A Non-Splitting Theorem in the Enumeration Degrees

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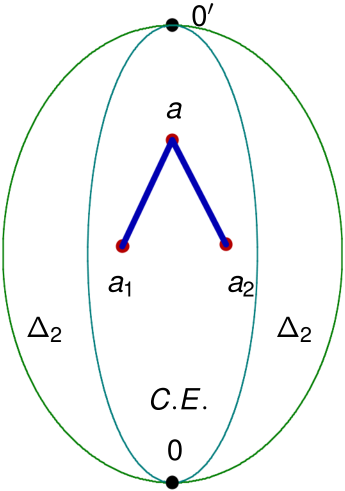
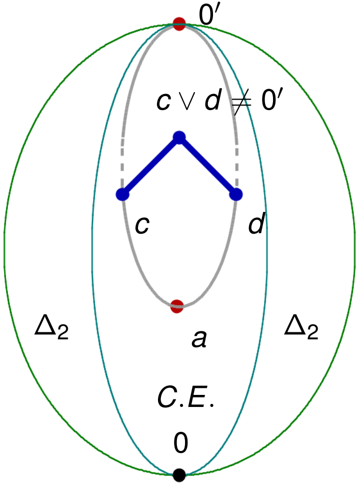
Definitions

We will say that a pair of degrees $\mathbf{a_1}$ and $\mathbf{a_2}$ form a splitting of $\mathbf{a}$ if $\mathbf{a_1} < \mathbf{a}$ and $\mathbf{a_2} < \mathbf{a}$ but $\mathbf{a_1} \cup \mathbf{a_2} = \mathbf{a}$.



Harrington's non-splitting theorem

There exists a c.e. degree $\mathbf{a} < \mathbf{0}'$ such that $\mathbf{0}'$ can not be split in the c.e. degrees above $\mathbf{a}$.



Δ_2

The semi-lattice of the enumeration degrees

Definition

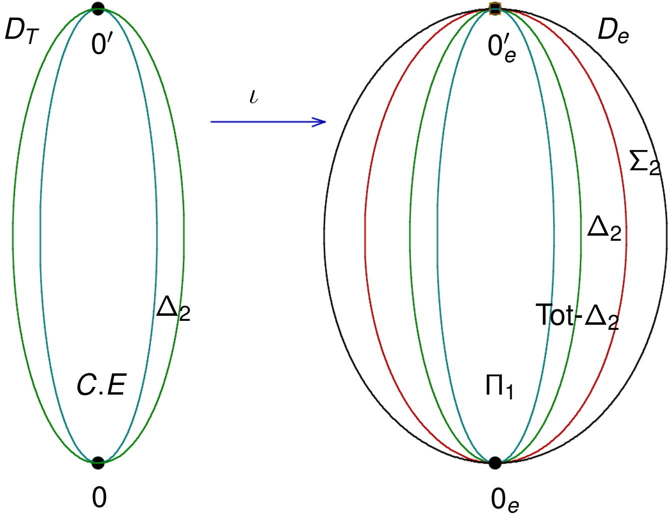
1. A set A is enumeration reducible to a set B ($A \leq_e B$), if there is a c.e. set Φ such that

$$n \in A \Leftrightarrow \exists D(\langle n, [D] \rangle \in \Phi \wedge D \subseteq B).$$

2. A is enumeration equivalent to B ($A \equiv_e B$) if $A \leq_e B$ and $B \leq_e A$.
3. Let $d_e(A) = \{B | A \equiv_e B\}$.
4. $(D_e, <, \cup, ', 0_e)$ is the semi-lattice of the enumeration degrees with jump operator.

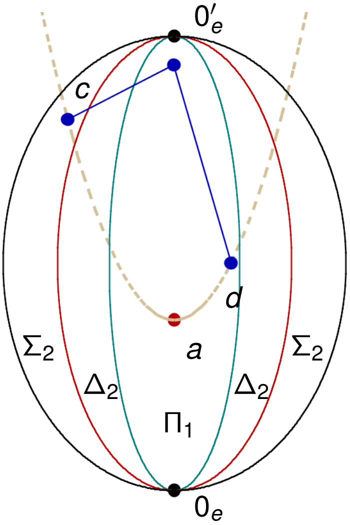
Embedding the Turing degrees into the enumeration degrees

There exists an order theoretic embedding $\iota : D_T \to D_e$.



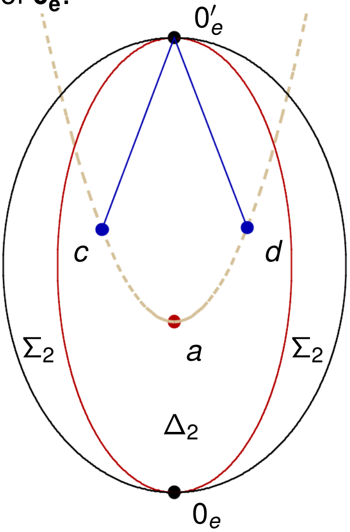
Known Results: Cooper and M.S.

There exists a Π_1 e-degree $\mathbf{a} < \mathbf{0_e'}$ such that there exist no nontrivial cuppings of Π_1 e-degrees in the Σ_2 e-degrees above $\mathbf{a}$.



Known Results: Arslanov and Sorbi

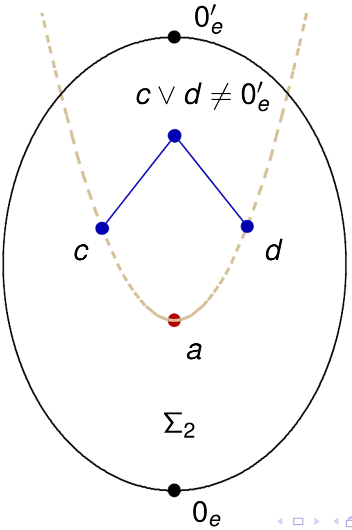
Above every Δ_2 e-degree **a** there exists a pair of Δ_2 e-degrees which form a splitting of $\mathbf{0}'_\mathbf{e}$.



Main Result

Theorem

There is a Σ_2 e-degree $\mathbf{a}$ *such that* $\mathbf{0'_e}$ *cannot be split in the Σ_2 e-degrees above* $\mathbf{a}$.



The requirements

We will construct a Σ_2 set A and a Π_1 set E such that:

- For all enumeration operators Ψ:

$$\mathcal{N}_\Psi : E \neq \Psi^A$$

- For each pair of a Σ_2 sets U and V and each enumeration operator Θ:

$$\mathcal{P}_{\Theta,U,V} : E = \Theta^{U,V} \Rightarrow (\exists \Gamma, \Lambda)[\overline{K} = \Gamma^{U,A} \vee \overline{K} = \Lambda^{V,A}]$$

The $\mathcal{P}$-strategy

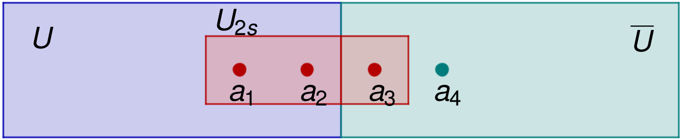
$$\mathcal{P}_{\Theta,U,V} : E = \Theta^{U,V} \Rightarrow (\exists \Gamma, \Lambda)[\overline{K} = \Gamma^{U,A} \vee \overline{K} = \Lambda^{V,A}]$$

- We monitor the length of agreement $l(E, \Theta^{U,V})$ and act only on expansionary stages.
- Construct an e-operator Γ so that $n \in \overline{K} \leftrightarrow \langle n, (U \oplus A) \upharpoonright \gamma_n \rangle \in \Gamma$.
- Correct errors in Γ by extracting $\gamma(n)$ from A.

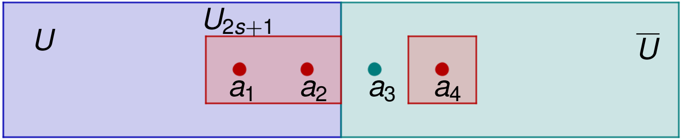
Σ_2 sets and their approximations

Consider a Σ_2 set U with approximating sequence $\{U_s\}_{s<\omega}$. If $n \notin U$ then $n \notin U_s$ for infinitely many s.

Even stages



Odd stages



Lachlan and Shore's Good approximations

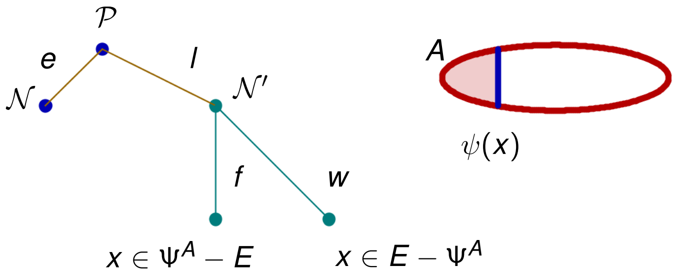
We define a good approximations to the sets U, V and $U \oplus V$ with following properties:

- Σ_2 Elements in the set are also in the approximations on all but finitely many stages
- Good Infinitely many good stages on which the approximation is a subset of the set.
- Exp If $\Theta^{U,V} = E$, then the length of agreement $l(\Theta^{U,V}, E)[s]$ is unbounded.

The $\mathcal{N}$-strategies

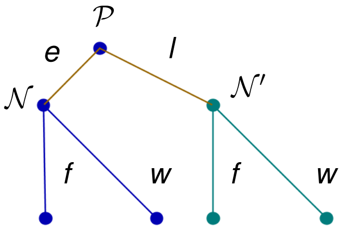
$$\mathcal{N}_\Psi : E \neq \Psi^A$$

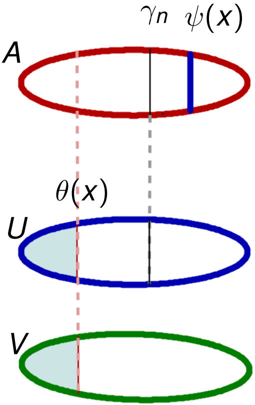
Below the l-outcome $\mathcal{N}'$ can follow a simple Friedberg-Muchnik strategy:



The $\mathcal{N}$-strategies

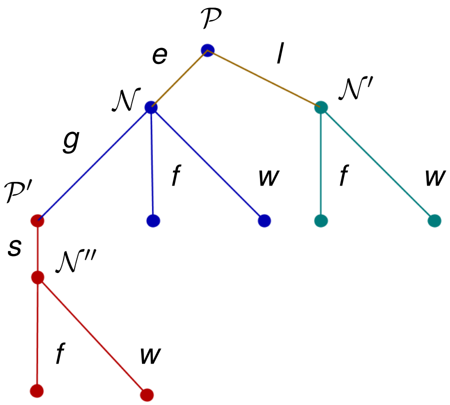
Activity at $\mathcal{P}$ may injure a restraint imposed by $\mathcal{N}$. The strategy $\mathcal{N}$ acts only after $x < l(\Theta^{U,V}, E)$. The extraction of x from E forces a change in $U \oplus V$.

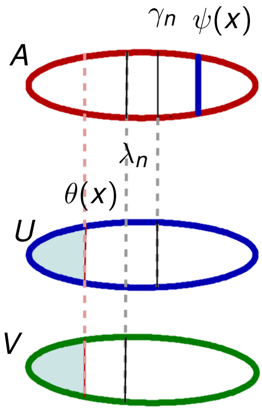




The backup strategy

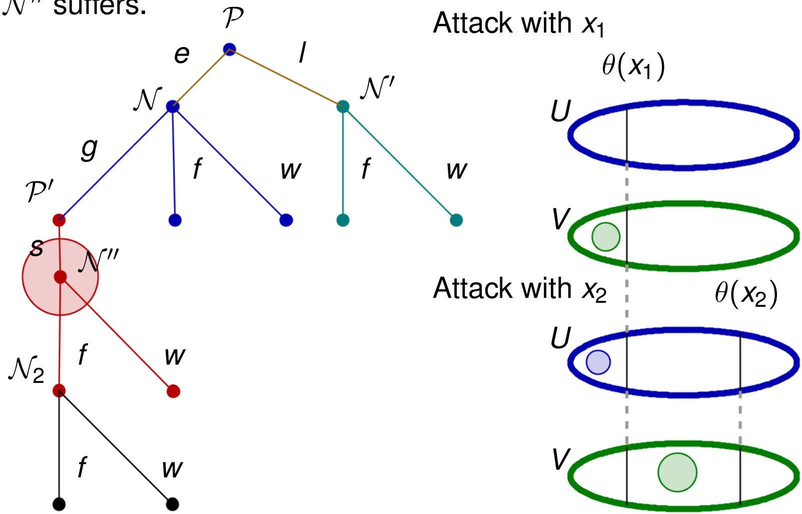
A backup strategy $\mathcal{P}$ constructs an operator Λ with $\Lambda^{V,A} = \overline{K}$. Strategy $\mathcal{N}$ will perform many attacks together with the backup strategies. An unsuccessful attack for $\mathcal{N}$ is successful for $\mathcal{N}''$.





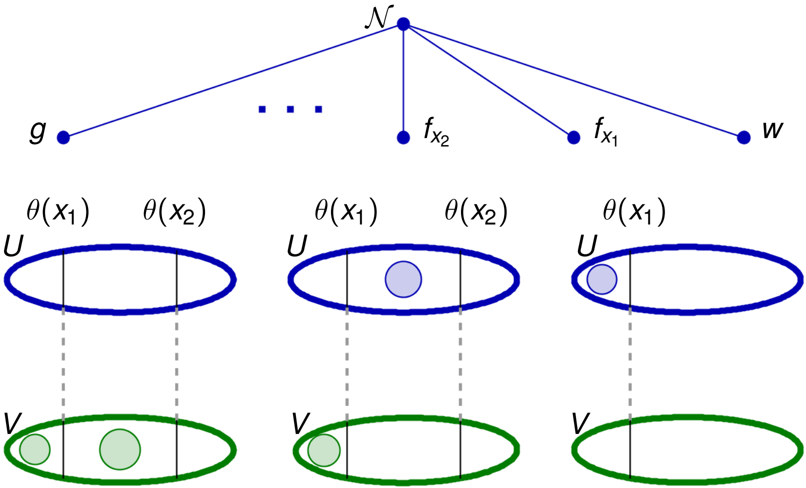
The Σ_2 sets gang up together

U and V can trick us to believe that an attack is unsuccessful.
$\mathcal{N}''$ suffers.



The main trick

Longer memory. Access the backup strategies only if all previous attacks are unsuccessful.



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