many automorphisms.
- A single total degree below $\mathbf{0}_e^{(5)}$ is an automorphism base of $\mathcal{D}_e$.

Problem

Does every automorphism of $\mathcal{D}_T$ extend to an automorphism of $\mathcal{D}_e$?

A positive answer would imply the first order definability (without parameters) of the relations "c.e. in" and "PA above" in $\mathcal{D}_T$.

Thank you!