Stratifying classes of enumeration degrees

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Outline

- Turing reducibility between sets of natural numbers allows us to gauge the algorithmic content of a mathematical object.
- We illustrate how to assign a Turing degree to an object in three different scenarios.
- The methodology in each case yields a Turing degree only for a small set of cases.
- We introduce enumeration reducibility and describe its relationship to Turing reducibility.
- We revisit each scenario and look at it from the enumeration degree perspective.
- This perspective gives rise to a zoo of classes of enumeration degrees whose interplay informs our understanding of the degree structure.

The real numbers

What is the effective content carried by a real number?
I can approximate a real number r using discrete objects: the rationals.

Definition

A *name* for a real number r is a function n_r that allows us to approximate r with arbitrary precision: on input $\varepsilon \in \mathbb{Q}^+$ the name outputs a rational so that

$$|r - n_r(\varepsilon)| < \varepsilon$$

Problem: A real r has infinitely many different names.

Solution: There is a computationally simplest name, because every name for r computes its Dedekind cut $C_r = \{q \in \mathbb{Q} \mid q < r\} \oplus \{q \in \mathbb{Q} \mid q > r\}$ and if a Turing degree computes C_r then it can compute a name for r.
So the Turing degree of a real r is $\deg_T(C_r)$.

Now we can talk about computable operations on real numbers: addition is an example.

Points in metric spaces

Definition

A *computable metric space* is a metric space (M, d), equipped with a countable dense sequence $\{m_i\}_{i<\omega}$ on which the metric is computable as a function on indices.

Examples: The following are computable metric spaces:

1. $\mathbb{R}$ with a listing of $\mathbb{Q}$.
2. $C[0, 1]$ with a listing of the stepwise linear functions on rational intervals.
3. $[0, 1]^\omega$ with a listing of the rational sequences with finite support.

Definition

A name for a point r in a computable metric space (M, d) is a function $n_r : \mathbb{Q}^+ \to \mathbb{N}$ such that $d(r, m_{n_r(\varepsilon)}) < \varepsilon$.

Theorem (Miller 04)

Not all $r \in M$ have a name of least Turing degree.

Symbolic dynamics

The *shift operator* on 2^ω erases the first bit of a sequence.

Definition

A *subshift* is a closed shift-invariant subspace X of 2^ω. X is a *minimal subshift* if no nonempty $Y \subset X$ is a subshift.

The *Turing degree* of the subshift X is the least Turing degree that computes a member of the subshift.

Example: The *language* L_X of a subshift X is the set of finite subwords of members of X. If X is minimal then every word in L_X appears along every member of X. If L_X is c.e. then we can compute a member of X and so the Turing degree of X is $\mathbf{0}$.

Theorem (Hochman, Vanier 2017)

There is a minimal subshift X with no member of least Turing degree.

Computable structure theory

Let $\mathcal{A}$ be a countable structure in a computable language L. An L-structure with universe ω that is isomorphic to $\mathcal{A}$ is called a *copy* of $\mathcal{A}$.

Definition

The *degree spectrum* of a countable structure $\mathcal{A}$ is the collection $\mathrm{Spec}(\mathcal{A})$ of the Turing degrees of the atomic diagrams of copies of $\mathcal{A}$.

When $\mathrm{Spec}(\mathcal{A})$ has a least element, we call it the *Turing degree* of $\mathcal{A}$.

Example: Consider a subgroup $\mathcal{G}$ of $(\mathbb{Q}, +)$. There is a set of natural numbers $S(\mathcal{G})$, *the standard type* of $\mathcal{G}$ that codes sufficiently much divisibility information in $\mathcal{G}$ to determine its isomorphism type. Every set of natural numbers is computably isomorphic to the standard type of some group $\mathcal{G}$.

Theorem (Downey, Jockusch 1997)

The degree spectrum of $\mathcal{G}$ is precisely $\{\deg_T(Y) \mid S(\mathcal{G}) \text{ is c.e. in } Y\}$.

Sets of this form do not always have a least element.

Enumeration reducibility

Friedberg and Rogers introduced enumeration reducibility in 1959.

Informally: $A \subseteq \omega$ is *enumeration reducible* to $B \subseteq \omega$ ($A \leqslant_e B$) if there is a uniform way to enumerate A from an enumeration of B.

Definition

$A \leqslant_e B$ if there is a c.e. set W such that

$$A = \{n\colon (\exists e)\, \langle n, e\rangle \in W \text{ and } D_e \subseteq B\},$$

where D_e is the eth finite set in a canonical enumeration.

Theorem (Selman 1971)

$A \leqslant_e B$ if and only if for every set X if B is c.e. in X then A is c.e. in X.

The degree structure $\mathcal{D}_e$ induced by $\leqslant_e$ is called the *enumeration degrees*. It is an upper semi-lattice with a least element (the degree of all c.e. sets).

The total enumeration degrees

$A \leqslant_T B \iff A \oplus \overline{A} \leqslant_e B \oplus \overline{B}$.

We can embed $\mathcal{D}_T$ in $\mathcal{D}_e$ by $\iota(d_T(A)) = d_e(A \oplus \overline{A})$.
This embedding preserves the order and the least upper bound.

Definition

$A \subseteq \omega$ is *total* if $\overline{A} \leqslant_e A$. A degree is *total* if it contains a total set.

The image of the Turing degrees under the embedding ι is exactly the set of total enumeration degrees.

Nontotal enumeration degrees exist: any generic $A \subseteq \omega$ has nontotal degree.

Theorem (Cai, Lempp, Ganchev, Miller, and S)

The total enumeration degree are first order definable in $\mathcal{D}_e$.

Characterizing points in computable metric spaces

Theorem (Miller 2004)

Let m be a point in a computable metric space. There is a point $\alpha \in [0,1]^\omega$ such that computing a name for m is uniformly effectively equivalent to computing a name for α.

For $\alpha \in [0,1]^\omega$, let $C_\alpha = \bigoplus_{i \in \omega} C_{\alpha(i)}$. Enumerating C_α is exactly as hard as computing a name for α and so $\deg_e(C_\alpha)$ captures the effective complexity of α.

Definition

The enumeration degree of a point m in a computable metric space is $\deg_e(X)$ if every enumeration of a X computes a name for m and every name for m computes an enumeration of X.

Corollary

Every point in a computable metric space has an enumeration degree.

The continuous enumeration degrees

Definition

An enumeration degree is called *continuous* if it is the degree of some point in a computable metric space.

- The degrees of points in $\mathbb{R}$ are the total enumeration degrees.
- And so the continuous enumeration degrees strictly extend the total enumeration degrees.
- Every proof that there are nontotal continuous degrees involves nontrivial topology.

Theorem (Andrews, Igusa, Miller, S)

A enumeration degree $\mathbf{a}$ is continuous if and only if it is *almost total*: if and only if for total $\mathbf{x} \nleqslant \mathbf{a}$ we have that $\mathbf{x} \vee \mathbf{a}$ is total.

Computing members of minimal subshifts

Recall, that to compute a member of a minimal subshift all we needed was a computable enumeration of its language.

Theorem (Jeandel)

A Turing degree $\mathbf{a}$ computes a member of the minimal subshift X if and only if $\mathbf{a}$ can enumerate L_X.

Once again we define the *enumeration degree* of X as the least degree $\mathbf{a}$ such that every enumeration of a member in $\mathbf{a}$ computes a member of X.

X has Turing degree $\deg_T(A)$ if and only if X has enumeration degree $\deg_e(A \oplus \overline{A})$.

If X is minimal then $\deg_e(L_X)$ is the enumeration degree of X.

The cototal enumeration degrees

Jeandel noticed something special: for a minimal subshift $L_X \leqslant_e \overline{L_X}$.

- An enumeration of $\overline{L_X}$ allows us to eliminate branches that do not belong to X in a stage by stage manner.
- If w is word that appears along every branch that remains at stage s, then $w \in L_X$.
- The compactness of 2^ω ensures that we won't miss any word from the language using this process of enumeration.

Definition

A set A is *cototal* if $A \leqslant_e \overline{A}$. An enumeration degree is *cototal* if it contains a cototal set.

The cototal enumeration degrees

Definition

A set A is *cototal* if $A \leqslant_e \overline{A}$. An enumeration degree is *cototal* if it contains a cototal set.

Examples:

- Total degrees are cototal, as they contain sets of the form $A \oplus \overline{A}$.
- Continuous degrees are cototal: recall that
  $C_\alpha = \bigoplus_i \{q \mid q < \alpha(i)\} \oplus \{q \mid q > \alpha(i)\}$.
  Note that $q < \alpha(i)$ if and only if $q < s$ for some $s \leqslant \alpha(i)$.
- There are quasiminimal cototal degrees.

Theorem (McCarthy)

Every cototal enumeration degree contain the language of some minimal subshift.

The enumeration degree of a structure

Definition

We say that a countable structure $\mathcal{A}$ has *enumeration degree* $\deg_e(X)$ if every enumeration of X computes a copy of $\mathcal{A}$ and every copy of $\mathcal{A}$ computes an enumeration of X.

If $\mathcal{A}$ has Turing degree $\deg_T(X)$ then $\mathcal{A}$ has enumeration degree $\deg_e(X \oplus \overline{X})$.

Theorem

Every structure in each of the following classes has an enumeration degree.

1. (Calvert, Harizanov, Shlapentokh 07) Torsion-free abelian groups of finite.
2. (Frolov, Kalimullin, Miller 09) Fields of finite transc. degree over $\mathbb{Q}$.
3. (Steiner 13) Graphs of finite valence with finitely many connected components.

Further, every e-degree is the degree of a structure in each of these classes.

Note! If the enumeration degree of a structure $\mathcal{A}$ is non-total then $\mathcal{A}$ does not have Turing degree.

Descriptive complexity of degree spectra

Richter proved that if $\mathcal{A}$ has the c.e. embeddability condition then $\mathcal{A}$ does not have enumeration degree unless it is $\mathbf{0}_e$.

Fix a structure $\mathcal{A}$ and consider the set
$D_{2^\omega(\mathcal{A})} = \{X \in 2^\omega | X \text{ computes a copy of } \mathcal{A}\}$.

Question

What is the descriptive complexity of $D_{2^\omega(\mathcal{A})}$?

We can ask the same question about
$D_{\omega^\omega(\mathcal{A})} = \{f \in \omega^\omega | f \text{ computes a copy of } \mathcal{A}\}$.

Theorem (Montalban)

$D_{2^\omega(\mathcal{A})}$ is never the upward closure of an F_σ set in Cantor space unless it is an enumeration cone.
$D_{\omega^\omega(\mathcal{A})}$ is never the upward closure of an F_σ set in Baire space unless it is an enumeration cone.

E-pointed trees

Definition

A tree $T \subseteq 2^{<\omega}$ (or $\subseteq \omega^{<\omega}$) with no dead ends is *e-pointed* if every branch in T enumerates T.

McCarthy's characterization of cototal degrees as degrees of minimal subshifts passes through the notion of *e-pointed trees* in Cantor space.

Theorem (McCarthy 2018)

The following are equivalent for an enumeration degree $\mathbf{a}$:

1. $\mathbf{a}$ is cototal.
2. $\mathbf{a}$ contains a *uniformly* e-pointed tree $T \subseteq 2^{<\omega}$, i.e. a tree T such that for some enumeration operator Γ we have that $T = \Gamma(f)$ for every branch f in $T \subseteq 2^{<\omega}$.
3. $\mathbf{a}$ contains a *uniformly* e-pointed tree $T \subseteq 2^{<\omega}$ *with dead ends*.
4. $\mathbf{a}$ contains an e-pointed tree $T \subseteq 2^{<\omega}$.
5. $\mathbf{a}$ contains an e-pointed tree $T \subseteq 2^{<\omega}$ with *dead ends*.

E-pointed trees in Baire space

We are left with the following question:

Question

What are the degrees of trees $T \subseteq \omega^{<\omega}$ that are:

1. uniformly e-pointed;
2. e-pointed;
3. uniformly e-pointed with dead ends;
4. e-pointed with dead ends?

We know that they contain the cototal degrees, but could there be non-cototal such degrees?

Hyper-enumeration reducibility

Let A and B be sets of natural numbers.

Definition (Sanchis 1978)

$A \leqslant_{he} B$ (A is *hyperenumeration reducible* to B) if and only if there is a c.e. set W such that

$$A = \{x \mid \forall f \in \omega^\omega \exists n \exists v [\langle f \upharpoonright n, x, v\rangle \in W \ \&\ D_v \subseteq B]\}.$$

Sanchis proved that he-reducibility has natural properties:

1. Hyperenumeration reducibility is a pre-order on $\mathcal{P}(\omega)$, and so it induces $\mathcal{D}_{he}$ the hyper enumeration degrees.
2. It extends enumeration reducibility: furthermore, if $A \leqslant_e B$ then $A \leqslant_{he} B$ and $\overline{A} \leqslant_{he} \overline{B}$.
3. A is Π^1_1 in B if and only if $A \leqslant_{he} B \oplus \overline{B}$.
4. $A \leqslant_h B$ if and only if $A \oplus \overline{A} \leqslant_{he} B \oplus \overline{B}$.
5. There are *non-hypertotal* degrees, i.e. not all hyper enumeration degrees contain a set of the form $A \oplus \overline{A}$.

Selman's theorem fails in the hyper enumeration degrees

Theorem (Jacobsen-Grocott (tba))

There are sets A and B such that $A \nleqslant_{he} B$ but for every X if B is $\Pi^1_1(X)$ then A is $\Pi^1_1(X)$.

Proof idea:

Josiah builds a uniformly e-pointed tree $T \subseteq \omega^{<\omega}$ such that $\overline{T} \nleqslant_{he} T$.

If T is $\Pi^1_1(X)$ then there is a branch f in T such that $f \leqslant_h X$.
But then $T \leqslant_e f$ and so

$$\overline{T} \leqslant_{he} \overline{f} \leqslant_{he} f \leqslant_{he} X \oplus \overline{X}.$$

In other words, $\overline{T}$ is $\Pi^1_1(X)$.

Hyper-cotototal sets

Definition

A set A is *hypercototal* if and only if $A \leqslant_{he} \overline{A}$.

- Every Π^1_1 set is hypercototal.
- $\overline{O}$ is not hypercototal because $\overline{O}$ is not Π^1_1.

Theorem (GJMS)

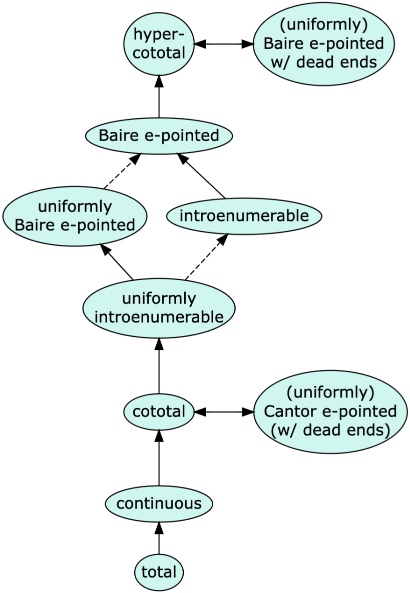
The following are equivalent for an enumeration degree $\mathbf{a}$

1. $\mathbf{a}$ contains a hypercototal set.
2. $\mathbf{a}$ contains a uniformly e-pointed tree $T \subseteq \omega^{<\omega}$ with dead ends.
3. $\mathbf{a}$ contains an e-pointed tree $T \subseteq \omega^{<\omega}$ with dead ends.

Theorem (GJMS)

If an enumeration degree $\mathbf{a}$ contains a 3-generic then $\mathbf{a}$ does not contain an e-pointed tree (without dead ends) in Baire space.

The complete picture



Introenumerable enumeration degrees

Definition

A set A is *introenumerable* if $A \leqslant_e S$ for every infinite set $S \subseteq A$.

A is *uniformly introenumerable* if there is an enumeration operator Γ such that $A = \Gamma(S)$ for every infinite set $S \subseteq A$.

- Note, Jockusch (1968) defined (uniform) intreonumerability differently: he asked that A is c.e. in each of its infinite subsets. The uniform versions coincide.
- Every cototal degree contains a uniformly introenumerable set: for example any uniformly e-pointed tree in Cantor space is uniformly introenumerable.
- If A is introenumerable then A is enumeration equivalent to the tree of all injective enumerations of infinite subsets of A. This is an e-pointed tree in Baire space with no dead ends.

The complete picture

Using forcing we can show that:

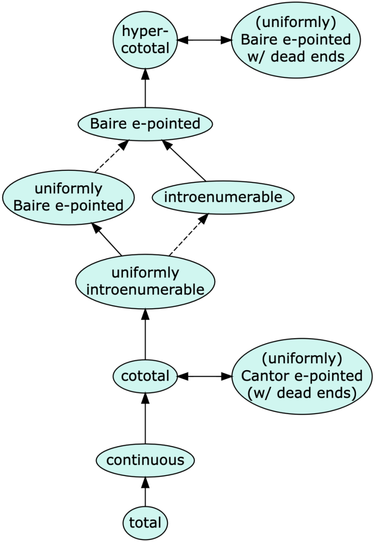
Theorem (GJMS)

There is a uniformly introenumerable set that does not have cototal degree.

Theorem (GJMS)

There is a uniformly Baire e-pointed tree that does not have introenumerable degree.

The remaining implications are open.



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Thank you!