The total degrees in the local structure of the enumeration degrees

Mariya I. Soskova[1]

Faculty of Mathematics and Informatics
Sofia University

joint work with H. Ganchev

[1] Research supported by BNSF Grant No. D002-258/18.12.08 and MC-ER Grant 239193 within the 7th European Community Framework Programme.

Preliminaries: The enumeration degrees

Definition

- $A \leq_e B$ iff there is a c.e. set W, such that $A = W(B) = \{x \mid \exists u(\langle x, u\rangle \in W \wedge D_u \subseteq B)\}$.
- $d_e(A) = \{B \mid A \leq_e B \;\&\; B \leq_e A\}$
- $d_e(A) \leq d_e(B)$ iff $A \leq_e B$.
- $\mathbf{0}_e = d_e(\emptyset) = \{W \mid W \text{ is c.e. }\}$.
- $d_e(A) \vee d_e(B) = d_e(A \oplus B)$.
- $d_e(A)' = d_e(A')$, where $A' = L_A \oplus \overline{L_A}$ and $L_A = \{x \mid x \in W_x(A)\}$.
- $\mathcal{D}_e = \langle D_e, \leq, \vee, ', \mathbf{0}_e\rangle$ is an upper semi-lattice with jump operation and least element.

Preliminaries: The enumeration degrees

Definition

- $A \leq_e B$ iff there is a c.e. set W, such that $A = W(B) = \{x \mid \exists u(\langle x, u\rangle \in W \wedge D_u \subseteq B)\}$.
- $d_e(A) = \{B \mid A \leq_e B \; \& \; B \leq_e A\}$
- $d_e(A) \leq d_e(B)$ iff $A \leq_e B$.
- $\mathbf{0}_e = d_e(\emptyset) = \{W \mid W \text{ is c.e. }\}$.
- $d_e(A) \vee d_e(B) = d_e(A \oplus B)$.
- $d_e(A)' = d_e(A')$, where $A' = L_A \oplus \overline{L_A}$ and $L_A = \{x \mid x \in W_x(A)\}$.
- $\mathcal{D}_e = \langle D_e, \leq, \vee, ', \mathbf{0}_e\rangle$ is an upper semi-lattice with jump operation and least element.

Preliminaries: The enumeration degrees

Definition

- $A \leq_e B$ iff there is a c.e. set W, such that $A = W(B) = \{x \mid \exists u(\langle x, u \rangle \in W \wedge D_u \subseteq B)\}$.
- $d_e(A) = \{B \mid A \leq_e B \;\&\; B \leq_e A\}$
- $d_e(A) \leq d_e(B)$ iff $A \leq_e B$.
- $\mathbf{0}_e = d_e(\emptyset) = \{W \mid W \text{ is c.e. }\}$.
- $d_e(A) \vee d_e(B) = d_e(A \oplus B)$.
- $d_e(A)' = d_e(A')$, where $A' = L_A \oplus \overline{L_A}$ and $L_A = \{x \mid x \in W_x(A)\}$.
- $\mathcal{D}_e = \langle D_e, \leq, \vee, ', \mathbf{0}_e \rangle$ is an upper semi-lattice with jump operation and least element.

Preliminaries: The enumeration degrees

Definition

- $A \leq_e B$ iff there is a c.e. set W, such that $A = W(B) = \{x \mid \exists u(\langle x, u\rangle \in W \wedge D_u \subseteq B)\}$.
- $d_e(A) = \{B \mid A \leq_e B \;\&\; B \leq_e A\}$
- $d_e(A) \leq d_e(B)$ iff $A \leq_e B$.
- $\mathbf{0}_e = d_e(\emptyset) = \{W \mid W \text{ is c.e. }\}$.
- $d_e(A) \vee d_e(B) = d_e(A \oplus B)$.
- $d_e(A)' = d_e(A')$, where $A' = L_A \oplus \overline{L_A}$ and $L_A = \{x \mid x \in W_x(A)\}$.
- $\mathcal{D}_e = \langle D_e, \leq, \vee, ', \mathbf{0}_e\rangle$ is an upper semi-lattice with jump operation and least element.

Preliminaries: The enumeration degrees

Definition

- $A \leq_e B$ iff there is a c.e. set W, such that $A = W(B) = \{x \mid \exists u(\langle x, u\rangle \in W \wedge D_u \subseteq B)\}$.
- $d_e(A) = \{B \mid A \leq_e B \ \&\ B \leq_e A\}$
- $d_e(A) \leq d_e(B)$ iff $A \leq_e B$.
- $\mathbf{0}_e = d_e(\emptyset) = \{W \mid W \text{ is c.e. }\}$.
- $d_e(A) \vee d_e(B) = d_e(A \oplus B)$.
- $d_e(A)' = d_e(A')$, where $A' = L_A \oplus \overline{L_A}$ and $L_A = \{x \mid x \in W_x(A)\}$.
- $\mathcal{D}_e = \langle D_e, \leq, \vee, ', \mathbf{0}_e\rangle$ is an upper semi-lattice with jump operation and least element.

Preliminaries: The enumeration degrees

Definition

- $A \leq_e B$ iff there is a c.e. set W, such that $A = W(B) = \{x \mid \exists u(\langle x, u \rangle \in W \wedge D_u \subseteq B)\}$.
- $d_e(A) = \{B \mid A \leq_e B \ \&\ B \leq_e A\}$
- $d_e(A) \leq d_e(B)$ iff $A \leq_e B$.
- $\mathbf{0}_e = d_e(\emptyset) = \{W \mid W \text{ is c.e. }\}$.
- $d_e(A) \vee d_e(B) = d_e(A \oplus B)$.
- $d_e(A)' = d_e(A')$, where $A' = L_A \oplus \overline{L_A}$ and $L_A = \{x \mid x \in W_x(A)\}$.
- $\mathcal{D}_e = \langle D_e, \leq, \vee, ', \mathbf{0}_e \rangle$ is an upper semi-lattice with jump operation and least element.

Preliminaries: The enumeration degrees

Definition

- $A \leq_e B$ iff there is a c.e. set W, such that $A = W(B) = \{x \mid \exists u(\langle x, u \rangle \in W \wedge D_u \subseteq B)\}$.
- $d_e(A) = \{B \mid A \leq_e B \;\&\; B \leq_e A\}$
- $d_e(A) \leq d_e(B)$ iff $A \leq_e B$.
- $\mathbf{0}_e = d_e(\emptyset) = \{W \mid W \text{ is c.e. }\}$.
- $d_e(A) \vee d_e(B) = d_e(A \oplus B)$.
- $d_e(A)' = d_e(A')$, where $A' = L_A \oplus \overline{L_A}$ and $L_A = \{x \mid x \in W_x(A)\}$.
- $\mathcal{D}_e = \langle D_e, \leq, \vee, ', \mathbf{0}_e \rangle$ is an upper semi-lattice with jump operation and least element.

The total degrees

Proposition

The embedding $\iota : \mathcal{D}_T \to \mathcal{D}_e$, defined by $\iota(d_T(A)) = d_e(A \oplus \overline{A})$, preserves the order, the least upper bound and the jump operation:

The sub structure of the total e-degrees is defined as $\mathcal{TOT} = \iota(D_T)$.

The total degrees above $\mathbf{0}'_e$

Theorem (Kalimullin)

There is a first order formula $\mathcal{J}$ in the language of a partial order, such that

$$\mathcal{D}_e \models \mathcal{J}(\mathbf{a}, \mathbf{b}) \Leftrightarrow \mathbf{b} = \mathbf{a}'.$$

Theorem

1. *If $\mathbf{b} = \mathbf{a}'$ the $\mathbf{b}$ is total.*
2. *Every total degree $\mathbf{b} \geq \mathbf{0}'_e$ is the jump of some $\mathbf{a} < \mathbf{b}$.*

Corollary

The total degrees above $\mathbf{0}'_e$ are first order definable in $\mathcal{D}_e$.

The total degrees above $\mathbf{0}_e'$

Theorem (Kalimullin)

There is a first order formula $\mathcal{J}$ in the language of a partial order, such that

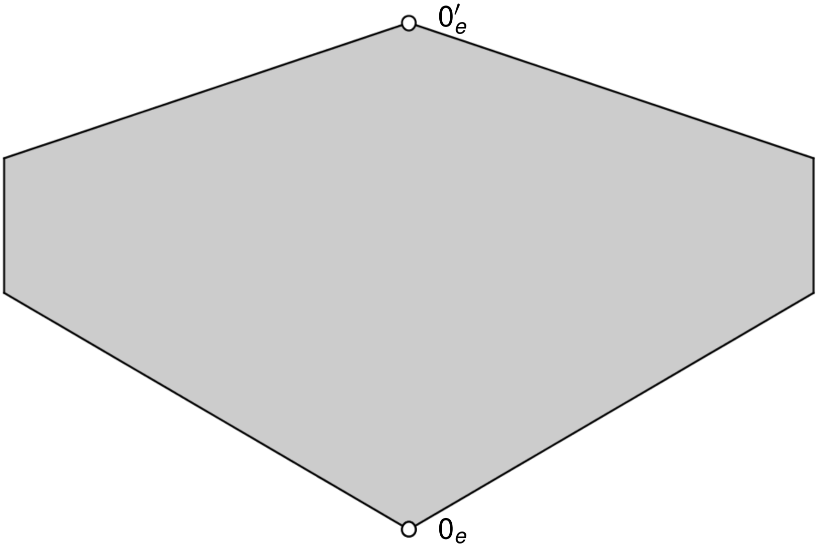
$$\mathcal{D}_e \models \mathcal{J}(\mathbf{a}, \mathbf{b}) \Leftrightarrow \mathbf{b} = \mathbf{a}'.$$

Theorem

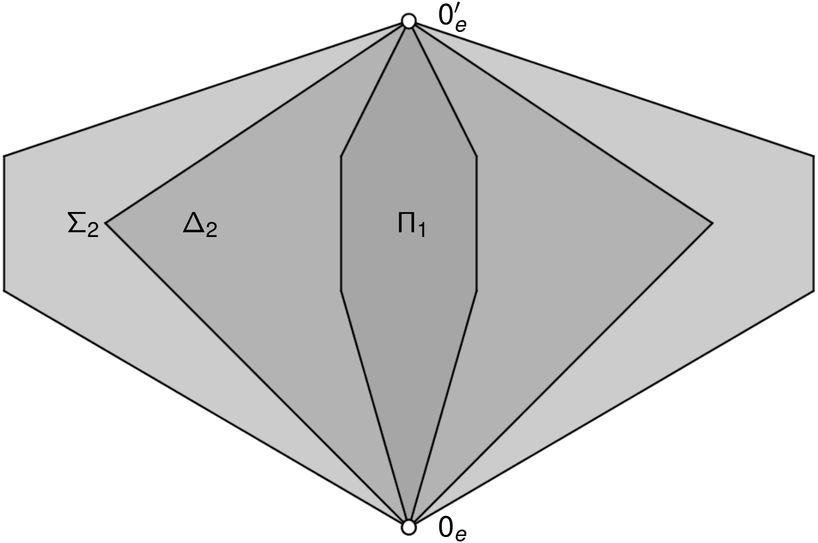
1. *If $\mathbf{b} = \mathbf{a}'$ the $\mathbf{b}$ is total.*
2. *Every total degree $\mathbf{b} \geq \mathbf{0}_e'$ is the jump of some $\mathbf{a} < \mathbf{b}$.*

Corollary

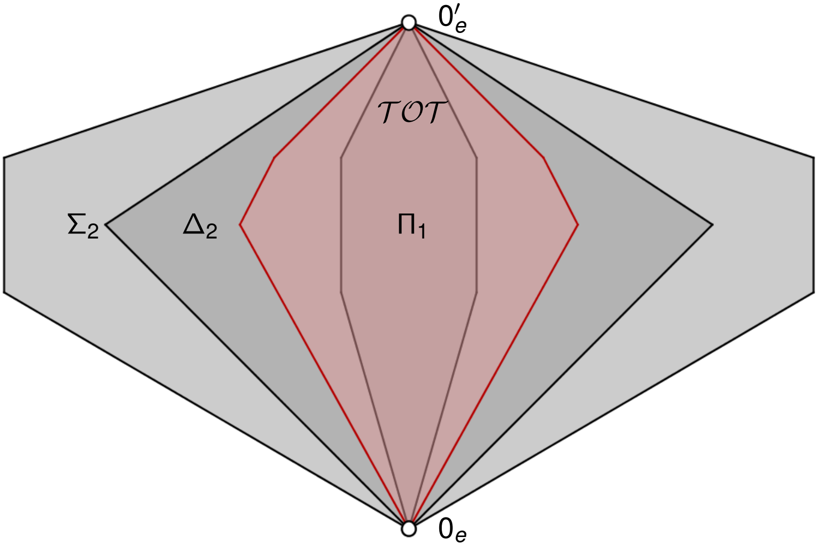
The total degrees above $\mathbf{0}_e'$ are first order definable in $\mathcal{D}_e$.



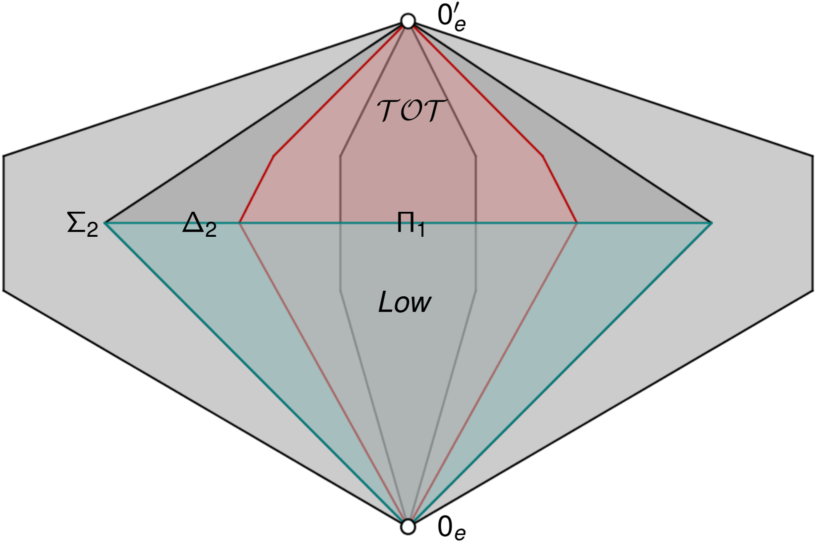
The local structure of the enumeration degrees $\mathcal{G}_e = \mathcal{D}_e(\leq 0'_e)$ consists of all degrees below $\mathbf{0}'_e$.



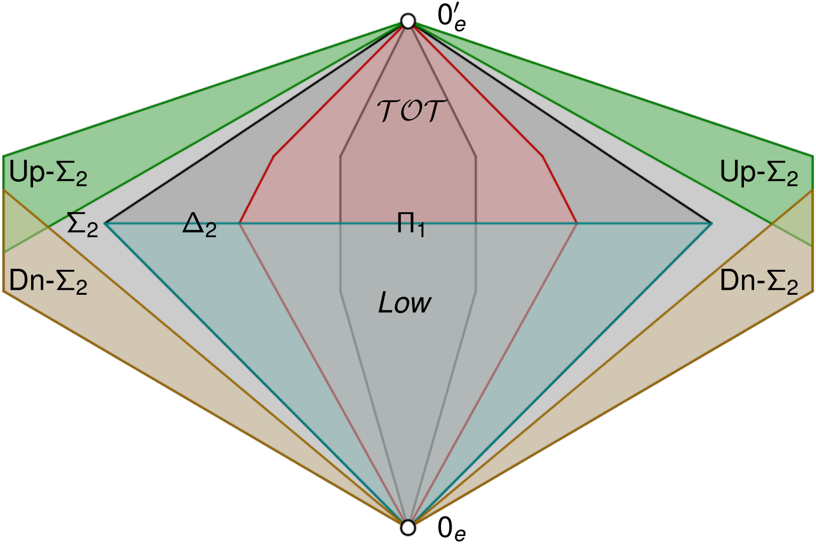
With respect to the arithmetic hierarchy the degrees can be partitioned into three classes.



The total degrees below $\mathbf{0}'_e$ are images of the Turing degrees below $\mathbf{0}'$.
Every total degree is Δ^0_2, but not all Δ^0_2 are total.



A degree is low if its jump is as low as possible: $\mathbf{0}'_e$. Every low degree is Δ^0_2.



The upwards properly Σ^0_2 have no incomplete Δ^0_2 above them. The downwards properly Σ^0_2 have no nonzero Δ^0_2 below them.

$\mathcal{K}$-pairs

Iskander Kalimullin: Definability of the jump operator in the enumeration degrees
Journal of Mathematical Logic (2003)

Definition

Let A and B be a pair sets of natural numbers. The pair (A, B) is a $\mathcal{K}$-pair (e-ideal) if there exists a c.e. set W, such that $A \times B \subseteq W$ and $\overline{A} \times \overline{B} \subseteq \overline{W}$.

$\mathcal{K}$-pairs: A trivial example

Example

Let V be a c.e set. Then (V, A) is a $\mathcal{K}$-pair for any set of natural numbers A.

Let $W = V \times \mathbb{N}$. Then $V \times A \subseteq W$ and $\overline{V} \times \overline{A} \subseteq \overline{W}$.

We will only be interested in non-trivial $\mathcal{K}$-pairs.

$\mathcal{K}$-pairs: A more interesting example

Definition (Jockusch)

A set of natural numbers A is semi-recursive if there is a computable function s_A such that for every pair of natural numbers (x, y):

1. $s_A(x, y) \in \{x, y\}$.
2. If $x \in A$ or $y \in A$ then $s_A(x, y) \in A$.

Example

Let A be a semi-recursive set. Then $(A, \overline{A})$ is a $\mathcal{K}$-pair.

Theorem (Jockusch)

For every noncomputable set B there is a semi recursive set $A \equiv_T B$ such that both A and $\overline{A}$ are not c.e.

$\mathcal{K}$-pairs: A more interesting example

Definition (Jockusch)

A set of natural numbers A is semi-recursive if there is a computable function s_A such that for every pair of natural numbers (x, y):

1. $s_A(x, y) \in \{x, y\}$.
2. If $x \in A$ or $y \in A$ then $s_A(x, y) \in A$.

Example

Let A be a semi-recursive set. Then $(A, \overline{A})$ is a $\mathcal{K}$-pair.

Theorem (Jockusch)

For every noncomputable set B there is a semi recursive set $A \equiv_T B$ such that both A and $\overline{A}$ are not c.e.

$\mathcal{K}$-pairs: A more interesting example

Definition (Jockusch)

A set of natural numbers A is semi-recursive if there is a computable function s_A such that for every pair of natural numbers (x, y):

1. $s_A(x, y) \in \{x, y\}$.
2. If $x \in A$ or $y \in A$ then $s_A(x, y) \in A$.

Example

Let A be a semi-recursive set. Then $(A, \overline{A})$ is a $\mathcal{K}$-pair.

Theorem (Jockusch)

For every noncomputable set B there is a semi recursive set $A \equiv_T B$ such that both A and $\overline{A}$ are not c.e.

An order theoretic characterization of $\mathcal{K}$-pairs

Kalimullin has proved that the property of being a $\mathcal{K}$-pair is degree theoretic and first order definable in $\mathcal{D}_e$.

Theorem (Kalimullin)

(A, B) is a $\mathcal{K}$-pair if and only if the degrees $\mathbf{a} = d_e(A)$ and $\mathbf{b} = d_e(B)$ have the following property:

$$\mathcal{K}(\mathbf{a}, \mathbf{b}) \leftrightharpoons (\forall \mathbf{x})((\mathbf{a} \vee \mathbf{x}) \wedge (\mathbf{b} \vee \mathbf{x}) = \mathbf{x})$$

Properties of $\mathcal{K}$-pairs in the local structure

1. The enumeration degrees of the elements of a $\mathcal{K}$ pair are quasi minimal, i.e. the only total degree bounded by either of them is $\mathbf{0}_e$.
2. The enumeration degrees of the elements of a $\mathcal{K}$ pair are low.
3. Every Δ^0_2 degree bounds a $\mathcal{K}$-pair.
4. The class of the enumeration degrees of sets that form a $\mathcal{K}$-pair with a fixed set A is an ideal.

Properties of $\mathcal{K}$-pairs in the local structure

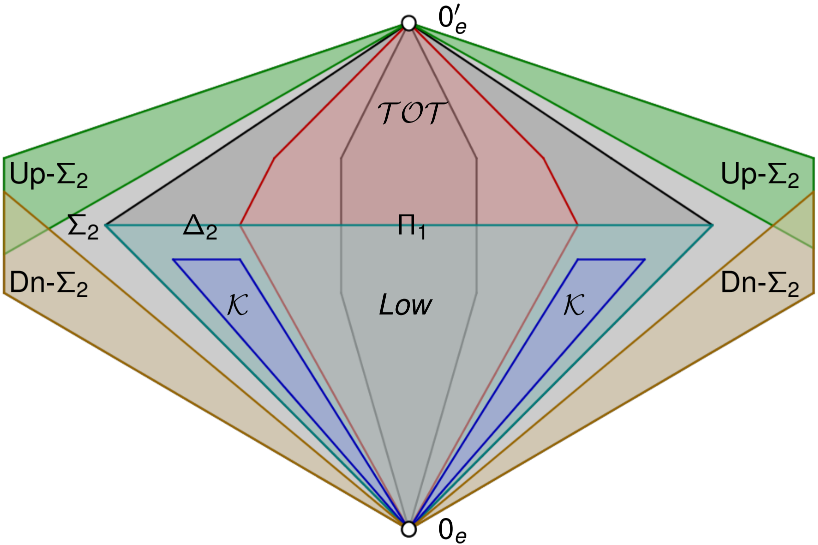
1. The enumeration degrees of the elements of a $\mathcal{K}$ pair are quasi minimal, i.e. the only total degree bounded by either of them is $\mathbf{0}_e$.
2. The enumeration degrees of the elements of a $\mathcal{K}$ pair are low.
3. Every Δ^0_2 degree bounds a $\mathcal{K}$-pair.
4. The class of the enumeration degrees of sets that form a $\mathcal{K}$-pair with a fixed set A is an ideal.

Properties of $\mathcal{K}$-pairs in the local structure

1. The enumeration degrees of the elements of a $\mathcal{K}$ pair are quasi minimal, i.e. the only total degree bounded by either of them is $\mathbf{0}_e$.
2. The enumeration degrees of the elements of a $\mathcal{K}$ pair are low.
3. Every Δ^0_2 degree bounds a $\mathcal{K}$-pair.
4. The class of the enumeration degrees of sets that form a $\mathcal{K}$-pair with a fixed set A is an ideal.

Properties of $\mathcal{K}$-pairs in the local structure

1. The enumeration degrees of the elements of a $\mathcal{K}$ pair are quasi minimal, i.e. the only total degree bounded by either of them is $\mathbf{0}_e$.
2. The enumeration degrees of the elements of a $\mathcal{K}$ pair are low.
3. Every Δ^0_2 degree bounds a $\mathcal{K}$-pair.
4. The class of the enumeration degrees of sets that form a $\mathcal{K}$-pair with a fixed set A is an ideal.



The $\mathcal{K}$-pairs in the local structure.

Local definability of $\mathcal{K}$-pairs

$$\mathcal{K}(\mathbf{a},\mathbf{b}) \leftrightharpoons (\forall \mathbf{x})((\mathbf{a} \vee \mathbf{x}) \wedge (\mathbf{b} \vee \mathbf{x}) = \mathbf{x})$$

Is it enough to require that this formula is satisfied by all Σ^0_2 e-degrees? Could there be a fake $\mathcal{K}$-pair $\{\mathbf{a},\mathbf{b}\}$, such that:

$$\mathcal{G}_e \models \mathcal{K}(\mathbf{a},\mathbf{b}) \;\;\&\;\; \neg(\mathcal{D}_e \models \mathcal{K}(\mathbf{a},\mathbf{b}))?$$

Theorem

There is a first order formula $\mathcal{LK}$, such that for any Σ^0_2 sets A and B, $\{A,B\}$ is a non-trivial $\mathcal{K}$-pair if and only if $\mathcal{G}_e \models \mathcal{LK}(\mathbf{d}_e(A),\mathbf{d}_e(B))$.

Local definability of $\mathcal{K}$-pairs

$$\mathcal{K}(\mathbf{a},\mathbf{b}) \leftrightharpoons (\forall \mathbf{x})((\mathbf{a} \vee \mathbf{x}) \wedge (\mathbf{b} \vee \mathbf{x}) = \mathbf{x})$$

Is it enough to require that this formula is satisfied by all Σ^0_2 e-degrees? Could there be a fake $\mathcal{K}$-pair $\{\mathbf{a},\mathbf{b}\}$, such that:

$$\mathcal{G}_e \models \mathcal{K}(\mathbf{a},\mathbf{b}) \ \ \& \ \ \neg(\mathcal{D}_e \models \mathcal{K}(\mathbf{a},\mathbf{b}))?$$

Theorem

There is a first order formula $\mathcal{LK}$, such that for any Σ^0_2 sets A and B, $\{A, B\}$ is a non-trivial $\mathcal{K}$-pair if and only if $\mathcal{G}_e \models \mathcal{LK}(\mathbf{d}_e(A), \mathbf{d}_e(B))$.

Local definability of $\mathcal{K}$-pairs

$$\mathcal{K}(\mathbf{a}, \mathbf{b}) \leftrightharpoons (\forall \mathbf{x})((\mathbf{a} \vee \mathbf{x}) \wedge (\mathbf{b} \vee \mathbf{x}) = \mathbf{x})$$

Is it enough to require that this formula is satisfied by all Σ^0_2 e-degrees? Could there be a fake $\mathcal{K}$-pair $\{\mathbf{a}, \mathbf{b}\}$, such that:

$$\mathcal{G}_e \models \mathcal{K}(\mathbf{a}, \mathbf{b}) \;\; \& \;\; \neg(\mathcal{D}_e \models \mathcal{K}(\mathbf{a}, \mathbf{b}))?$$

Theorem

There is a first order formula $\mathcal{LK}$, such that for any Σ^0_2 sets A and B, $\{A, B\}$ is a non-trivial $\mathcal{K}$-pair if and only if $\mathcal{G}_e \models \mathcal{LK}(\mathbf{d}_e(A), \mathbf{d}_e(B))$.

Cupping properties

Definition

A Σ^0_2 enumeration degree $\mathbf{a}$ is called *cuppable* if there is an incomplete Σ^0_2 e-degree $\mathbf{b}$, such that $\mathbf{a} \vee \mathbf{b} = \mathbf{0}'_e$.
If furthermore $\mathbf{b}$ is low, then $\mathbf{a}$ will be called *low-cuppable*.

Theorem

If $\mathbf{u}$ *and* $\mathbf{v}$ *are* Σ^0_2 *enumeration degrees such that* $\mathbf{u} \vee \mathbf{v} = \mathbf{0}'_e$ *then* $\mathbf{u}$ *is low-cuppable or* $\mathbf{v}$ *is low-cuppable.*

Theorem

For every nonzero Δ^0_2 *degree* $\mathbf{b}$ *there is a nontrivial* $\mathcal{K}$*-pair,* $(\mathbf{c}, \mathbf{d})$*, such that*

$$\mathbf{b} \vee \mathbf{c} = \mathbf{c} \vee \mathbf{d} = \mathbf{0}'_e.$$

Cupping properties

Definition

A Σ^0_2 enumeration degree $\mathbf{a}$ is called *cuppable* if there is an incomplete Σ^0_2 e-degree $\mathbf{b}$, such that $\mathbf{a} \vee \mathbf{b} = \mathbf{0}'_e$.
If furthermore $\mathbf{b}$ is low, then $\mathbf{a}$ will be called *low-cuppable*.

Theorem

If $\mathbf{u}$ *and* $\mathbf{v}$ *are* Σ^0_2 *enumeration degrees such that* $\mathbf{u} \vee \mathbf{v} = \mathbf{0}'_e$ *then* $\mathbf{u}$ *is low-cuppable or* $\mathbf{v}$ *is low-cuppable.*

Theorem

For every nonzero Δ^0_2 *degree* $\mathbf{b}$ *there is a nontrivial* $\mathcal{K}$*-pair,* $(\mathbf{c}, \mathbf{d})$*, such that*

$$\mathbf{b} \vee \mathbf{c} = \mathbf{c} \vee \mathbf{d} = \mathbf{0}'_e.$$

Cupping properties

Definition

A Σ^0_2 enumeration degree $\mathbf{a}$ is called *cuppable* if there is an incomplete Σ^0_2 e-degree $\mathbf{b}$, such that $\mathbf{a} \vee \mathbf{b} = \mathbf{0}'_e$.
If furthermore $\mathbf{b}$ is low, then $\mathbf{a}$ will be called *low-cuppable*.

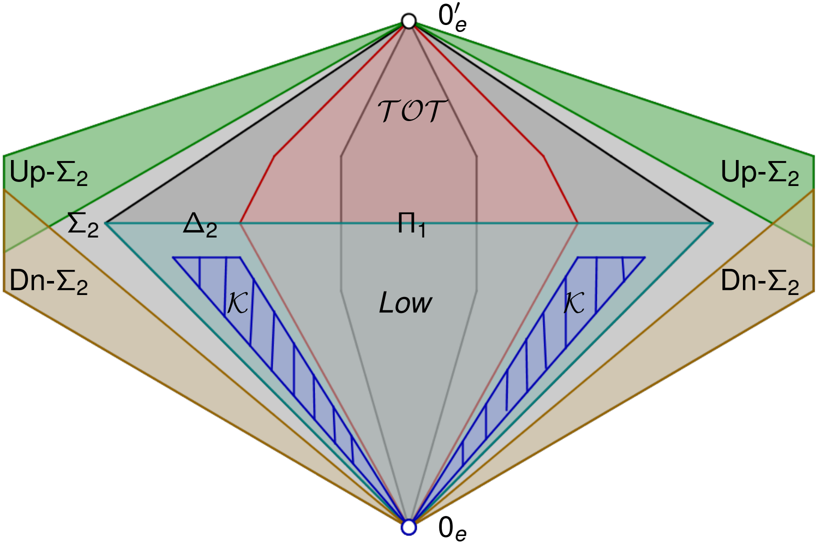
Theorem

If $\mathbf{u}$ *and* $\mathbf{v}$ *are* Σ^0_2 *enumeration degrees such that* $\mathbf{u} \vee \mathbf{v} = \mathbf{0}'_e$ *then* $\mathbf{u}$ *is low-cuppable or* $\mathbf{v}$ *is low-cuppable.*

Theorem

For every nonzero Δ^0_2 *degree* $\mathbf{b}$ *there is a nontrivial* $\mathcal{K}$*-pair,* $(\mathbf{c}, \mathbf{d})$*, such that*

$$\mathbf{b} \vee \mathbf{c} = \mathbf{c} \vee \mathbf{d} = \mathbf{0}'_e.$$



The first example of a definable class of degrees in the local structure: $\mathcal{K}$-pairs.

An easy consequence

If **a** bounds a nonzero Δ_2^0 degree then it bounds a nontrivial $\mathcal{K}$-pair.

If **a** is a downwards properly Σ_2^0 degree, then it bounds no $\mathcal{K}$-pair.

Theorem

The class of downwards properly Σ_2^0 is first order definable in $\mathcal{G}_e$ by the formula:

$$\mathcal{DP}\Sigma_2^0(\mathbf{x}) \rightleftharpoons \forall \mathbf{b}, \mathbf{c}[(\mathbf{b} \leq \mathbf{x} \,\&\, \mathbf{c} \leq \mathbf{x}) \Rightarrow \neg \mathcal{LK}(\mathbf{b}, \mathbf{c})].$$

An easy consequence

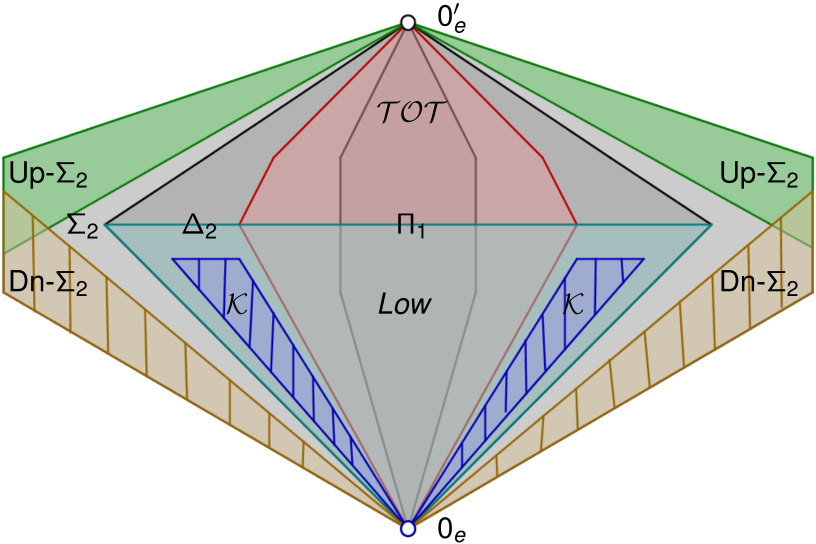
If **a** bounds a nonzero Δ_2^0 degree then it bounds a nontrivial $\mathcal{K}$-pair.

If **a** is a downwards properly Σ_2^0 degree, then it bounds no $\mathcal{K}$-pair.

Theorem

The class of downwards properly Σ_2^0 is first order definable in $\mathcal{G}_e$ by the formula:

$$\mathcal{DP}\Sigma_2^0(\mathbf{x}) \rightleftharpoons \forall \mathbf{b}, \mathbf{c}[(\mathbf{b} \leq \mathbf{x} \ \& \ \mathbf{c} \leq \mathbf{x}) \Rightarrow \neg \mathcal{LK}(\mathbf{b}, \mathbf{c})].$$



The second example of a definable class of degrees in the local structure: Downwards properly Σ^0_2 degrees.

The upwards properly Σ^0_2 degrees

Definition

$\mathbf{x}$ is upwards properly Σ^0_2 every $\mathbf{y} \in [\mathbf{x}, \mathbf{0}'_e)$ is properly Σ^0_2.

Theorem (Jockusch)

For every noncomputable set B there is a semi recursive set $A \equiv_T B$ such that both A and $\overline{A}$ are not c.e.

Corollary

Every nonzero total enumeration degree can be represented as the least upper bound of a nontrivial $\mathcal{K}$-pair.

The upwards properly Σ_2^0 degrees

Definition

$\mathbf{x}$ is upwards properly Σ_2^0 every $\mathbf{y} \in [\mathbf{x}, \mathbf{0}_e')$ is properly Σ_2^0.

Theorem (Jockusch)

For every noncomputable set B there is a semi recursive set $A \equiv_T B$ such that both A and $\overline{A}$ are not c.e.

Corollary

Every nonzero total enumeration degree can be represented as the least upper bound of a nontrivial $\mathcal{K}$-pair.

The upwards properly Σ^0_2 degrees

Definition

$\mathbf{x}$ is upwards properly Σ^0_2 every $\mathbf{y} \in [\mathbf{x}, \mathbf{0}'_e)$ is properly Σ^0_2.

Theorem (Jockusch)

For every noncomputable set B there is a semi recursive set $A \equiv_T B$ such that both A and $\overline{A}$ are not c.e.

Corollary

Every nonzero total enumeration degree can be represented as the least upper bound of a nontrivial $\mathcal{K}$-pair.

The upwards properly Σ^0_2 degrees

Theorem (Arslanov, Cooper, Kalimullin)

For every Δ^0_2 enumeration degree $\mathbf{a} < \mathbf{0}'_e$ there is a total enumeration degree $\mathbf{b}$ such that $\mathbf{a} \leq \mathbf{b} < \mathbf{0}'_e$.

So a degree $\mathbf{a}$ is upwards properly Σ^0_2 if and only if no element above it other than $\mathbf{0}'_e$ can be represented as the least upper bound of a nontrivial $\mathcal{K}$-pair.

Theorem

The class of upwards properly Σ^0_2 is first order definable in $\mathcal{G}_e$ by the formula :

$$\mathcal{U}P\Sigma^0_2(\mathbf{x}) \rightleftharpoons \forall \mathbf{c}, \mathbf{d}(\mathcal{LK}(\mathbf{c}, \mathbf{d}) \;\&\; \mathbf{x} \leq \mathbf{c} \vee \mathbf{d} \Rightarrow \mathbf{c} \vee \mathbf{d} = \mathbf{0}'_e).$$

The upwards properly Σ^0_2 degrees

Theorem (Arslanov, Cooper, Kalimullin)

For every Δ^0_2 enumeration degree $\mathbf{a} < \mathbf{0}'_e$ there is a total enumeration degree $\mathbf{b}$ such that $\mathbf{a} \leq \mathbf{b} < \mathbf{0}'_e$.

So a degree $\mathbf{a}$ is upwards properly Σ^0_2 if and only if no element above it other than $\mathbf{0}'_e$ can be represented as the least upper bound of a nontrivial $\mathcal{K}$-pair.

Theorem

The class of upwards properly Σ^0_2 is first order definable in $\mathcal{G}_e$ by the formula :

$$\mathcal{UP}\Sigma^0_2(\mathbf{x}) \rightleftharpoons \forall \mathbf{c}, \mathbf{d}(\mathcal{LK}(\mathbf{c}, \mathbf{d}) \;\&\; \mathbf{x} \leq \mathbf{c} \vee \mathbf{d} \Rightarrow \mathbf{c} \vee \mathbf{d} = \mathbf{0}'_e).$$

The upwards properly Σ_2^0 degrees

Theorem (Arslanov, Cooper, Kalimullin)

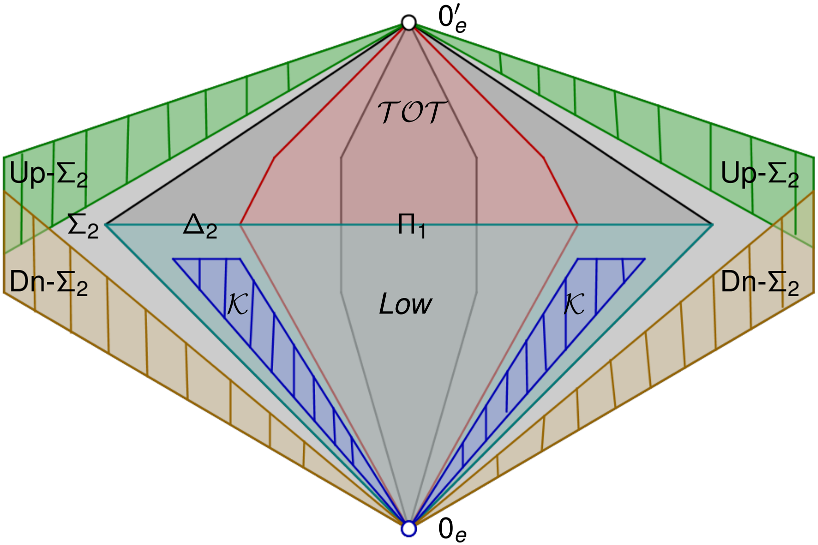
For every Δ_2^0 enumeration degree $\mathbf{a} < \mathbf{0}_e'$ there is a total enumeration degree $\mathbf{b}$ such that $\mathbf{a} \leq \mathbf{b} < \mathbf{0}_e'$.

So a degree $\mathbf{a}$ is upwards properly Σ_2^0 if and only if no element above it other than $\mathbf{0}_e'$ can be represented as the least upper bound of a nontrivial $\mathcal{K}$-pair.

Theorem

The class of upwards properly Σ_2^0 is first order definable in $\mathcal{G}_e$ by the formula :

$$\mathcal{U}P\Sigma_2^0(\mathbf{x}) \rightleftharpoons \forall \mathbf{c}, \mathbf{d}(\mathcal{LK}(\mathbf{c}, \mathbf{d}) \ \& \ \mathbf{x} \leq \mathbf{c} \vee \mathbf{d} \Rightarrow \mathbf{c} \vee \mathbf{d} = \mathbf{0}_e').$$



The third example of a definable class of degrees in the local structure: Upwards properly Σ^0_2 degrees.

Semi-recursive sets revisited

Theorem (Kalimullin)

If A and B form a nontrivial Δ^0_2 $\mathcal{K}$-pair then $A \leq_e \overline{B}$ and $B \leq_e \overline{A}$.

Consider a nontrivial $\mathcal{K}$-pair of a semi recursive set and its complement: $\{A, \overline{A}\}$.
Assume that there is a $\mathcal{K}$-pair $\{C, D\}$ such that $A <_e C$ and $\overline{A} <_e D$.
By the ideal property A forms a $\mathcal{K}$-pair with D.
Hence $D \leq_e \overline{A}$.

Semi-recursive sets revisited

Theorem (Kalimullin)

If A and B form a nontrivial Δ^0_2 $\mathcal{K}$-pair then $A \leq_e \overline{B}$ and $B \leq_e \overline{A}$.

Consider a nontrivial $\mathcal{K}$-pair of a semi recursive set and its complement: $\{A, \overline{A}\}$.

Assume that there is a $\mathcal{K}$-pair $\{C, D\}$ such that $A <_e C$ and $\overline{A} <_e D$.

By the ideal property A forms a $\mathcal{K}$-pair with D.

Hence $D \leq_e \overline{A}$.

Semi-recursive sets revisited

Theorem (Kalimullin)

If A and B form a nontrivial Δ^0_2 $\mathcal{K}$-pair then $A \leq_e \overline{B}$ and $B \leq_e \overline{A}$.

Consider a nontrivial $\mathcal{K}$-pair of a semi recursive set and its complement: $\{A, \overline{A}\}$.

Assume that there is a $\mathcal{K}$-pair $\{C, D\}$ such that $A <_e C$ and $\overline{A} <_e D$.

By the ideal property A forms a $\mathcal{K}$-pair with D.

Hence $D \leq_e \overline{A}$.

Semi-recursive sets revisited

Theorem (Kalimullin)

If A and B form a nontrivial Δ^0_2 $\mathcal{K}$-pair then $A \leq_e \overline{B}$ and $B \leq_e \overline{A}$.

Consider a nontrivial $\mathcal{K}$-pair of a semi recursive set and its complement: $\{A, \overline{A}\}$.
Assume that there is a $\mathcal{K}$-pair $\{C, D\}$ such that $A <_e C$ and $\overline{A} <_e D$.
By the ideal property A forms a $\mathcal{K}$-pair with D.
Hence $D \leq_e \overline{A}$.

Semi-recursive sets revisited

Theorem (Kalimullin)

If A and B form a nontrivial Δ^0_2 $\mathcal{K}$-pair then $A \leq_e \overline{B}$ and $B \leq_e \overline{A}$.

Consider a nontrivial $\mathcal{K}$-pair of a semi recursive set and its complement: $\{A, \overline{A}\}$.
Assume that there is a $\mathcal{K}$-pair $\{C, D\}$ such that $A <_e C$ and $\overline{A} <_e D$.
By the ideal property A forms a $\mathcal{K}$-pair with D.
Hence $D \leq_e \overline{A}$.

Semi-recursive sets revisited

Theorem (Kalimullin)

If A and B form a nontrivial Δ^0_2 $\mathcal{K}$-pair then $A \leq_e \overline{B}$ and $B \leq_e \overline{A}$.

Consider a nontrivial $\mathcal{K}$-pair of a semi recursive set and its complement: $\{A, \overline{A}\}$.
Assume that there is a $\mathcal{K}$-pair $\{C, D\}$ such that $A <_e C$ and $\overline{A} <_e D$.
By the ideal property A forms a $\mathcal{K}$-pair with D.
Hence $D \leq_e \overline{A}$.

Maximal $\mathcal{K}$-pairs

Definition

We say that $\{A, B\}$ is a maximal $\mathcal{K}$-pair if for every $\mathcal{K}$-pair $\{C, D\}$, such that $A \leq_e C$ and $B \leq_e D$, we have $A \equiv_e C$ and $B \equiv_e D$.

Corollary

Every nonzero total set is enumeration equivalent to the join of a maximal $\mathcal{K}$-pair.

Goal

Prove that the join of every maximal $\mathcal{K}$-pair is a total degree.

Maximal $\mathcal{K}$-pairs

Definition

We say that $\{A, B\}$ is a maximal $\mathcal{K}$-pair if for every $\mathcal{K}$-pair $\{C, D\}$, such that $A \leq_e C$ and $B \leq_e D$, we have $A \equiv_e C$ and $B \equiv_e D$.

Corollary

Every nonzero total set is enumeration equivalent to the join of a maximal $\mathcal{K}$-pair.

Goal

Prove that the join of every maximal $\mathcal{K}$-pair is a total degree.

Maximal $\mathcal{K}$-pairs

Definition

We say that $\{A, B\}$ is a maximal $\mathcal{K}$-pair if for every $\mathcal{K}$-pair $\{C, D\}$, such that $A \leq_e C$ and $B \leq_e D$, we have $A \equiv_e C$ and $B \equiv_e D$.

Corollary

Every nonzero total set is enumeration equivalent to the join of a maximal $\mathcal{K}$-pair.

Goal

Prove that the join of every maximal $\mathcal{K}$-pair is a total degree.

Extending $\mathcal{K}$-pairs to maximal

Theorem

For every nontrivial Δ^0_2 $\mathcal{K}$-pair $\{A, B\}$ there is a $\mathcal{K}$-pair $\{C, \overline{C}\}$, such that $A \leq_e C$ and $B \leq_e \overline{C}$.

Proof Sketch: Fix a nontrivial $\mathcal{K}$-pair $\{A, B\}$. We construct sets C and D satisfying following requirements:

(E) $A \leq_e C$, $B \leq_e D$;

(Δ^0_2) C and D are Δ^0_2;

(C) $\overline{C} = D$;

(K) $\{C, D\}$ is a $\mathcal{K}$-pair.

Extending $\mathcal{K}$-pairs to maximal

Theorem

For every nontrivial Δ^0_2 $\mathcal{K}$-pair $\{A, B\}$ there is a $\mathcal{K}$-pair $\{C, \overline{C}\}$, such that $A \leq_e C$ and $B \leq_e \overline{C}$.

Proof Sketch: Fix a nontrivial $\mathcal{K}$-pair $\{A, B\}$. We construct sets C and D satisfying following requirements:

- (E) $A \leq_e C$, $B \leq_e D$;
- (Δ^0_2) C and D are Δ^0_2;
- (C) $\overline{C} = D$;
- (K) $\{C, D\}$ is a $\mathcal{K}$-pair.

Extending $\mathcal{K}$-pairs to maximal

Lemma (Kalimullin)

A pair of non-c.e. Δ^0_2 sets A, B is a $\mathcal{K}$-pair if and only if there are Δ^0_2 approximations $\{A_i\}_{i<\omega}$ to A and $\{B_i\}_{i<\omega}$ to B, such that:

$$\forall i(A_i \subseteq A \ \vee \ B_i \subseteq B).$$

Let $\{A_i\}_{i<\omega}$ and $\{B_i\}_{i<\omega}$ be $\mathcal{K}$-pair approximations to A and B. We construct Σ^0_2 approximations to sets C and D.

Extending $\mathcal{K}$-pairs to maximal

(P1) $A_i = \{x \mid \exists j[2\langle x, j\rangle \in C_i]\}$ and $B_i = \{x \mid \exists j[2\langle x, j\rangle + 1 \in D_i]\}$.

(P2) $\forall i[A_i \not\subseteq A \Rightarrow D_i \subseteq D]$ and $\forall i[B_i \not\subseteq B \Rightarrow C_i \subseteq C]$.

(P3) $\forall i[C_i \cap D_i = \emptyset]$ and every natural number is eventually enumerated in one of the sets.

(P4) $\forall i[C_i \subseteq C \vee D_i \subseteq D]$.

(P2) implies that C and D are Δ^0_2.

Assume that there is an $x \notin D$ and infinitely many stages i such that $x \in D_i$.

By (P2) at all such stages $A_i \subseteq A$.

Hence we have a c.e. approximation to the set A.

Extending $\mathcal{K}$-pairs to maximal

(P1) $A_i = \{x \mid \exists j[2\langle x, j\rangle \in C_i]\}$ and $B_i = \{x \mid \exists j[2\langle x, j\rangle + 1 \in D_i]\}$.

(P2) $\forall i[A_i \not\subseteq A \Rightarrow D_i \subseteq D]$ and $\forall i[B_i \not\subseteq B \Rightarrow C_i \subseteq C]$.

(P3) $\forall i[C_i \cap D_i = \emptyset]$ and every natural number is eventually enumerated in one of the sets.

(P4) $\forall i[C_i \subseteq C \vee D_i \subseteq D]$.

(P2) implies that C and D are Δ^0_2.

Assume that there is an $x \notin D$ and infinitely many stages i such that $x \in D_i$.

By (P2) at all such stages $A_i \subseteq A$.

Hence we have a c.e. approximation to the set A.

Extending $\mathcal{K}$-pairs to maximal

(P1) $A_i = \{x \mid \exists j[2\langle x, j\rangle \in C_i]\}$ and $B_i = \{x \mid \exists j[2\langle x, j\rangle + 1 \in D_i]\}$.

(P2) $\forall i[A_i \not\subseteq A \Rightarrow D_i \subseteq D]$ and $\forall i[B_i \not\subseteq B \Rightarrow C_i \subseteq C]$.

(P3) $\forall i[C_i \cap D_i = \emptyset]$ and every natural number is eventually enumerated in one of the sets.

(P4) $\forall i[C_i \subseteq C \vee D_i \subseteq D]$.

(P2) implies that C and D are Δ^0_2.

Assume that there is an $x \notin D$ and infinitely many stages i such that $x \in D_i$.

By (P2) at all such stages $A_i \subseteq A$.

Hence we have a c.e. approximation to the set A.

Extending $\mathcal{K}$-pairs to maximal

(P1) $A_i = \{x \mid \exists j[2\langle x,j\rangle \in C_i]\}$ and $B_i = \{x \mid \exists j[2\langle x,j\rangle + 1 \in D_i]\}$.

(P2) $\forall i[A_i \not\subseteq A \Rightarrow D_i \subseteq D]$ and $\forall i[B_i \not\subseteq B \Rightarrow C_i \subseteq C]$.

(P3) $\forall i[C_i \cap D_i = \emptyset]$ and every natural number is eventually enumerated in one of the sets.

(P4) $\forall i[C_i \subseteq C \vee D_i \subseteq D]$.

(P2) implies that C and D are Δ^0_2.

Assume that there is an $x \notin D$ and infinitely many stages i such that $x \in D_i$.

By (P2) at all such stages $A_i \subseteq A$.

Hence we have a c.e. approximation to the set A.

Extending $\mathcal{K}$-pairs to maximal

(P1) $A_i = \{x \mid \exists j[2\langle x, j\rangle \in C_i]\}$ and $B_i = \{x \mid \exists j[2\langle x, j\rangle + 1 \in D_i]\}$.

(P2) $\forall i[A_i \not\subseteq A \Rightarrow D_i \subseteq D]$ and $\forall i[B_i \not\subseteq B \Rightarrow C_i \subseteq C]$.

(P3) $\forall i[C_i \cap D_i = \emptyset]$ and every natural number is eventually enumerated in one of the sets.

(P4) $\forall i[C_i \subseteq C \vee D_i \subseteq D]$.

(P2) implies that C and D are Δ^0_2.

Assume that there is an $x \notin D$ and infinitely many stages i such that $x \in D_i$.

By (P2) at all such stages $A_i \subseteq A$.

Hence we have a c.e. approximation to the set A.

Extending $\mathcal{K}$-pairs to maximal

(P1) $A_i = \{x \mid \exists j[2\langle x, j\rangle \in C_i]\}$ and $B_i = \{x \mid \exists j[2\langle x, j\rangle + 1 \in D_i]\}$.

(P2) $\forall i[A_i \not\subseteq A \Rightarrow D_i \subseteq D]$ and $\forall i[B_i \not\subseteq B \Rightarrow C_i \subseteq C]$.

(P3) $\forall i[C_i \cap D_i = \emptyset]$ and every natural number is eventually enumerated in one of the sets.

(P4) $\forall i[C_i \subseteq C \vee D_i \subseteq D]$.

(P2) implies that C and D are Δ^0_2.

Assume that there is an $x \notin D$ and infinitely many stages i such that $x \in D_i$.

By (P2) at all such stages $A_i \subseteq A$.

Hence we have a c.e. approximation to the set A.

Extending $\mathcal{K}$-pairs to maximal

(P1) $A_i = \{x \mid \exists j[2\langle x, j\rangle \in C_i]\}$ and $B_i = \{x \mid \exists j[2\langle x, j\rangle + 1 \in D_i]\}$.

(P2) $\forall i[A_i \not\subseteq A \Rightarrow D_i \subseteq D]$ and $\forall i[B_i \not\subseteq B \Rightarrow C_i \subseteq C]$.

(P3) $\forall i[C_i \cap D_i = \emptyset]$ and every natural number is eventually enumerated in one of the sets.

(P4) $\forall i[C_i \subseteq C \vee D_i \subseteq D]$.

(P2) implies that C and D are Δ^0_2.

Assume that there is an $x \notin D$ and infinitely many stages i such that $x \in D_i$.

By (P2) at all such stages $A_i \subseteq A$.

Hence we have a c.e. approximation to the set A.

Extending $\mathcal{K}$-pairs to maximal

(P1) $A_i = \{x \mid \exists j[2\langle x, j\rangle \in C_i]\}$ and $B_i = \{x \mid \exists j[2\langle x, j\rangle + 1 \in D_i]\}$.

(P2) $\forall i[A_i \not\subseteq A \Rightarrow D_i \subseteq D]$ and $\forall i[B_i \not\subseteq B \Rightarrow C_i \subseteq C]$.

(P3) $\forall i[C_i \cap D_i = \emptyset]$ and every natural number is eventually enumerated in one of the sets.

(P4) $\forall i[C_i \subseteq C \vee D_i \subseteq D]$.

(P2) implies that C and D are Δ^0_2.

Assume that there is an $x \notin D$ and infinitely many stages i such that $x \in D_i$.

By (P2) at all such stages $A_i \subseteq A$.

Hence we have a c.e. approximation to the set A.

Extending $\mathcal{K}$-pairs to maximal

(P1) $A_i = \{x \mid \exists j[2\langle x, j\rangle \in C_i]\}$ and $B_i = \{x \mid \exists j[2\langle x, j\rangle + 1 \in D_i]\}$.

(P2) $\forall i[A_i \not\subseteq A \Rightarrow D_i \subseteq D]$ and $\forall i[B_i \not\subseteq B \Rightarrow C_i \subseteq C]$.

(P3) $\forall i[C_i \cap D_i = \emptyset]$ and every natural number is eventually enumerated in one of the sets.

(P4) $\forall i[C_i \subseteq C \vee D_i \subseteq D]$.

(Δ^0_2)+(P1) imply (E):

$$A = \{x \mid \exists i[2\langle x, i\rangle \in C]\} \text{ and } B = \{x \mid \exists i[2\langle x, i\rangle + 1 \in D]\}.$$

(Δ^0_2)+(P3) imply (C): $\overline{C} = D$

(P4) implies (K): $\{C, D\}$ is a $\mathcal{K}$-pair.

Extending $\mathcal{K}$-pairs to maximal

(P1) $A_i = \{x \mid \exists j[2\langle x, j\rangle \in C_i]\}$ and $B_i = \{x \mid \exists j[2\langle x, j\rangle + 1 \in D_i]\}$.

(P2) $\forall i[A_i \not\subseteq A \Rightarrow D_i \subseteq D]$ and $\forall i[B_i \not\subseteq B \Rightarrow C_i \subseteq C]$.

(P3) $\forall i[C_i \cap D_i = \emptyset]$ and every natural number is eventually enumerated in one of the sets.

(P4) $\forall i[C_i \subseteq C \vee D_i \subseteq D]$.

(Δ^0_2)+(P1) imply (E):

$$A = \{x \mid \exists i[2\langle x, i\rangle \in C]\} \text{ and } B = \{x \mid \exists i[2\langle x, i\rangle + 1 \in D]\}.$$

(Δ^0_2)+(P3) imply (C): $\overline{C} = D$

(P4) implies (K): $\{C, D\}$ is a $\mathcal{K}$-pair.

Extending $\mathcal{K}$-pairs to maximal

(P1) $A_i = \{x \mid \exists j[2\langle x, j\rangle \in C_i]\}$ and $B_i = \{x \mid \exists j[2\langle x, j\rangle + 1 \in D_i]\}$.

(P2) $\forall i[A_i \not\subseteq A \Rightarrow D_i \subseteq D]$ and $\forall i[B_i \not\subseteq B \Rightarrow C_i \subseteq C]$.

(P3) $\forall i[C_i \cap D_i = \emptyset]$ and every natural number is eventually enumerated in one of the sets.

(P4) $\forall i[C_i \subseteq C \vee D_i \subseteq D]$.

(Δ^0_2)+(P1) imply (E):

$$A = \{x \mid \exists i[2\langle x, i\rangle \in C]\} \text{ and } B = \{x \mid \exists i[2\langle x, i\rangle + 1 \in D]\}.$$

(Δ^0_2)+(P3) imply (C): $\overline{C} = D$

(P4) implies (K): $\{C, D\}$ is a $\mathcal{K}$-pair.

Extending $\mathcal{K}$-pairs to maximal

(P1) $A_i = \{x \mid \exists j[2\langle x, j\rangle \in C_i]\}$ and $B_i = \{x \mid \exists j[2\langle x, j\rangle + 1 \in D_i]\}$.

(P2) $\forall i[A_i \not\subseteq A \Rightarrow D_i \subseteq D]$ and $\forall i[B_i \not\subseteq B \Rightarrow C_i \subseteq C]$.

(P3) $\forall i[C_i \cap D_i = \emptyset]$ and every natural number is eventually enumerated in one of the sets.

(P4) $\forall i[C_i \subseteq C \vee D_i \subseteq D]$.

Note that (P1) + (P4) ensure (P2).

If $A_i \not\subseteq A$ then by (P1) $C_i \not\subseteq C$.

By (P4) we have $D_i \subseteq D$.

Extending $\mathcal{K}$-pairs to maximal

(P1) $A_i = \{x \mid \exists j[2\langle x, j\rangle \in C_i]\}$ and $B_i = \{x \mid \exists j[2\langle x, j\rangle + 1 \in D_i]\}$.

(P2) $\forall i[A_i \not\subseteq A \Rightarrow D_i \subseteq D]$ and $\forall i[B_i \not\subseteq B \Rightarrow C_i \subseteq C]$.

(P3) $\forall i[C_i \cap D_i = \emptyset]$ and every natural number is eventually enumerated in one of the sets.

(P4) $\forall i[C_i \subseteq C \vee D_i \subseteq D]$.

Note that (P1) + (P4) ensure (P2).

If $A_i \not\subseteq A$ then by (P1) $C_i \not\subseteq C$.

By (P4) we have $D_i \subseteq D$.

Extending $\mathcal{K}$-pairs to maximal

(P1) $A_i = \{x \mid \exists j[2\langle x, j\rangle \in C_i]\}$ and $B_i = \{x \mid \exists j[2\langle x, j\rangle + 1 \in D_i]\}$.

(P2) $\forall i[A_i \not\subseteq A \Rightarrow D_i \subseteq D]$ and $\forall i[B_i \not\subseteq B \Rightarrow C_i \subseteq C]$.

(P3) $\forall i[C_i \cap D_i = \emptyset]$ and every natural number is eventually enumerated in one of the sets.

(P4) $\forall i[C_i \subseteq C \vee D_i \subseteq D]$.

Note that (P1) + (P4) ensure (P2).

If $A_i \not\subseteq A$ then by (P1) $C_i \not\subseteq C$.

By (P4) we have $D_i \subseteq D$.

Main Property

$$(P4): \forall i[C_i \subseteq C \vee D_i \subseteq D]$$

Suppose that $x \notin C$, but $x \in C_i$.

Then $C_i \not\subseteq C$, and we must ensure that for every $y \in D_i$ ultimately $y \in D$.

If $x \in C_i$ and $y \in D_i$, we say that x is **connected** to y at stage i.

(MP) If x is connected to y at stage i then for all $j \geq i$

$$x \in D_j \implies y \in D_j \quad \text{and} \quad y \in C_j \implies x \in C_j.$$

Main Property

$$(P4) : \forall i[C_i \subseteq C \vee D_i \subseteq D]$$

Suppose that $x \notin C$, but $x \in C_i$.

Then $C_i \not\subseteq C$, and we must ensure that for every $y \in D_i$ ultimately $y \in D$.

If $x \in C_i$ and $y \in D_i$, we say that x is **connected** to y at stage i.

(MP) If x is connected to y at stage i then for all $j \geq i$

$$x \in D_j \implies y \in D_j \quad \text{and} \quad y \in C_j \implies x \in C_j.$$

Main Property

$$(P4) : \forall i [C_i \subseteq C \vee D_i \subseteq D]$$

Suppose that $x \notin C$, but $x \in C_i$.

Then $C_i \not\subseteq C$, and we must ensure that for every $y \in D_i$ ultimately $y \in D$.

If $x \in C_i$ and $y \in D_i$, we say that x is **connected** to y at stage i.

(MP) If x is connected to y at stage i then for all $j \geq i$

$$x \in D_j \implies y \in D_j \quad \text{and} \quad y \in C_j \implies x \in C_j.$$

Main Property

$$(P4): \forall i[C_i \subseteq C \vee D_i \subseteq D]$$

Suppose that $x \notin C$, but $x \in C_i$.

Then $C_i \nsubseteq C$, and we must ensure that for every $y \in D_i$ ultimately $y \in D$.

If $x \in C_i$ and $y \in D_i$, we say that x is **connected** to y at stage i.

(MP) If x is connected to y at stage i then for all $j \geq i$

$$x \in D_j \Longrightarrow y \in D_j \quad \text{and} \quad y \in C_j \Longrightarrow x \in C_j.$$

Main Property

$$(P4) : \forall i[C_i \subseteq C \vee D_i \subseteq D]$$

Suppose that $x \notin C$, but $x \in C_i$.

Then $C_i \nsubseteq C$, and we must ensure that for every $y \in D_i$ ultimately $y \in D$.

If $x \in C_i$ and $y \in D_i$, we say that x is **connected** to y at stage i.

(MP) If x is connected to y at stage i then for all $j \geq i$

$$x \in D_j \implies y \in D_j \quad \text{and} \quad y \in C_j \implies x \in C_j.$$

The construction

(P1) $A_i = \{x \mid \exists j[2\langle x, j\rangle \in C_i]\}$ and $B_i = \{x \mid \exists j[2\langle x, j\rangle + 1 \in D_i]\}$.

(P3) $\forall i[C_i \cap D_i = \emptyset]$ and every natural number is eventually enumerated in one of the sets.

(MP) If x is connected to y at stage i then for all $j \geq i$

$$x \in D_j \implies y \in D_j \text{ and } y \in C_j \implies x \in C_j.$$

Local definability of the total degrees

Denote by $\mathcal{MK}(\mathbf{x}, \mathbf{y})$ the first order formula that defines in $\mathcal{G}_e$ the set of degrees of maximal $\mathcal{K}$-pairs.

Corollary

The class of total degrees is first order definable in $\mathcal{G}_e$ by the formula:

$$\mathcal{TOT}(\mathbf{x}) \rightleftharpoons \mathbf{x} = \mathbf{0}_e \ \vee \ \exists \mathbf{c} \exists \mathbf{d} [\mathcal{MK}(\mathbf{c}, \mathbf{d}) \ \& \ \mathbf{x} = \mathbf{c} \vee \mathbf{d}.]$$

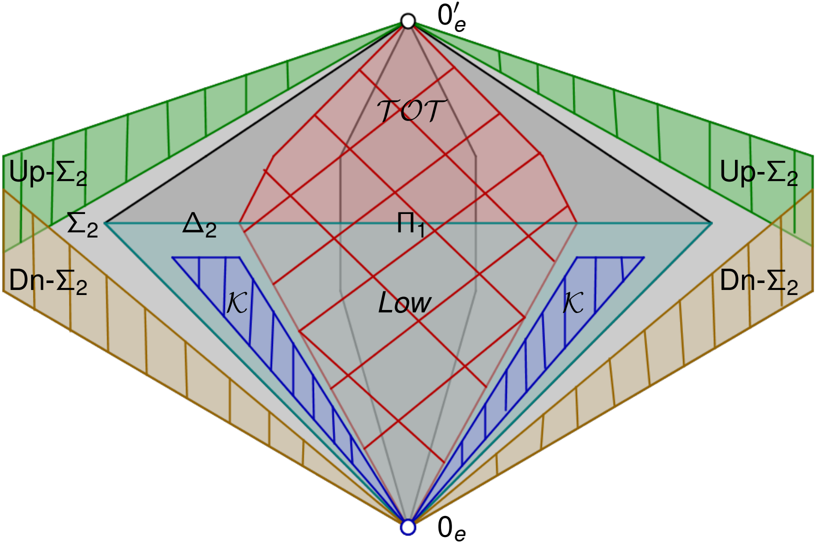
Local definability of the total degrees

Denote by $\mathcal{MK}(\mathbf{x}, \mathbf{y})$ the first order formula that defines in $\mathcal{G}_e$ the set of degrees of maximal $\mathcal{K}$-pairs.

Corollary

The class of total degrees is first order definable in $\mathcal{G}_e$ by the formula:

$$\mathcal{TOT}(\mathbf{x}) \rightleftharpoons \mathbf{x} = \mathbf{0}_e \ \vee \ \exists \mathbf{c} \exists \mathbf{d} [\mathcal{MK}(\mathbf{c}, \mathbf{d}) \ \& \ \mathbf{x} = \mathbf{c} \vee \mathbf{d}.]$$



The fourth example of a definable class of degrees in the local structure: The total degrees.

One final consequence

Theorem (Giorgi, Sorbi, Yang)

Every non-low total degree bounds a downwards properly Σ^0_2 enumeration degree.

Corollary

The class of low total e-degrees is first order definable in $\mathcal{G}_e$ by the formula:

$$\mathcal{TL}(\mathbf{x}) \rightleftharpoons \mathcal{TOT}(\mathbf{x}) \ \&\ \forall \mathbf{c} \leq \mathbf{x}[\neg \mathcal{DP}\Sigma^0_2(\mathbf{c})]$$

One final consequence

Theorem (Giorgi, Sorbi, Yang)

Every non-low total degree bounds a downwards properly Σ^0_2 enumeration degree.

Corollary

The class of low total e-degrees is first order definable in $\mathcal{G}_e$ by the formula:

$$\mathcal{TL}(\mathbf{x}) \rightleftharpoons \mathcal{TOT}(\mathbf{x}) \ \&\ \forall \mathbf{c} \leq \mathbf{x}[\neg \mathcal{D}P\Sigma^0_2(\mathbf{c})]$$

One final consequence

Theorem (Soskov)

For every enumeration degree $\mathbf{x}$ there is a total enumeration degree $\mathbf{y}$, such that $\mathbf{x} < \mathbf{y}$ and $\mathbf{x}' = \mathbf{y}'$.

Thus a Σ^0_2 enumeration degree is low if and only if there is a low total Σ^0_2 enumeration degree above it.

Theorem

The class of low e-degrees is first order definable in $\mathcal{G}_e$ by the formula:

$$\mathcal{LOW}(\mathbf{x}) \rightleftharpoons \exists \mathbf{y}[\mathbf{x} \leq \mathbf{y} \ \& \ \mathcal{TL}(\mathbf{y})]$$

One final consequence

Theorem (Soskov)

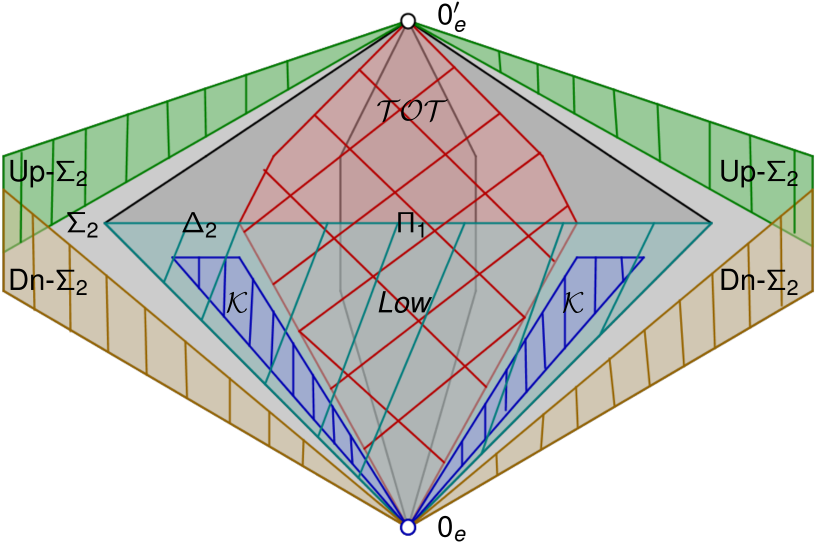
For every enumeration degree $\mathbf{x}$ *there is a total enumeration degree* $\mathbf{y}$*, such that* $\mathbf{x} < \mathbf{y}$ *and* $\mathbf{x}' = \mathbf{y}'$*.*

Thus a Σ^0_2 enumeration degree is low if and only if there is a low total Σ^0_2 enumeration degree above it.

Theorem

The class of low e-degrees is first order definable in $\mathcal{G}_e$ *by the formula:*

$$\mathcal{LOW}(\mathbf{x}) \rightleftharpoons \exists \mathbf{y}[\mathbf{x} \leq \mathbf{y} \ \& \ \mathcal{TL}(\mathbf{y})]$$



The fifth example of a definable class of degrees in the local structure: The low degrees.

The end

Thank you!