Introducing a New Characterization of the Low_n Degrees
A look inside Kenneth Harris' PhD Thesis
Part 1

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Introduction

Definition

Let $n \geq 1$. A set A is low_n if $A^{(n)} \equiv_T \emptyset^{(n)}$. A Turing degree d is low_n if it contains a low_n set.

- Aim: Find some property that characterizes the low_n degrees, which is easier to work with.

The strong quantifiers

Definition

1. $(\exists^\infty x)P \Leftrightarrow (\forall y \exists x > y)P$ - for infinitely many x.
2. $(\forall^\infty x)P \Leftrightarrow (\exists y \forall x > y)P$ - for almost all (all but finitely many) x.

- $\forall \Rightarrow \forall^\infty \Rightarrow \exists^\infty \Rightarrow \exists$

Dart Vader vs Luke Skywalker

- For f, g- total functions, f dominates g if

$$(\forall^{\infty} x)[f(x) > g(x)].$$

- For f, g- total functions, g escapes f

$$(\exists^{\infty} x)[f(x) \leq g(x)].$$

The starting Point

Theorem

Martin's High Domination Theorem.
A Turing degree $\mathbf{a}$ *is high iff*

$$(\exists f \leq \mathbf{a})(\forall g \leq \mathbf{0})[f \text{ dominates } g].$$

Corollary

A Turing degree $\mathbf{a}$ *is not high iff*

$$(\forall f \leq \mathbf{a})(\exists g \leq \mathbf{0})[g \text{ escapes } f].$$

- Can we use this to characterize the low_n degrees?
- What additional properties should the escape functions have?

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A closer look at the Arithmetical Hierarchy

We will fix some standard effective coding of all finite tuples.

- E.g suppose we have some pairing function $\pi : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$. Then we will code $n_1, \dots n_k$ by $\langle n_1, \dots, n_k \rangle = \pi(k, \pi(n_1, \dots \pi(n_{k-1}, n_k)))$.
- This gives us the means to consider only 1-ary relations. Any any k-ary relation $P(x_1, \dots x_k)$ will be represented by the relation $P'(n) \Leftrightarrow n = \langle n_1, \dots, n_k \rangle \wedge P(n_1, \dots, n_k)$.

A closer look at the Arithmetical Hierarchy

Definition

Let P be any relation.

1. P is Σ_0 (Π_0) if it is computable.
2. P is Σ_{n+1} if there is a Π_n relation Q such that:

$$P(x) \Leftrightarrow \exists \overline{y} Q(\langle x, \overline{y} \rangle)$$

3. P is Π_{n+1} if there is a Σ_n relation Q such that:

$$P(x) \Leftrightarrow \forall \overline{y} Q(\langle x, \overline{y} \rangle)$$

4. P is Δ_{n+1} iff P is Σ_{n+1} and Π_{n+1}

A closer look at the Arithmetical Hierarchy

Definition

Let P be any relation and A be any set.

1. P is Σ^A_0 (Π^A_0) if it is A-computable.
2. P is Σ^A_{n+1} if there is a Π^A_n relation Q such that:

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A closer look at the Arithmetical Hierarchy

Mariya I. Soskova

- Connect with a set A the relation $A(x) \Leftrightarrow x \in A$.
- Note that A is c.e. iff the relation $A(x)$ is Σ_1.
- If A is a Σ_{2n+1} set, then there is a c.e. set W_e, s.t.

  $$A(x) \Leftrightarrow (\exists y_{2n-1})(\forall y_{2n-2})(\exists y_1)(\forall z)[\langle x, z, y_1 \dots, y_{2n+1}\rangle \in W_e]$$

  We will say that A has Σ_{2n+1} index e.
  If $B \in \Pi_{2n+1}$ then a Π_{2n+1} index for B is any Σ_{2n+1} index for $\overline{B}$.
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Additional Tools

- Post's Theorem: For every $n \geq 1$ and every A
  1. $A^{(n)}$ is Σ_n^A- complete.
  2. $X \in \Delta_{n+1}^A$ iff $X \leq_T A^{(n)}$.
- Theorem. The following are equivalent
  1. A is low_n.
  2. $\Sigma_n^A \subset \Pi_{n+1}$.
  3. $\Pi_n^A \subset \Sigma_{n+1}$.

  Proof: $1 \Rightarrow 2$. A is low_n, hence $A^{(n)} \leq_T 0^{(n)}$. Then A is Δ_{n+1}. If $B \in \Sigma_n^A$, then $B \leq_m A$, hence B is $\Delta_{n+1} \subset \Pi_{n+1}$.

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The Low_1 Degrees

Definition

A set A has the uniform escape property (UEP) if there is a partial computable function $h(e, x)$ such that whenever ϕ_e^A is total, then $h(e, x)$ is total and escapes domination from ϕ_e^A, i.e:

$$(\exists^\infty x)[\phi_e^A(x) \leq h(e, x)].$$

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A is low if and only if A has UEP.

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Strong Quantifier Normal Form

Theorem

There is a computable function g such that for any Π_2 set A with Π_2 index e:

1. $A(x) \Leftrightarrow W_{g(e,x)} = \omega$
2. $\neg A(x) \Leftrightarrow W_{g(e,x)}$ *is finite.*

Proof.

- $A \in \Pi_2$ with index e, hence $A(x) \Leftrightarrow (\forall z)[\langle x, z\rangle \in W_e]$
- Use the S^m_n-Theorem: $A(x) \Leftrightarrow (\forall z)[z \in W_{h(e,x)}]$
- Define a g so that:
  $y \in W_{g(e,x)} \Leftrightarrow (\forall z \leq y)[z \in W_{h(e,x)}]$

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Every low set has UEP

Basic Tools

- The Recursion Thoerem
  Let s be a total computable function. Then there is an e such that $W_e = W_{s(e)}$.
- Settling functions.
  Let W_e be any c.e. set with standard approximation $W_{e,s}$. The settling function for W_e is the denoted by m_e and defined by

$$m_e(x) = (\mu s)[x \in W_{e,s}].$$

  - m_e is a partial computable function, uniformly in e.
  - m_e is total if and only if $W_e = \omega$

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Every low set has UEP

Let A be a low set. We will show that there is a total computable function k, such that if ϕ_e^A is total, then $W_{k(e)} = \omega$ and the settling function $m_{k(e)}$ escapes domination from ϕ_e^A. Then we will define $h(e, x) = m_{k(e)}(x)$.

Will define $k(e)$ so that for all e if ϕ_e^A is total then $k(e)$ has the following properties:

Esc $(\exists^\infty x)(\exists s)[\phi_{e,s}^A(x) \downarrow < s \wedge x \notin W_{k(e),s}]$

Tot $W_{k(e)} = \omega$.

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The main idea

Esc $(\exists^\infty x)(\exists s)[\phi^A_{e,s}(x) < s \downarrow \wedge x \notin W_{k(e),s}]$

Tot $W_{k(e)} = \omega$.

- (Esc) is a Π^A_2 predicate $V^A(e)$
- A is low_1, hence $\Sigma^A_2 \subset \Pi_2$. It follows that $\Pi^A_2 \subset \Pi_2$.
- Use the strong quantifier normal form theorem.

$$V^A(e) \Leftrightarrow W_{g(u,e)} = \omega$$

Every low set has UEP

Let $V^A(e,i)$ be the A-predicate expressing Esc:

$$V^A(e,i) \Leftrightarrow (\exists^\infty x)(\exists s)[\phi^A_{e,s}(x)\downarrow < s \wedge x \notin W_{g(i,e),s}]$$

- $V^A(e,i)$ is Π^A_2. But A is low, hence $\Sigma^A_1 \subseteq \Pi_2$, and $\Pi^A_2 \subseteq \Pi_2$.
- Let $V^A(e,i)$ have Π_2-index v:

$$V^A(e,i) \Leftrightarrow (\forall z)[\langle i,e,z\rangle \in W_v]$$

- Apply the S^m_n Theorem:

$$V^A(e,i) \Leftrightarrow (\forall z)[\langle e,z\rangle \in W_{s(v,i)}]$$

- Apply the recursion theorem $\Rightarrow$ there is some u, such that $W_{s(v,u)} = W_u$
- $V^A(e,u)$ has Π_2-index u.

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Every low set has UEP

Now let $V^A(e) = V^A(e,u)$.

- By definition

$$V^A(e) \Leftrightarrow (\exists^\infty x)(\exists s)[\phi^A_{e,s}(x)\downarrow < s \wedge x \notin W_{g(u,e),s}].$$

- By the strong quantifier normal form theorem:

$$V^A(e) \Leftrightarrow W_{g(u,e)} = \omega.$$

Every low set has UEP

Lemma

If ϕ_e^A is total, then $V^A(e)$.

Proof.

Suppose not. Let e be such that ϕ_e^A is total and $\neg V^A(e)$.

- By (TSQNF) $W_{g(u,e)}$ is finite.
- Let $M = \max(W_{g(u,e)})$. Let $x > M$. Then by the totality of ϕ_e^A there is a stage s, such that $\phi_{e,s}^A(x) \downarrow < s$ and $x \notin W_{g(u,e),s}$.
- $(\exists^\infty x)(\exists s)[\phi_{e,s}^A(x) \downarrow < s \wedge x \notin W_{g(u,e),s}]$, hence V_e^A.

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- $(\exists^\infty x)(\exists s)[\phi_{e,s}^A(x) \downarrow < s \wedge x \notin W_{g(u,e),s}]$, hence V_e^A.

□

Every low set has UEP

Mariya I. Soskova

Lemma

If ϕ_e^A is total, then $V^A(e)$.

Proof.

Suppose not. Let e be such that ϕ_e^A is total and $\neg V^A(e)$.

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Every low set has UEP

Define $h(e,x) = m_{g(u,e)}(x)$.

Corollary

If ϕ_e^A is total. Then $h(e,x)$ is total and escapes domination from ϕ_e^A.

Proof.

- ϕ_e^A is total $\Rightarrow V^A(e)$
- $V^A(e) \Rightarrow W_{g(u,e)} = \omega$, hence $h(e,x)$ is total.
- $V^A(e)$ and $h(e,x)$ is total $\Rightarrow h(e,x)$ escapes domination from ϕ_e^A.

□

Tot and Fin

Definition

$Tot = \{e | \phi_e \text{ is total }\}$.

- Tot is Π_2:

$$Tot(e) \Leftrightarrow (\forall x)(\exists s)[\phi_{e,s}(x) \downarrow].$$

- Tot is Π_2 - complete.
  Let A be a Π_2- set with Π_2 index e, then from the (SQNF):

$$A(x) \Leftrightarrow W_{g(e,x)} = \omega \Leftrightarrow g(e,x) \in Tot.$$

Tot and Fin

Definition

$Fin = \{e | dom(\phi_e) \text{ is finite }\}$.

- Fin is Σ_2:

$$Fin(e) \Leftrightarrow (\exists x)(\forall y > x)(\forall s)[\phi_{e,s}(y) \uparrow].$$

- Fin is Σ_2 - complete.
  Let A be a Σ_2- set with Σ_2 index e, hence $\overline{A}$ has Π_2 index e then from the (SQNF):

$$A(x) \Leftrightarrow \neg\overline{A}(x) \Leftrightarrow$$

$$W_{g(e,x)} \text{ is finite } \Leftrightarrow g(e,x) \in Fin.$$