Degree structures and Arithmetic

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    - Slaman and Woodin (1991) conjectured: There are no non-trivial automorphisms of $\mathcal{D}_T$.

Slaman and Woodin's automorphism analysis

Theorem (Slaman and Woodin)

There is an element $\mathbf{g} \leq \mathbf{0}^{(5)}$ such that $\{\mathbf{g}\}$ is an automorphism base for the structure of the Turing degrees $\mathcal{D}_T$.

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Definition

Let $\mathcal{A}$ be a structure. A set $B \subseteq |\mathcal{A}|$ is an automorphism base for $\mathcal{A}$ if whenever f and g are automorphisms of $\mathcal{A}$ such that $(\forall x \in B)(f(x) = g(x))$, then $f = g$.

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Equivalently if f is an automorphism of $\mathcal{A}$ and $(\forall x \in B)(f(x) = x)$ then f is the identity.

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1. Shore (1981) proved that the theory of $\mathcal{D}_T(\leq \mathbf{0}')$ is computably isomorphic to the theory of first order arithmetic.
2. Harrington and Slaman proved that the theory of $\mathcal{R}$ is computably isomorphic to the theory of first order arithmetic.

The local coding theorem

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A set of degrees $\mathcal{Z}$ contained in $\mathcal{D}_T(\leq \mathbf{0}')$ is *uniformly low* if it is bounded by a low degree and there is a sequence $\{Z_i\}_{i<\omega}$, representing the degrees in $\mathcal{Z}$, and a computable function f such that $\{f(i)\}^{\emptyset'}$ is the Turing jump of $\bigoplus_{j<i} Z_j$.

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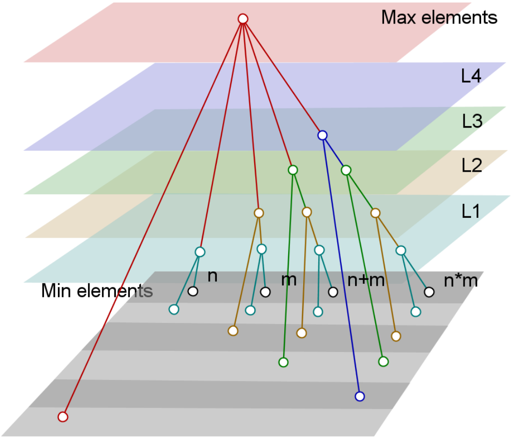
Theorem (Slaman and Woodin)

If $\mathcal{Z}$ is a uniformly low subset of $\mathcal{D}_T(\leq \mathbf{0}')$ then $\mathcal{Z}$ is definable from finitely many parameters in $\mathcal{D}_T(\leq \mathbf{0}')$.

Applications of the coding theorem

We can represent a model of as a partial order $\mathcal{M}$ and embed it in $\mathcal{D}_T(\leq \mathbf{0}')$:



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So in $\mathcal{D}_T(\leq \mathbf{0}')$ we can represent a model of arithmetic $\mathcal{M}$, coded by finitely many parameters $\vec{p}$, so that every formula φ in the language of arithmetic has an effective translation into a formula φ' in the language of partial orders and

$$\mathbb{N} \models \varphi \iff \mathcal{D}_T(\leq \mathbf{0}') \models \varphi'(\vec{p}).$$

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If $\mathcal{Z} \subseteq \mathcal{D}_T(\leq \mathbf{0}')$ is uniformly low and represented by the sequence $\{Z_i\}_{i<\omega}$ then there are parameters that code a model of arithmetic $\mathcal{M}$ and a function $\varphi : \mathbb{N}^{\mathcal{M}} \to \mathcal{D}_T(\leq \mathbf{0}')$ such that $\varphi(i^{\mathcal{M}}) = d_T(Z_i)$.

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1. The sequence $\mathcal{C} = \{d_T(Z_i) \oplus i^{\mathcal{M}}\}_{i<\omega}$ is a uniformly low antichain.

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Start the construction of your coded model of arithmetic $\mathcal{M}$ with a 1-generic relative to $\bigoplus_i Z_i$.

1. The sequence $\mathcal{C} = \{d_T(Z_i) \oplus i^{\mathcal{M}}\}_{i<\omega}$ is a uniformly low antichain.
2. For all $\mathbf{z} \in \mathcal{Z}$ and $n^{\mathcal{M}} \in \mathbb{N}^{\mathcal{M}}$ we have that $\mathbf{z} \vee n^{\mathcal{M}} \in \mathcal{C}$ if and only if $Z_n \in \mathbf{z}$.

Applications of the coding theorem

Theorem (Slaman and Woodin)

There are finitely many Δ^0_2 parameters which code a model of arithmetic $\mathcal{M}$ and an indexing of the c.e. degrees: a function $\psi : \mathbb{N}^{\mathcal{M}} \to \mathcal{D}_T(\leq \mathbf{0}')$, such that $\psi(e^{\mathcal{M}}) = d_T(W_e)$.

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Then $\mathcal{A} = \{d_T(A_e) \mid e < \omega\}$ and $\mathcal{B} = \{d_T(B_e) \mid e < \omega\}$ are uniformly low and hence can be indexed and $d_T(W_e) = d_T(A_e) \vee d_T(B_e)$.

□

Biinterpretability

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Proposition

If $\mathcal{R}$ is biinterpretable with first order arithmetic then:

1. $\mathcal{R}$ has no nontrivial automorphisms.
2. The definable relations in $\mathcal{R}$ are exactly the ones induced by definable relations in arithmetic that are closed under Turing equivalence.

Biinterpretability for $\mathcal{D}_T(\leq \mathbf{0}')$

Let X_e be $\Phi_e^{\emptyset'}$ if this is a total binary function and $\emptyset$ otherwise.

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Definition

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$\mathcal{D}_T(\leq \mathbf{0}')$ is biinterpretable using parameters with first order arithmetic

1. $\mathcal{D}_T(\leq \mathbf{0}')$ has only countably many automorphisms.
2. Every relation on $\mathcal{D}_T(\leq \mathbf{0}')$, induced by an arithmetical relation, closed under Turing reducibility, is definable in $\mathcal{D}_T(\leq \mathbf{0}')$ with parameters.

Step 1

Theorem

For every Δ^0_2 Turing degrees $\mathbf{a}$ there are low Turing degrees $\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3, \mathbf{g}_4$ such that

$$\mathbf{a} = (\mathbf{g}_1 \vee \mathbf{g}_2) \wedge (\mathbf{g}_3 \vee \mathbf{g}_4).$$

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Question

Is every low Δ^0_2 Turing degree uniquely positioned relative to the c.e. Turing degrees?

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If $\mathbf{x}$ is low and $\mathbf{y}$ is another Δ^0_2 degree such that $\mathbf{y} \not\leq \mathbf{x}$ then there are c.e. degrees $\mathbf{a}_1, \mathbf{a}_2$ and Δ^0_2 degrees $\mathbf{b}_1, \mathbf{b}_2, \mathbf{c}_1, \mathbf{c}_2$ in the set $\mathcal{Z}$ and Δ^0_2 degrees $\mathbf{g}_1$ and $\mathbf{g}_2$, such that:

1. $\mathbf{g}_i$ is the least element below $\mathbf{a}_i$ which joins $\mathbf{b}_i$ above $\mathbf{c}_i$.
2. $\mathbf{x} \leq \mathbf{g}_1 \vee \mathbf{g}_2$.
3. $\mathbf{y} \not\leq \mathbf{g}_1 \vee \mathbf{g}_2$.

The enumeration degrees

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$$A = W(B) = \{x \mid \exists D(\langle x, D\rangle \in W \; \& \; D \subseteq B)\}.$$

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- The least upper bound: $d_e(A) \vee d_e(B) = d_e(A \oplus B)$.

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- $d_e(A) \leq d_e(B)$ iff $A \leq_e B$.
- The least element: $\mathbf{0_e} = d_e(\emptyset)$, the set of all c.e. sets.
- The least upper bound: $d_e(A) \vee d_e(B) = d_e(A \oplus B)$.
- The enumeration jump: $d_e(A)' = d_e(K_A \oplus \overline{K_A})$, where $K_A = \{\langle e, x\rangle \mid x \in W_e(A)\}$.

What connects $\mathcal{D}_T$ and $\mathcal{D}_e$

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$$A \leq_T B \Leftrightarrow A \oplus \overline{A} \leq_e B \oplus \overline{B}.$$

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If $\mathbf{x} \in \mathcal{D}_T$ then we will call $\iota(\mathbf{x})$ *the image of* $\mathbf{x}$.

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The total degrees as an automorphism base

Theorem (Selman)

A is enumeration reducible to B if and only if
$\{\mathbf{x} \in \mathcal{TOT} \mid d_e(A) \leq \mathbf{x}\} \supseteq \{\mathbf{x} \in \mathcal{TOT} \mid d_e(B) \leq \mathbf{x}\}$.

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The total enumeration degrees form a definable automorphism basis of the enumeration degrees.

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The total enumeration degrees form a definable automorphism basis of the enumeration degrees.

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Goal:

Exploit the definability in the e-degrees and in particular of the relation "c.e. in", to 'extend' any indexing of the image of the c.e. degrees to an indexing of all total degrees below $\mathbf{0}_e^{(5)}$.

The power of the c.e. degrees in $\mathcal{D}_e(\leq \mathbf{0}'_e)$

Given parameters $\vec{p}$ that code in $\mathcal{D}_e(\leq \mathbf{0}'_e)$ a standard model of arithmetic and an indexing of the image of the c.e. degrees, we must show that there is an indexing of the total enumeration degrees in $\mathcal{D}_e(\leq \mathbf{0}'_e)$, definable from $\vec{p}$.

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1. Every total e-degree in $\mathcal{D}_e(\leq \mathbf{0}'_e)$ is the join of two low enumeration degrees.
2. The low enumeration degrees are uniquely positioned with respect to the image of the c.e. degrees and the low co-d.c.e e-degrees.

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An enumeration degree is co-d.c.e if it contains a set of the form $\overline{W_e \setminus W_i}$.

The power of the c.e. degrees in $\mathcal{D}_e(\leq \mathbf{0}_e')$

Given parameters $\vec{p}$ that code in $\mathcal{D}_e(\leq \mathbf{0}_e')$ a standard model of arithmetic and an indexing of the image of the c.e. degrees, we must show that there is an indexing of the total enumeration degrees in $\mathcal{D}_e(\leq \mathbf{0}_e')$, definable from $\vec{p}$.

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3. The low co-d.c.e. degrees are uniquely positioned with respect to the c.e. degrees.

Moving outside of local territory

From an indexing of the total enumeration degrees below $\mathbf{0}'_e$, we must define an indexing of the total enumeration degrees in:

$$\mathcal{I} = \bigcup_{\mathbf{x} \in \mathcal{TOT} \cap \mathcal{D}_e(\leq \mathbf{0}'_e)} [\mathbf{x}, \mathbf{x}'].$$

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1. First we will define an indexing that maps $e^{\mathcal{M}}$ to the image of $d_T(U_e)$.
2. Then using a relativized version of the previous step we will identify all total enumeration degrees in $\mathcal{I}$.

Identifying the sets that are c.e. in and above some total $\mathbf{x}$.

Let $\mathbf{x} \leq \mathbf{0}'$ and $\mathbf{a}$ be c.e. in and above $\mathbf{x}$.

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3. Suppose that $\mathbf{0}' \not\leq \mathbf{a}$ and $\mathbf{a}' = \mathbf{x}'$ and let $L(\mathbf{x})$ contain all degrees that have these properties: c.e. in $\mathbf{x}$, low over $\mathbf{x}$ and avoid the cone above $\mathbf{0}'$.

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Theorem (Slaman, S)

If $\mathbf{a}, \mathbf{y}$ are in $L(\mathbf{x})$ and $\mathbf{a} \not\leq \mathbf{y}$ then there are $\mathbf{u}, \mathbf{v} \leq \mathbf{0}'$ such that $\mathbf{a} \vee \mathbf{u} \geq \mathbf{v}$ and $\mathbf{y} \vee \mathbf{u} \not\geq \mathbf{v}$.

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There are high degrees $\mathbf{h}_1$ and $\mathbf{h}_2$, such that for every 2-generic $\mathbf{g}$ we have that

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If $\mathbf{g} \leq \mathbf{0}''$ then $\mathbf{g} \vee \mathbf{h}_i \in [\mathbf{h}_i, \mathbf{h}_i']$.

And now we iterate!

Theorem

Let n be a natural number and $\vec{p}$ be parameters that index the image of the c.e. Turing degrees. There is a definable from $\vec{p}$ indexing of the total Δ^0_{n+1} sets.

Consequences

1. There is a finite automorphism base for the enumeration degrees consisting of total Δ_2^0 enumeration degrees.

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Question

1. Can we carry out this proof in the Turing degrees assuming that the relation c.e. in is definable?
2. What are the automorphism bases of the local structure $\mathcal{D}_e(\leq \mathbf{0}'_e)$?