One point extensions of antichains in the local structure of the enumeration degrees

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The theory of a degree structure

Let $\mathcal{D}$ be a degree structure: $\mathcal{D} \in \{\mathcal{D}_T, \mathcal{D}_e, \mathcal{D}_T(\leqslant \mathbf{0}'), \mathcal{R}, \mathcal{D}_e(\leqslant \mathbf{0}')\}$.

Question

Is the theory of the structure in the language of partial orders decidable?

The answer in all of these cases is "No". In fact each of these structures has maximally complex theory.

Question

What about the existential theory?

To understand what existential sentences are true in $\mathcal{D}$ we need to understand what finite partial orders can be embedded into $\mathcal{D}$;

In every case the answer is: "All" and so each structure has a decidable existential theory.

Question

How many quantifiers does it take to break decidability?

A summary of the known results

Degree structure	Complexity of $Th(\mathcal{D})$	$\exists\forall\exists$-$Th(\mathcal{D})$	$\forall\exists$-$Th(\mathcal{D})$
$\mathcal{D}_T$	Simpson 77	Lerman-Schmerl 83	Shore 78; Lerman 83
$\mathcal{D}_T(\leqslant \mathbf{0}')$	Shore 81	Lerman-Schmerl 83	Lerman-Shore 88
$\mathcal{R}$	Slaman-Harrington 80s	Lempp-Nies-Slaman 98	Open
$\mathcal{D}_e$	Slaman-Woodin 97	Lempp-Slaman-S 21	Open
$\mathcal{D}_e(\leqslant \mathbf{0}')$	Ganchev-Soskova 12	Kent 06	Open

One main difference between the structures for which the problem is solved and for which it is not is density:

1. $\mathcal{D}_T$ and $\mathcal{D}_T(\leqslant \mathbf{0}')$ have minimal degrees (Spector 1956, Sacks 1963);
2. $\mathcal{D}_e$ is downwards dense (Gutteridge 1971), while $\mathcal{R}$ and $\mathcal{D}_e(\leqslant \mathbf{0}')$ are dense (Sacks 1964, Cooper 1984);

Extensions of embeddings

To understand what $\forall\exists$ sentences are true in $\mathcal{D}$ we need to understand the following problem:

Problem

Given a finite partial order P and finite extensions of P, say $Q_1, \ldots Q_n$, does every embedding of P in $\mathcal{D}$ extend to an embedding of one of the Q_i?

When $n = 1$ we call this the *Extension of embeddings problem.*

1. For $\mathcal{D}_T$ and $\mathcal{D}_T(\leqslant_T)$ the case $n = 1$ is decidable (Lerman, Shore 78/88). The case $n > 1$ can be reduced to the case $n = 1$: every embedding of P extends to an embedding of Q_1 or $\ldots$ or Q_n if and only if for some i every embedding of P extends to an embedding of Q_i.
2. The extension of embeddings problem is decidable for each of the remaining structures:
   - Slaman and Soare (2001) prove it for $\mathcal{R}$.
   - Lempp, Slaman, and Sorbi (2005) for $\mathcal{D}_e(\leqslant_e \mathbf{0}')$.
   - Lempp, Slaman, and S (2021) for $\mathcal{D}_e$.

A subproblem

We focus on $\mathcal{D}_e(\leqslant \mathbf{0}')$ and isolate a subproblem that we hope is more approachable.

Problem

Given a finite antichain P and finite one point extensions of P, say $Q_1, \ldots Q_n$ such that every new element is below some element in P. Does every embedding of P in $\mathcal{D}_e(\leqslant \mathbf{0}')$ extend to an embedding of one of the Q_i in $\mathcal{D}_e(\leqslant \mathbf{0}') \smallsetminus \{\mathbf{0}_e, \mathbf{0}_e'\}$?

- The problem when P is a finite chain and Q_i are one point extensions can be solved easily using variations on Cooper's density construction.
- If P is an antichain and Q_i adds one new point that is incomparable to P then the answer is 'Yes'. This follows from the extension of embeddings theorem.
- The dual problem of placing one additional point above some members of P is easily seen to be decidable: the methods used here are density, every pair of degrees has a least upper bound and we can make that bound equal to $\mathbf{0}_e'$.

Representing an instance

Problem

Given a finite antichain P and finite one point extensions of P, say $Q_1, \dots Q_n$ such that every new element is below some element in P. Does every embedding of P in $\mathcal{D}_e(\leqslant \mathbf{0}')$ extend to an embedding of one of the Q_i in $\mathcal{D}_e(\leqslant \mathbf{0}') \smallsetminus \{\mathbf{0}_e, \mathbf{0}'_e\}$?

- We represent an antichain of size n by $0, 1, \dots n-1$.
- We represent an extension Q_i by the subset $i_0, \dots i_k$ of points that bound the new element in Q_i.
- An instance of the problem is a subset $S = \{Q_1, \dots Q_k\}$ of $\mathcal{P}(n) \smallsetminus \{\varnothing\}$.
- Suppose $S_1 \subseteq S_2$ are two instances.
  - if S_1 has a positive answer then so does S_2;
  - if S_2 has a negative answer then so does S_1;
  - if S_2 can be obtained from S_1 by a permutation of $0, 1, \dots n$ then S_1 and S_2 have the same answer.

Density, Ahmad and minimal pairs

Consider the case when $P = \{0, 1\}$. Below are all instances of the problem:

$$\{0, 1, 01\}$$

$$\{0, 01\}\{0, 1\}\{1, 01\}$$

$$\{0\}\{01\}\{1\}$$

We are looking for a dividing line between the 'yes' and 'no' answers.
Fix an embedding of P to $\mathbf{a}_0, \mathbf{a}_1$ incomparable Σ^0_2 enumeration degrees.

- Consider $\{0, 01\}$. By density there is some nonzero degree $\mathbf{x}$ such that $\mathbf{0}_e < \mathbf{x} < \mathbf{a}_0$. This degree is either below $\mathbf{a}_1$ or not. The answer is 'yes'.
- Consider $\{01\}$. If our embedding of P is such that $\mathbf{a}_0, \mathbf{a}_1$ form a minimal pair then there is no extension of this embedding to $\{01\}$. So the answer is 'no'.

Density, Ahmad and minimal pairs

Consider the case when $P = \{0, 1\}$. Below are all instances of the problem:

$$\{0, 1, 01\}$$

$$\{0, 01\}\{0, 1\}\{1, 01\}$$

$$\{0\}\{01\}\{1\}$$

We are looking for a dividing line between the 'yes' and 'no' answers.
Fix an embedding of P to $\mathbf{a}_0, \mathbf{a}_1$ incomparable Σ^0_2 enumeration degrees.

- Consider $\{0\}$. Suppose the embedding of P is such that every degrees $\mathbf{x} < \mathbf{a}_0$ is also below $\mathbf{a}_1$. We call such degrees $\{\mathbf{a}_0, \mathbf{a}_1\}$ an *Ahmad pair*. Ahmad proved that such Σ^0_2 degrees exists. The answer is 'no'.

- Ahmad 1998 proved that there are no Σ^0_2 symmetric Ahmad pairs: i.e. if every degrees $\mathbf{x} < \mathbf{a}_0$ is also below $\mathbf{a}_1$ then there must be some degree $\mathbf{y} < \mathbf{a}_1$ such that $\mathbf{y} \not\leq \mathbf{a}_0$. This means that the answer for $\{0, 1\}$ is 'yes'.

The case $n = 2$ solved

Consider the case when $P = \{0, 1\}$. We have identified the dividing line:

$$\{0, 1, 01\}$$

$$\{0, 01\}\{0, 1\}\{1, 01\}$$

$$\{0\}\{01\}\{1\}$$

Note that this illustrates a difference between the extension of embedding problem and the more general problem we are considering!

Ahmad's construction of her pairs

Definition

Degrees $\mathbf{a}$ and $\mathbf{b}$ are an *Ahmad pair* (we write $A(\mathbf{a}, \mathbf{b})$) if $\mathbf{a} \nleqslant \mathbf{b}$ and for all $\mathbf{x}$ if $\mathbf{x} < \mathbf{a}$ then $\mathbf{x} < \mathbf{b}$.

Note that $A(\mathbf{a}, \mathbf{b})$ implies that $\mathbf{a}$ is *nonsplittable*: $\mathbf{a} \neq \mathbf{x} \vee \mathbf{y}$ for $\mathbf{x}, \mathbf{y} < \mathbf{a}$. Nonsplittable degrees are easier to build: Kent and Sorbi 2007 show that there is one below any nonzero Σ^0_2 enumeration degree.

To build a Σ^0_2 Ahmad pair:

1. Start with a low non-splittable degree.
2. If A has low nonsplittable degree then there is a uniformly Σ^0_2 sequence $\{B_i\}_{i<\omega}$ such that for all $X <_e A$ there is some B_i such that $X \leqslant_e B_i$.
3. If A is a Σ^0_2 set and $\{B_i\}_{i<\omega}$ is a uniformly Σ^0_2 sequence of sets such that $A \nleqslant_e B_i$ for all i then there is a set B such that $A \nleqslant_e B$ and for all i $B_i \leqslant_e B$.

Lempp and Sorbi 2002 give a direct construction of Ahmad pairs on a priority tree.

No Symmetric Ahmad pair

Theorem (Ahmad, Lachlan 1998)

There are no incomparable Σ^0_2 degrees $\mathbf{a}$ and $\mathbf{b}$ such that $A(\mathbf{a}, \mathbf{b})$ and $A(\mathbf{b}, \mathbf{a})$.

Proof sketch:

1. Gutteridge built an enumeration operator Θ such that
    - Every column of $\Theta(X)$ is finite.
    - $\mathbf{0}'$ can compute uniformly in n a bound b_n on the n-th column: if $x < b_n$ then $x \in \Theta(X)^{[n]}$ and if $x > b_n$ then $x \notin \Theta(X)^{[n]}$.
    - For every n we have that $n \in X$ if and only if $b_n \in \Theta(X)^{[n]}$.
    - If $\Theta(X) \equiv_e X$ then X is c.e.
2. If $\Theta(\bigoplus_e \Gamma_e(A)) \leqslant_e B$ then we can approximate each set $\Gamma_e(A)$ with a Σ^0_2 approximation that behaves nicely relative to B uniformly in e.
3. Having nice approximations to sets $\Gamma_e(A)$ allows us to execute Cooper's construction used to show density: we can build a set $\Psi(B) <_e B$ such that $\Psi(B) \nleqslant_e A$.

No Symmetric Ahmad pair directly

Theorem (Ahmad, Lachlan 1998)

There are no Σ^0_2 degrees $\mathbf{a}$ and $\mathbf{b}$ such that $A(\mathbf{a}, \mathbf{b})$ and $A(\mathbf{b}, \mathbf{a})$.

Proof sketch by GNLS: Fix Σ^0_2 sets $A \not\leqslant_e B$. We build an enumeration operator Φ_1 so that $X_1 = \Phi_1(B)$ satisfies for every e:

- $\mathcal{R}_e : X_1 \neq \Gamma_e(A)$
- $\mathcal{S}_e : B \neq \Delta_e(X_1)$.

If some $\mathcal{R}_e$ fails then we build an enumeration operator Φ_0 so that $X_0 = \Phi_0(A)$ satisfies for every i:

- $\mathcal{R}_{e,i} : X_0 = \Gamma_i(B) \implies A = \Psi(B)$ (for a Ψ built by us)
- $\mathcal{S}_{e,i} : A \neq \Delta_i(X_0)$.

To satisfy $\mathcal{R}_{e,i}$ we code A into X_0: To every $a \in A \setminus \Psi(B)$ we assign a marker x_a targeted for X_0 and a marker m_a targeted for X_1. We enumerate x_a in X_0 with an axiom that includes the Γ_e valid axiom for each m_b where b is an older number we have dealt with. Then enumerate axioms for m_a in Φ_1 and a in Ψ using a valid Γ_i axiom for x_a.

Ahmad triples

Problem

Are there Σ_2^0 degrees $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ such that $A(\mathbf{a}, \mathbf{b})$ and $A(\mathbf{b}, \mathbf{c})$?

Definition

Let $\mathbf{a}$ be a degree and F be a finite set of degrees such that $\mathbf{a} \nleqslant \mathbf{b}$ for all $\mathbf{b} \in F$. If for every degree $\mathbf{x} < \mathbf{a}$ there is some $\mathbf{b} \in F$ such that $\mathbf{x} \leqslant \mathbf{b}$ we say that $\mathbf{a}$ and F are a *generalized Ahmad pair* and write $A(\mathbf{a}, F)$.

Theorem (GLNS)

There are degrees $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ such that $A(\mathbf{a}, \{\mathbf{b}, \mathbf{c}\})$ but $\neg A(\mathbf{a}, \mathbf{b})$ and $\neg A(\mathbf{a}, \mathbf{c})$.

Theorem (GLNS)

There are no degrees $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}_1$ and $\mathbf{c}_2$ such that $A(\mathbf{a}, \mathbf{b})$ and $A(\mathbf{b}, \{\mathbf{c}_1, \mathbf{c}_2\})$.

Note, that when $\mathbf{c}_1 = \mathbf{c}_2$ we get 'no Ahmad triple'; when $\mathbf{c}_1 = \mathbf{c}_2 = \mathbf{a}$ we get 'no symmetric Ahmad pair.'

One point extension when $n = 3$

Theorem (GLNS)

In the Σ_2^0 enumeration degrees, every embedding of the 3-element antichain $P = \{0, 1, 2\}$ can be extended to an embedding of some ordering in S if S is a superset of one of the following, up to permutation.

1. $\{0, 01, 02, 012\}$;
2. $\{0, 1, 02\}$;
3. $\{0, 1, 2\}$.

There is an embedding of P which cannot be extended to an embedding of any ordering in S if S is a subset of one of the following, up to permutation:

1. $\{01, 02, 12, 012\}$;
2. $\{0, 01, 02, 12\}$;
3. $\{0, 02, 12, 012\}$;
4. $\{0, 1, 01, 012\}$.

This accounts for all cases.

The 'yes' side: extensions

In the Σ^0_2 enumeration degrees, every embedding of the 3-element antichain $P = \{0, 1, 2\}$ can be extended to an embedding of some ordering in S if S is a superset of one of the following, up to permutation.

1. $\{0, 01, 02, 012\}$: This is once again just by density. There is some nonzero $\mathbf{x} < \mathbf{a}_0$ and we are given all possibilities for what other members of the antichain could bound $\mathbf{x}$.

2. $\{0, 1, 02\}$: This is exactly the previous theorem we discussed. If $\neg A(\mathbf{a}_0, \mathbf{a}_1)$ then pick $\mathbf{x} < \mathbf{a}_0$ such that $\mathbf{x} \nleq \mathbf{a}_1$ it is either just below $\mathbf{a}_0$ or also below $\mathbf{a}_2$.

   If $A(\mathbf{a}_0, \mathbf{a}_1)$ then by the previous theorem we know that $\neg A(\mathbf{a}_1, \{\mathbf{a}_0, \mathbf{a}_2\})$ and so there is some $\mathbf{x} < \mathbf{a}_1$ such that $\mathbf{x} \nleq \mathbf{a}_0, \mathbf{a}_2$.

3. $\{0, 1, 2\}$: for this we need to expand the no symmetric Ahmad pairs proof: we need to show that it is not possible to have degrees $\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2$ such that $A(\mathbf{a}_0, \{\mathbf{a}_1, \mathbf{a}_2\})$, $A(\mathbf{a}_1, \{\mathbf{a}_0, \mathbf{a}_2\})$, and $A(\mathbf{a}_2, \{\mathbf{a}_0, \mathbf{a}_1\})$.

Gutteridge sets

Definition

Let A be a given set.We say that $G(A)$ is an A-*Gutteridge* set if there is a computable function f such that for every n $\lim_s f(n,s) = n_f$ exists and

1. If $x < n_f$ then $x \in G(A)^{[n]}$ and if $x > n_f$ then $x \notin G(A)^{[n_f]}$.
2. $n_f \in G(A)^{[n]}$ if and only if $n_f \in K_A = \bigoplus_e \Gamma_e(A)$.

Note that $G(A) \leqslant_e A$.

Furthermore if $G(A)$ is c.e. then K_A is Δ^0_2, i.e. A is low.

Theorem (Gutteridge)

For every non-c.e. set A there is a Gutteridge set $G(A) <_e A$.

Gutteridge sets

Theorem (GLNS)

Suppose $G(A)$ is an A-Gutteridge set and $G(A) \leqslant_e B$. Then for every B-Gutteridge set $G(B)$ there is an A-Gutteridge set $\hat{G}(A) \leqslant_e G(B)$.

Theorem (GLNS)

Fix Σ^0_2 sets A and $B_1, \dots B_n$ such that $A \nleqslant B_i$ for all $i \leqslant n$. Suppose that A bounds a B_i-Gutteridge set for every i. For any $Y <_e A$ there is a set X such that $Y \leqslant_e X <_e A$ and $X \nleqslant_e B_i$ for every i.

Theorem (GLNS)

There are no Σ^0_2 degrees $\mathbf{a}_0, \dots \mathbf{a}_n$ such that $A(\mathbf{a}_i, \{\mathbf{a}_j \mid j \neq i\})$ for every i .

Proof sketch: Assume towards a contradiction that $A_0, \dots A_n$ represent degrees that violate this. Fix a Gutteridge set $G(A_0) <_e A_0$. It must be below one of the other sets, say A_1. So we can find X_1 such that $G(A_1) \leqslant_e X_1 < A_1$ and $X_1 \nleqslant_e A_0$. Now X_1 bound both an A_0- and an A_1-Gutteridge set. It must be below one of the other sets, say A_2. We get X_2 such that $G(A_2) \leqslant_e X_2 < A_2$ and X_2 is not below A_0, A_1. Etc...

The ‘no’ side: no extensions

There is an embedding of P which cannot be extended to an embedding of any ordering in S if S is a subset of one of the following, up to permutation:

1. $\{01, 02, 12, 012\}$: Embed P so that $M(\mathbf{a}_0, \mathbf{a}_1)$, $M(\mathbf{a}_1, \mathbf{a}_2)$, and $M(\mathbf{a}_0, \mathbf{a}_2)$.
2. $\{0, 02, 12, 012\}$: Embed P so that $A(\mathbf{a}_0, \mathbf{a}_1)$ and $M(\mathbf{a}_1, \mathbf{a}_2)$. This can be done using an easy direct construction.
3. $\{0, 1, 01, 012\}$: Embed P so that $A(\mathbf{a}_0, \mathbf{a}_2)$, $A(\mathbf{a}_1, \mathbf{a}_2)$ and $M(\mathbf{a}_0, \mathbf{a}_1)$. Start with a minimal pair of low degrees, find non-splittable degrees $\mathbf{a}_0$, $\mathbf{a}_1$ below each member. Modify the last step in Ahmad's proof to get $\mathbf{a}_2$ with the requested properties.
4. $\{0, 01, 02, 12\}$; Embed P so that $A(\mathbf{a}_0, \mathbf{a}_1)$, $A(\mathbf{a}_0, \mathbf{a}_2)$ and $\mathbf{a}_1$ and $\mathbf{a}_2$ form an exact pair for the ideal of degrees strictly below $\mathbf{a}_0$. We show that this is possible using a (complicated) direct construction.

General conjecture

Conjecture

Let $n > 1$ and $\mathcal{S}$ be a family of non-empty subsets of $\{0, \cdots, n\}$. Let $S_0 \subseteq \{0, \cdots, n\}$ be the collection of all singletons in $\mathcal{S}$ and $S_1 = \{0, 1, \cdots, n\} \smallsetminus S_0$.
Then $\mathcal{S}$ is blockable if and only if $S_0 = \varnothing$, or there exists a function $\nu : S_0 \to \mathcal{P}(S_1) - \{\varnothing\}$, called an *assignment on* $\mathcal{S}$, with the following properties:

1. For every $i \in S_0$, $\{i\} \cup \nu(i) \notin \mathcal{S}$.
2. For each $F \subseteq S_0$ where $|F| > 1$ we have $\bigcap\{\nu(i) \mid i \in F\} \notin \mathcal{S}$.

Example: For the blockable set $S = \{0, 1, 01, 012\}$ we have

- $S_0 = \{0, 1\}$; $S_1 = \{2\}$
- $\nu(0) = \nu(1) = \{2\}$.
    1. We have that $02 \notin S$ and $12 \notin S$.
    2. For $F = \{0, 1\}$ we have that $\bigcap\{\nu(i) \mid i \in F\} = \{2\} \notin S$.

The ultimate extension of no Ahmad triple

Theorem (GLNS)

There are no Σ_2^0 degrees $\mathbf{a}$, $\mathbf{b}_i$ where $i < k$ and $\mathbf{c}_{i,j}$ where $i < n$ and $j < n_i$ such that $\mathbf{a} \not\leqslant \mathbf{b}_i$ for all $i < n$ and $\mathbf{b}_i \not\leqslant \mathbf{c}_{i,j}$ for all $i < n, j < n_i$ and $A(\mathbf{a}, \{\mathbf{b}_i\}_{i<n})$ and $A(\mathbf{b}_i, \{\mathbf{c}_{i,j}\}_{j<n_i})$for every i.

This proves the 'only if' direction of our conjecture.

We are working on the 'if' direction.

Lempp and Ng are working on:

Question

Is there an Ahmad pair $A(\mathbf{a}, \mathbf{b})$ such that $\mathbf{a} \vee \mathbf{b} = \mathbf{0}_e'$?

Thank you!