Local definability of $\mathcal{K}$-pairs in the enumeration degrees

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joint work with H. Ganchev

Preliminaries: The enumeration degrees

Definition

- $A \leq_e B$ iff there is a c.e. set W, such that
  $A = W(B) = \{x \mid \exists u(\langle x, u \rangle \in W \wedge D_u \subseteq B)\}$.

- $d_e(A) = \{B \mid A \leq_e B \ \& \ B \leq_e A\}$

- $d_e(A) \leq d_e(B)$ iff $A \leq_e B$.

- $\mathbf{0}_e = d_e(\emptyset) = \{W \mid W \text{ is c.e. }\}$.

- $d_e(A) \vee d_e(B) = d_e(A \oplus B)$.

- $d_e(A)' = d_e(A')$, where $A' = L_A \oplus \overline{L_A}$ and $L_A = \{x \mid x \in W_x(A)\}$.

- $\mathcal{D}_e = \langle D_e, \leq, \vee, ', \mathbf{0}_e \rangle$ is an upper semi-lattice with jump operation and least element.

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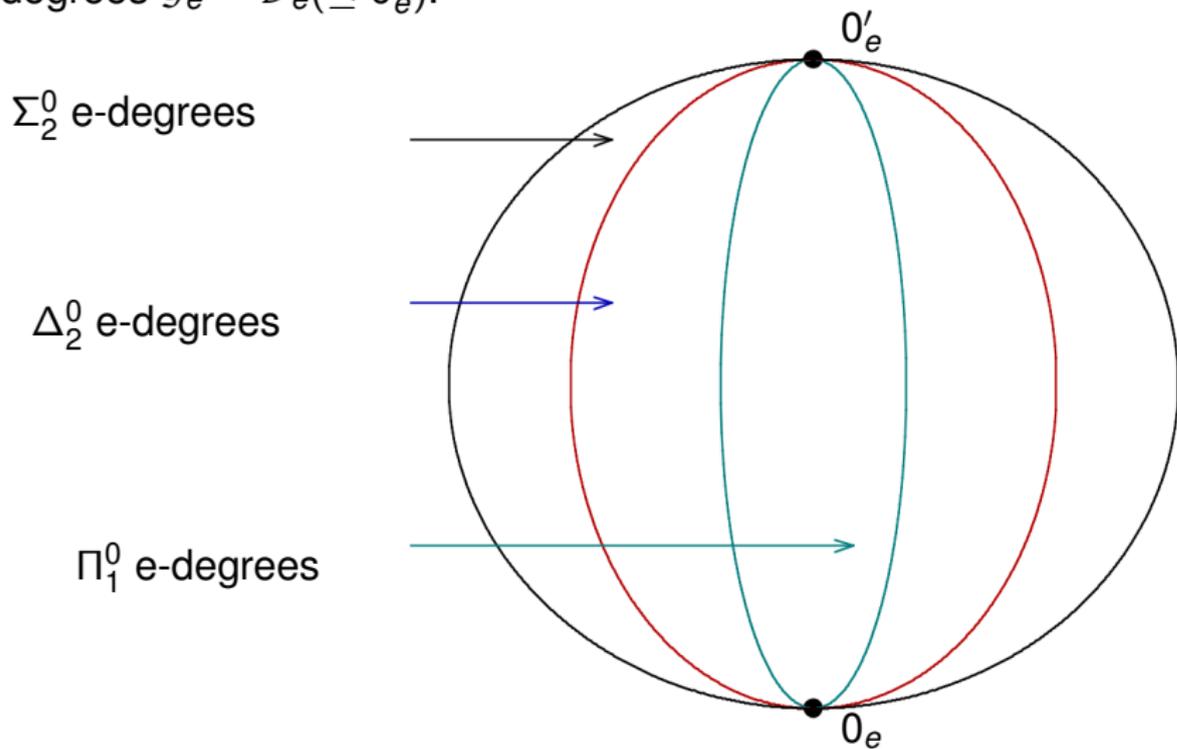
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Preliminaries: The local structure

The jump operation gives rise to the local structure of the enumeration degrees $\mathcal{G}_e = \mathcal{D}_e(\leq 0'_e)$.

Σ^0_2 e-degrees

Δ^0_2 e-degrees

Π^0_1 e-degrees

Preliminaries: The local structure

The local structure $\mathcal{G}_e$ can be partitioned into classes with respect to the jump hierarchy:

Definition

A degree $\mathbf{a} \in \mathcal{G}_e$ is low if $\mathbf{a}' = \mathbf{0}'_e$.

Or in terms of its relation ship to the Turing degrees.

Proposition

The embedding $\iota : \mathcal{D}_T \to \mathcal{D}_e$, defined by $\iota(d_T(A)) = d_e(A \oplus \overline{A})$, preserves the order, the least upper bound and the jump operation:

The sub structure of the total e-degrees is defined as $\mathcal{TOT} = \iota(D_T)$.

- Every low e-degree is Δ_2^0.
- Every total e-degree in $\mathcal{G}_e$ is Δ_2^0.
- There are properly Σ_2^0 degrees.

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$\mathcal{K}$-pairs

Iskander Kalimullin: Definability of the jump operator in the
enumeration degrees
Journal of Mathematical Logic (2003)

Definition

Let A and B be a pair sets of natural numbers. The pair (A, B) is a
$\mathcal{K}$-pair (e-ideal) if there exists a c.e. set W, such that $A \times B \subseteq W$ and
$\overline{A} \times \overline{B} \subseteq \overline{W}$.

$\mathcal{K}$-pairs: A trivial example

Example

Let V be a c.e set. Then (V, A) is a $\mathcal{K}$-pair for any set of natural numbers A.

Let $W = V \times \mathbb{N}$. Then $V \times A \subseteq W$ and $\overline{V} \times \overline{A} \subseteq \overline{W}$.

We will only be interested in non-trivial $\mathcal{K}$-pairs.

$\mathcal{K}$-pairs: A more interesting example

Definition (Jockusch)

A set of natural numbers A is semi-recursive if there is a computable function s_A such that for every pair of natural numbers (x, y):

1. $s_A(x, y) \in \{x, y\}$.
2. If $x \in A$ or $y \in A$ then $s_A(x, y) \in A$.

Example

Let A be a semi-recursive set. Then $(A, \overline{A})$ is a $\mathcal{K}$-pair.

Theorem (Jockusch)

For every noncomputable set B there is a semi recursive set $A \equiv_T B$ such that both A and $\overline{A}$ are not c.e.

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An order theoretic characterization of $\mathcal{K}$-pairs

Theorem (Kalimullin)

(A, B) *is a* $\mathcal{K}$*-pair if and only if the degrees* $\mathbf{a} = d_e(A)$ *and* $\mathbf{b} = d_e(B)$ *have the following property:*

$$\mathcal{K}(\mathbf{a}, \mathbf{b}) \rightleftharpoons (\forall \mathbf{x})((\mathbf{a} \vee \mathbf{x}) \wedge (\mathbf{b} \vee \mathbf{x}) = \mathbf{x})$$

Question

Is the property "$\mathbf{a}$ *and* $\mathbf{b}$ *form a* $\mathcal{K}$*-pair" first order definable in the local structure?*

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The problem

Theorem (Kalimullin)

If (A, B) is not a $\mathcal{K}$-pair then there is a witness C computable from $A \oplus B \oplus K$ such that:
$$(d_e(A) \vee d_e(C)) \wedge (d_e(B) \vee d_e(C)) \neq d_e(C)$$

- If **a** and **b** are Δ_2^0 then C is also Δ_2^0 and $\mathcal{K}(\mathbf{a}, \mathbf{b})$ ensures "**a** and **b** are a true $\mathcal{K}$-pair".
- Every $\mathcal{K}$-pair in $\mathcal{G}_e$ consists of low (hence Δ_2^0) e-degrees.
- If **a** and **b** are properly Σ_2^0 then C is at best Δ_3^0. So it is possible that there is a fake $\mathcal{K}$-pair **a** and **b** such that

$$\mathcal{G}_e \models \mathcal{K}(\mathbf{a}, \mathbf{b}), \text{ but } \mathcal{D}_e \models \neg\mathcal{K}(\mathbf{a}, \mathbf{b})$$

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Cupping properties

Definition

A Σ_2^0 enumeration degree **a** is called *cuppable* if there is an incomplete Σ_2^0 e-degree **b**, such that $\mathbf{a} \vee \mathbf{b} = \mathbf{0}'_e$.
If furthermore **b** is low, then **a** will be called *low-cuppable*.

Proposition (The $\mathcal{K}$-cupping property)

Let **a** *and* **b** *are* Σ_2^0 *degrees such that* $\mathcal{G}_e \models \mathcal{K}(\mathbf{a}, \mathbf{b})$.
If **c** *is a* Σ_2^0 *degree, such that* $\mathbf{c} \vee \mathbf{b} = \mathbf{0}'_e$ *then* $\mathbf{a} \leq \mathbf{c}$.

Proof:

$$\mathbf{c} = (\mathbf{a} \vee \mathbf{c}) \wedge (\mathbf{b} \vee \mathbf{c}) = (\mathbf{a} \vee \mathbf{c}) \wedge \mathbf{0}'_e = \mathbf{a} \vee \mathbf{c}$$

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Theorem (S,Wu)

Every nonzero Δ_2^0 enumeration degree **a** *is low-cuppable, i.e. there is a low* **b** *such that* $\mathbf{a} \vee \mathbf{b} = \mathbf{0}'_e$.

Theorem (Cooper, Sorbi, Yi)

There are non-cuppable nonzero Σ_2^0 enumeration degrees.

Question

Are all cuppable degrees also low-cuppable?

Cupping $\mathbf{0}'_e$-splittings

Theorem

If $\mathbf{u}$ and $\mathbf{v}$ are Σ^0_2 enumeration degrees such that $\mathbf{u} \vee \mathbf{v} = \mathbf{0}'_e$ then $\mathbf{u}$ is low-cuppable or $\mathbf{v}$ is low-cuppable.

Proof:
Uses a construction very similar to the construction of a non-splitting enumeration degree.

A non-splitting theorem

Theorem (S)

There is a degree $\mathbf{a} < \mathbf{0}'_e$ such that no pair of incomplete Σ^0_2 degrees $\mathbf{u}$ and $\mathbf{v}$ above $\mathbf{a}$ splits $\mathbf{0}'_e$.

We build a Σ^0_2 set A and an auxiliary Π^0_1 set E so that:

$$\mathcal{N}_\Phi : \Phi(A) \neq E$$

$$\mathcal{P}_{\Theta,U,V} : \Theta(U \oplus V) = E \Rightarrow \exists \Gamma, \Lambda(\Gamma(U \oplus A) = \overline{K} \vee \Lambda(V \oplus A) = \overline{K})$$

Corollary

There exists an incomplete Σ^0_2 e-degree $\mathbf{a}$, such that for every pair of Σ^0_2 enumeration degrees $\mathbf{u}$ and $\mathbf{v}$ with $\mathbf{u} \vee \mathbf{v} = \mathbf{0}'_e$ either $\mathbf{u} \vee \mathbf{a} = \mathbf{0}'_e$ or $\mathbf{v} \vee \mathbf{a} = \mathbf{0}'_e$

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$$\mathcal{P}_{\Theta, U, V} : \Theta(U \oplus V) = E \Rightarrow \exists \Gamma, \Lambda(\Gamma(U \oplus A) = \overline{K} \vee \Lambda(V \oplus A) = \overline{K})$$

Corollary

There exists an incomplete Σ^0_2 e-degree $\mathbf{a}$, such that for every pair of Σ^0_2 enumeration degrees $\mathbf{u}$ and $\mathbf{v}$ with $\mathbf{u} \vee \mathbf{v} = \mathbf{0}'_e$ either $\mathbf{u} \vee \mathbf{a} = \mathbf{0}'_e$ or $\mathbf{v} \vee \mathbf{a} = \mathbf{0}'_e$

Cupping $\mathbf{0}'_e$-splittings

Theorem

If $\mathbf{u}$ and $\mathbf{v}$ are Σ^0_2 enumeration degrees such that $\mathbf{u} \vee \mathbf{v} = \mathbf{0}'_e$ then $\mathbf{u}$ is low-cuppable or $\mathbf{v}$ is low-cuppable.

Proof:
Fix U, V such that $U \oplus V \equiv_e \overline{K}$.

We construct an auxiliary Π^0_1 set E and find an e-operator Θ such that $\Theta(U \oplus V) = E$.

First we try to construct a 1-generic Δ^0_2 set A such that $A \oplus U \equiv_e \overline{K}$.

If this plan fails we have acquired sufficient information to construct a 1-generic Δ^0_2 set B such that $B \oplus V \equiv_e \overline{K}$.

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Defining true $\mathcal{K}$-pairs: Step 1

Corollary

If **a**, **b** *are nonzero* Σ_2^0 *degrees such that* $\mathcal{G}_e \models \mathcal{K}(\mathbf{a}, \mathbf{b})$ *and* $\mathbf{a} \vee \mathbf{b} = \mathbf{0}_e'$ *then* $(\mathbf{a}, \mathbf{b})$ *is a true* $\mathcal{K}$-*pair.*

Proof:

By the previous theorem **a** is low-cuppable or **b** is low-cuppable.

b is low-cuppable $\Rightarrow$ **a** is low $\Rightarrow$ **a** is $\Delta_2^0 \Rightarrow$
a is low cuppable $\Rightarrow$ **b** is low $\Rightarrow$ **b** is $\Delta_2^0 \Rightarrow$ **b** is low cuppable.

In either case both **a** and **b** are Δ_2^0 and hence $\mathcal{K}(\mathbf{a}, \mathbf{b})$ ensures that they form a true $\mathcal{K}$-pair.

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Theorem

The formula

$$\mathcal{L}(\mathbf{a}) \Leftrightarrow \mathbf{a} > \mathbf{0}_e \,\&\, (\exists \mathbf{b} > \mathbf{0}_e)(\mathcal{K}(\mathbf{a}, \mathbf{b}) \& (\mathbf{a} \vee \mathbf{b} = \mathbf{0}'_e))$$

defines in $\mathcal{G}_e$ a nonempty set of true halves of nontrivial $\mathcal{K}$-pairs.

Defining true $\mathcal{K}$-pairs: Step 2

Denote by $\mathcal{L}$ the definable set of all degrees **a**, such that

$$\mathcal{G}_e \models \mathcal{L}(\mathbf{a}).$$

Definition

x is downwards properly Σ_2^0 every $\mathbf{y} \in (\mathbf{0}_e, \mathbf{x}]$ is properly Σ_2^0.

Example

If **x** is not low cuppable then it is downwards properly Σ_2^0.

If $(\mathbf{a}, \mathbf{b})$ is a fake $\mathcal{K}$-pair then i.e.:

$$\mathcal{G}_e \models \mathcal{K}(\mathbf{a}, \mathbf{b}), \text{ but } \mathcal{D}_e \models \neg\mathcal{K}(\mathbf{a}, \mathbf{b})$$

then **a** and **b** are non-low cuppable, hence downwards properly Σ_2^0, hence incomparable with every member of $\mathcal{L}$.

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For every nonzero Δ_2^0 degree $\mathbf{b}$ there is a nontrivial $\mathcal{K}$-pair, $(\mathbf{c}, \mathbf{d})$, such that

$$\mathbf{b} \vee \mathbf{c} = \mathbf{c} \vee \mathbf{d} = \mathbf{0}'_e.$$

Hence if $(\mathbf{a}, \mathbf{b})$ is a true $\mathcal{K}$-pair of Σ_2^0 e-degrees (hence low and Δ_2^0) we apply this theorem to get a $\mathcal{K}$-pair $(\mathbf{c}, \mathbf{d})$ such that:

- $\mathbf{b} \vee \mathbf{c} = \mathbf{0}'_e$ and hence $\mathbf{a} \leq \mathbf{c}$.

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Let $\mathcal{L}\mathcal{K}(\mathbf{a}, \mathbf{b}) \Leftrightarrow \mathcal{K}(\mathbf{a}, \mathbf{b}) \,\&\, \mathbf{a} > \mathbf{0}_e \,\&\, \mathbf{b} > \mathbf{0}_e \,\&\, \exists \mathbf{c}(\mathbf{c} \geq \mathbf{a} \,\&\, \mathcal{L}(\mathbf{c}))$

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A pair of Σ_2^0 enumeration degrees $\mathbf{a}$, $\mathbf{b}$ forms a nontrivial $\mathcal{K}$-pair if and only if:

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Application I: The complexity of $Th(\mathcal{G}_e)$

Theorem

The first order theory of $\mathcal{G}_e$ is computably isomorphic to first order arithmetic.

Given a sentence in the langauge of true arithmetic φ we want to be able to computably translate it into a sentence φ_e in the langauge of the $\mathcal{G}_e$ so that:

$$\langle \mathbb{N}, +, * \rangle \vDash \varphi \text{ iff } \mathcal{G}_e \vDash \varphi_e$$

  I Represent $\langle \mathbb{N}, +, * \rangle$ as a partial order (PO).

 II Embed this partial order in $\mathcal{G}_e$ and code it with a finite number of parameters.

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A special type of partial order

We can represent an SMA $\langle \mathbb{N}, +, * \rangle$ as follows:
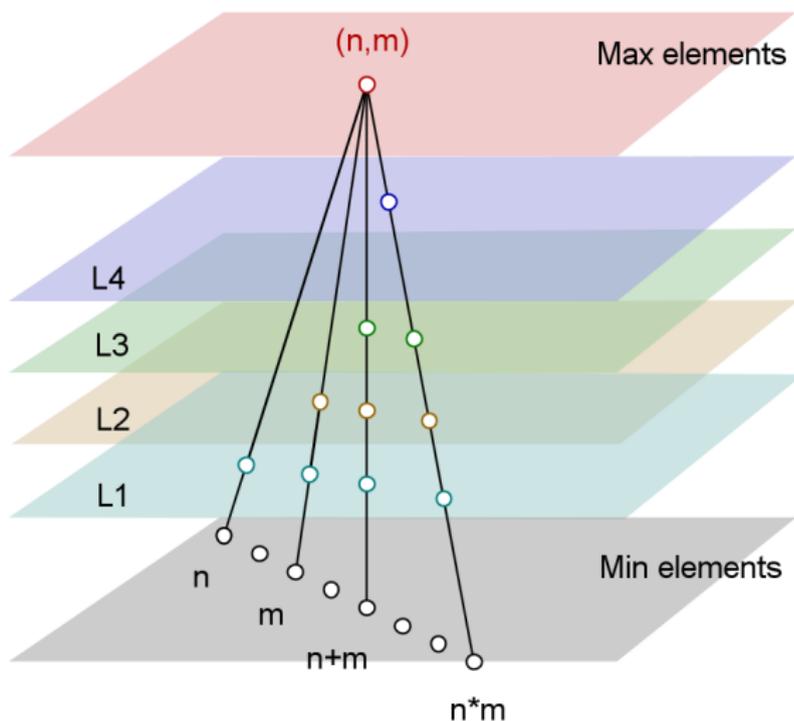
First tool: Coding antichains

$$\varphi_{SW}(\boldsymbol{x}, \boldsymbol{a}, \boldsymbol{p}, \boldsymbol{q}) \iff \boldsymbol{x} \leq \boldsymbol{a} \text{ is a minimal solution to}$$

$$\boldsymbol{x} \neq (\boldsymbol{x} \vee \boldsymbol{p}) \wedge (\boldsymbol{x} \vee \boldsymbol{q}).$$

Theorem (Slaman, Woodin)

If $\{\mathbf{x}_i \mid i \in \mathbb{N}\}$ is an antichain, uniformly enumeration reducible to a low $\boldsymbol{a}$ then there are Σ_2^0 e-degrees $\mathbf{p}$ and $\mathbf{q}$, such that for arbitrary Σ_2^0 degree $\mathbf{x}$

$$\mathcal{G}_e \models \varphi_{SW}(\mathbf{x}, \mathbf{a}, \mathbf{p}, \mathbf{q}) \iff \exists i[\mathbf{x}_i \in \mathbf{x}].$$

Goal: Embed the PO so that each level is *well presented*.

Second tool: $\mathcal{K}$-systems

Definition

We shall say that a system of nonzero degrees $\{\mathbf{a}_i \mid i \in I\}$ ($|I| \geq 2$) is a $\mathcal{K}$-system, if $\mathcal{K}(\mathbf{a}_i, \mathbf{a}_j)$ for each $i, j \in I$, such that $i \neq j$.

- Every $\mathcal{K}$-pair is a minimal pair, hence every $\mathcal{K}$-system is an antichain.

- The degrees that form a $\mathcal{K}$-pair with a fixed degree form an ideal. Hence if $\{\mathbf{a}_i \mid i \in I\}$ is a $\mathcal{K}$-system and $i_1 \neq i_2 \in I$ then $\{\mathbf{a}_{i_1} \vee \mathbf{a}_{i_2}\} \cup \{\mathbf{a}_i \mid i \in I, i \neq i_1, i_2\}$ is a $\mathcal{K}$-system .

Theorem

Every non-zero Δ_2^0 e-degree bounds a well-presented $\mathcal{K}$-system.

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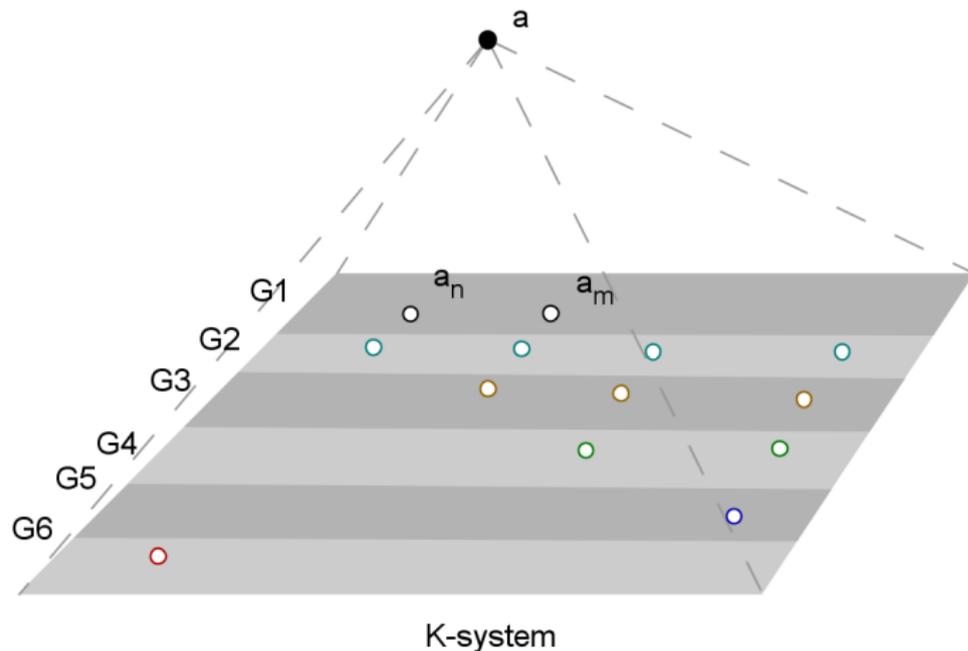
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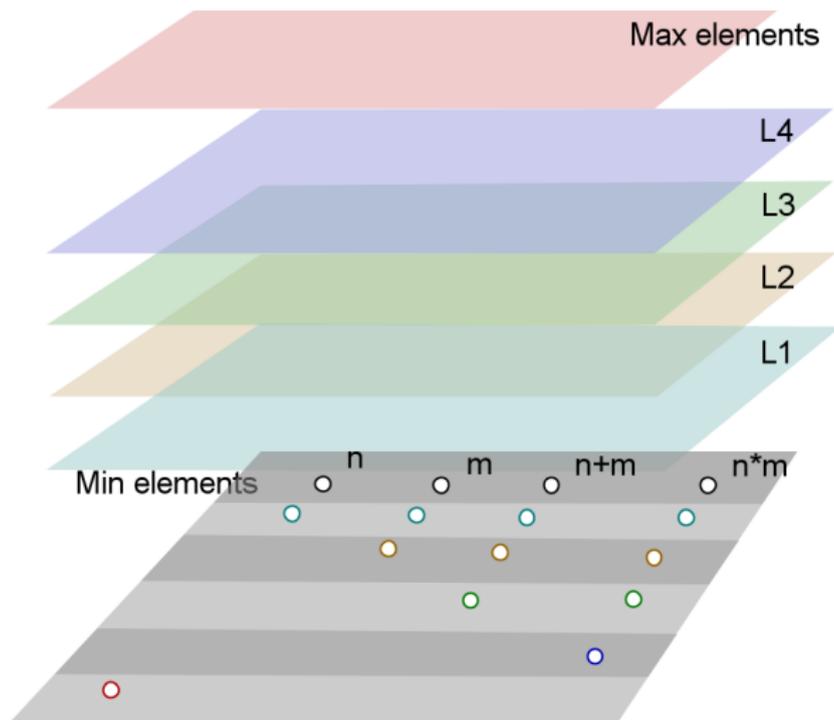*Every non-zero Δ_2^0 e-degree bounds a well-presented $\mathcal{K}$-system.*

Coding an SMA below any half of a $\mathcal{K}$-pair

Fix a half of a nontrivial $\mathcal{K}$-pair **a** and a well presented $\mathcal{K}$-system below it.

We computably divide the system $\{\mathbf{a}_i\}_{i<\omega}$ into six infinite groups.
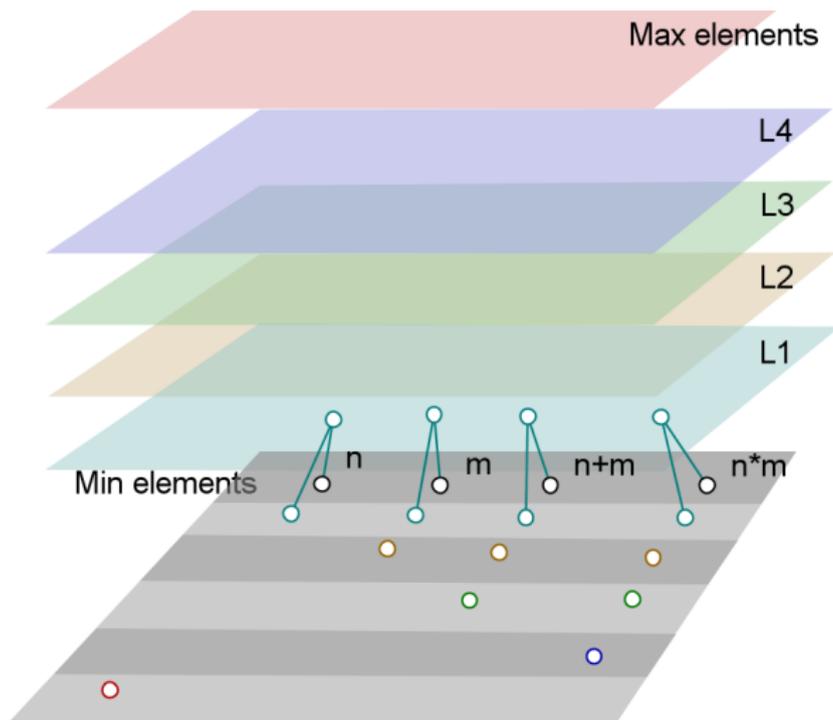

K-system

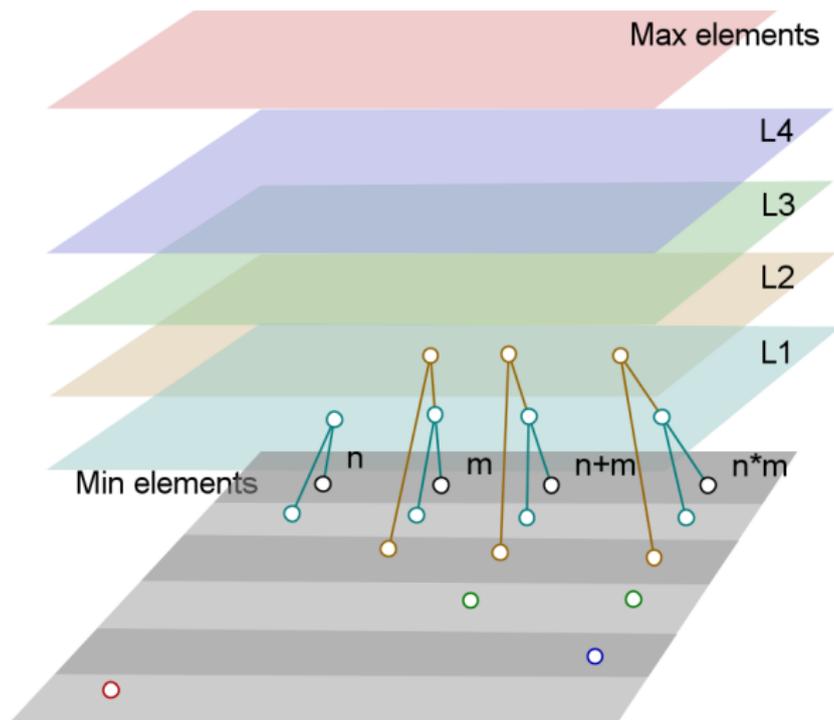# Coding an SMA below any half of a $\mathcal{K}$-pair

The elements of G1 will represent the natural numbers. There are parameters $\mathbf{p}_0$ and $\mathbf{q}_0$ such that $\varphi_{SW}(\mathbf{x}, \mathbf{a}, \mathbf{p}_0, \mathbf{q}_0)$ defines them.

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L1 is constructed from lub's of elements from G1 and G2. There are parameters $\mathbf{p}_1$ and $\mathbf{q}_1$ such that $\varphi_{SW}(\mathbf{x}, \mathbf{a}, \mathbf{p}_1, \mathbf{q}_1)$ defines them.
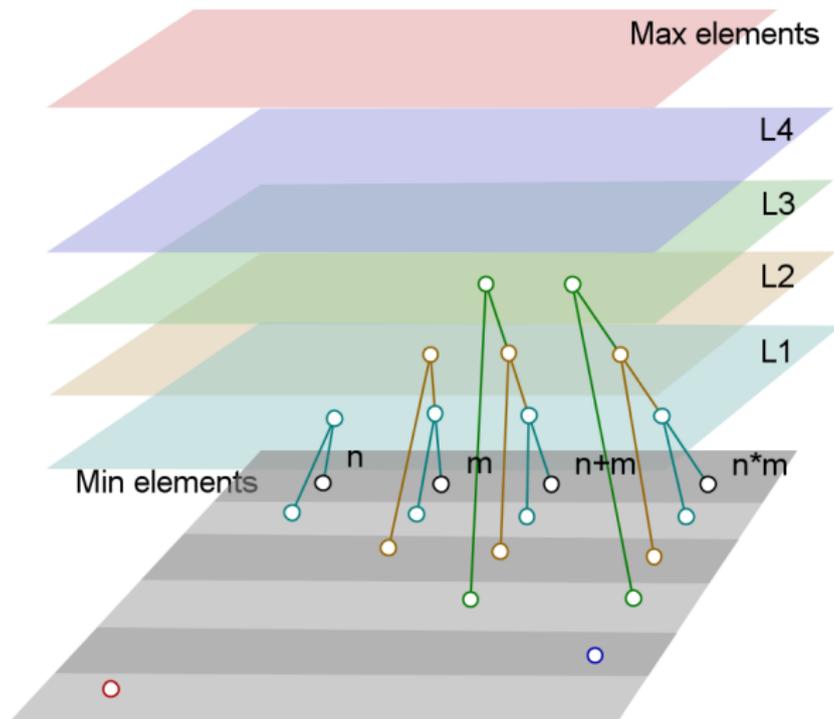
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L2 is constructed from lub's of elements from L1 and G3. There are parameters $\mathbf{p}_2$ and $\mathbf{q}_2$ such that $\varphi_{SW}(\mathbf{x}, \mathbf{a}, \mathbf{p}_2, \mathbf{q}_2)$ defines them.

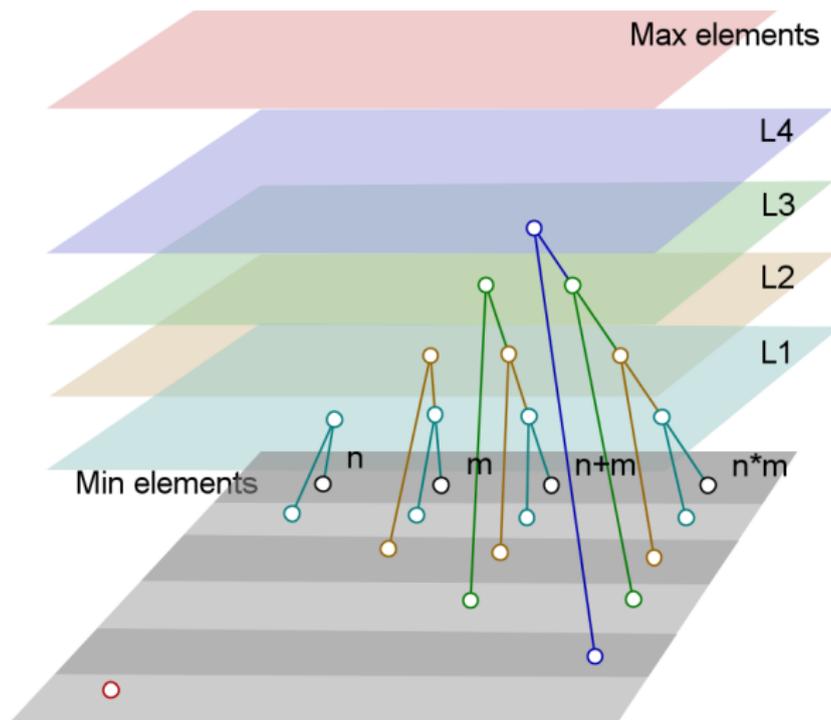# Coding an SMA below any half of a $\mathcal{K}$-pair

L3 is constructed from lub's of elements from L2 and G4. There are parameters $\mathbf{p}_3$ and $\mathbf{q}_3$ such that $\varphi_{SW}(\mathbf{x}, \mathbf{a}, \mathbf{p}_3, \mathbf{q}_3)$ defines them.

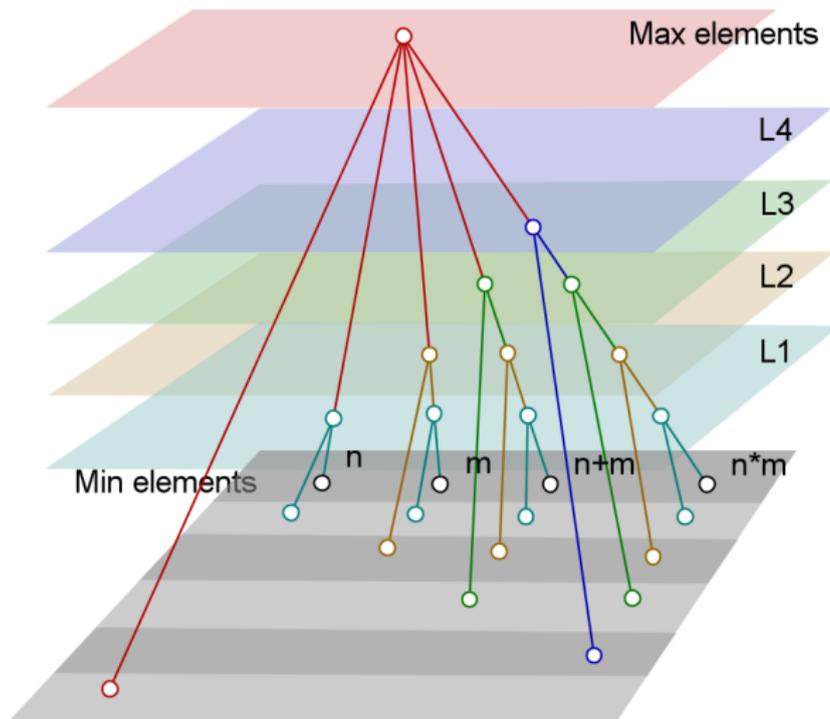# Coding an SMA below any half of a $\mathcal{K}$-pair

L4 is constructed from lub's of elements from L3 and G5. There are parameters $\mathbf{p}_4$ and $\mathbf{q}_4$ such that $\varphi_{SW}(\mathbf{x}, \mathbf{a}, \mathbf{p}_4, \mathbf{q}_4)$ defines them.

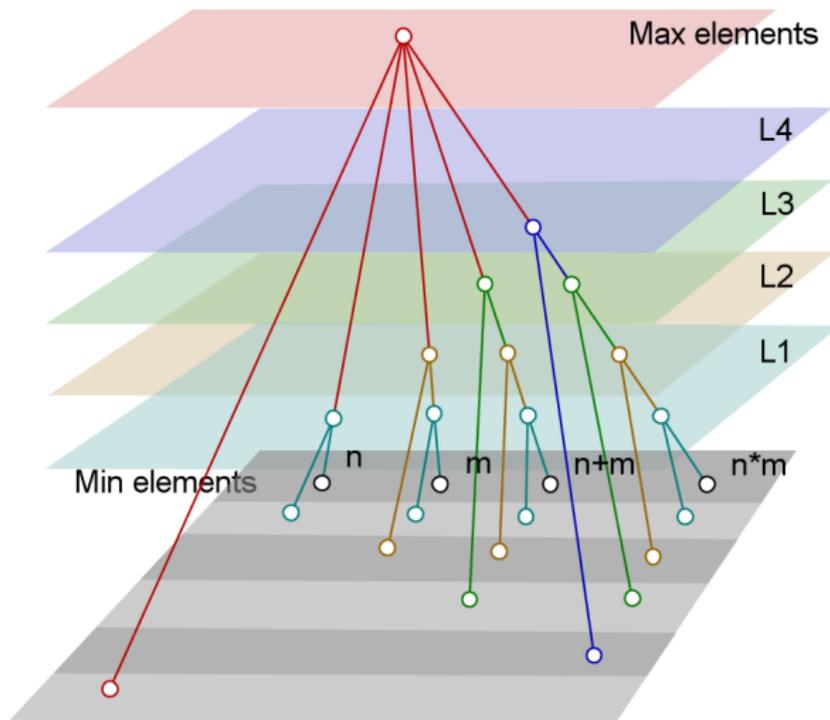# Coding an SMA below any half of a $\mathcal{K}$-pair

Finally the maximal elements are constructed from lub's of elements from L1, L2, L3, L4 and G6. $\varphi_{SW}(\mathbf{x}, \mathbf{a}, \mathbf{p}_5, \mathbf{q}_5)$ defines them.

Coding an SMA below any half of a $\mathcal{K}$-pair

So the parameters $\mathbf{a}$, $\mathbf{p_0}$, $\mathbf{p_1}$, $\mathbf{p_2}$ $\mathbf{p_3}$, $\mathbf{p_4}$, $\mathbf{p_5}$, $\mathbf{q_0}$, $\mathbf{q_1}$, $\mathbf{q_2}$, $\mathbf{q_3}$, $\mathbf{q_4}$, $\mathbf{q_5}$ code a partial order, which represents a standard model of arithmetic $\mathfrak{A}(\mathbf{a}, \overline{\mathbf{p}}, \overline{\mathbf{q}})$.

The other direction

Given parameters **a**, $\overline{p}$, $\overline{q}$, let we can define a first order condition $ST_0(\mathbf{a}, \overline{\mathbf{p}}, \overline{\mathbf{q}})$ which ensures that the parameters code a structure $\mathfrak{A}(\mathbf{a}, \overline{\mathbf{p}}, \overline{\mathbf{q}})$ which is is a model of arithmetic which contains a standard part.

We ask additionally that $ST_0(\mathbf{a}, \overline{\mathbf{p}}, \overline{\mathbf{q}})$ ensures:

- **a** is half of a nontrivial $\mathcal{K}$-pair;
- The domain of $\mathfrak{A}(\mathbf{a}, \overline{\mathbf{p}}, \overline{\mathbf{q}})$ is a $\mathcal{K}$-system.

Let **b** be such that **a** and **b** are a $\mathcal{K}$-pair.
If the model coded below **a** is embedded in all models coded below **b**, then $\mathfrak{A}(\mathbf{a}, \overline{\mathbf{p}}, \overline{\mathbf{q}})$ will be embedded into a SMA and hence will be itself a SMA.

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Comparison maps

For every model $\mathfrak{A}(\mathbf{b}, \overline{\mathbf{p}}', \overline{\mathbf{q}}')$ we ask that
$\forall m_a \in \mathfrak{A}(\mathbf{a}, \overline{\mathbf{p}}, \overline{\mathbf{q}})$ there is an $m_b \in \mathfrak{A}(\mathbf{b}, \overline{\mathbf{p}}', \overline{\mathbf{q}}')$ and an antichain
$(y_0, y_1, \ldots y_m)$ coded by parameters $\mathbf{c}$, $\mathbf{p}''$ and $\mathbf{q}''$ such that:
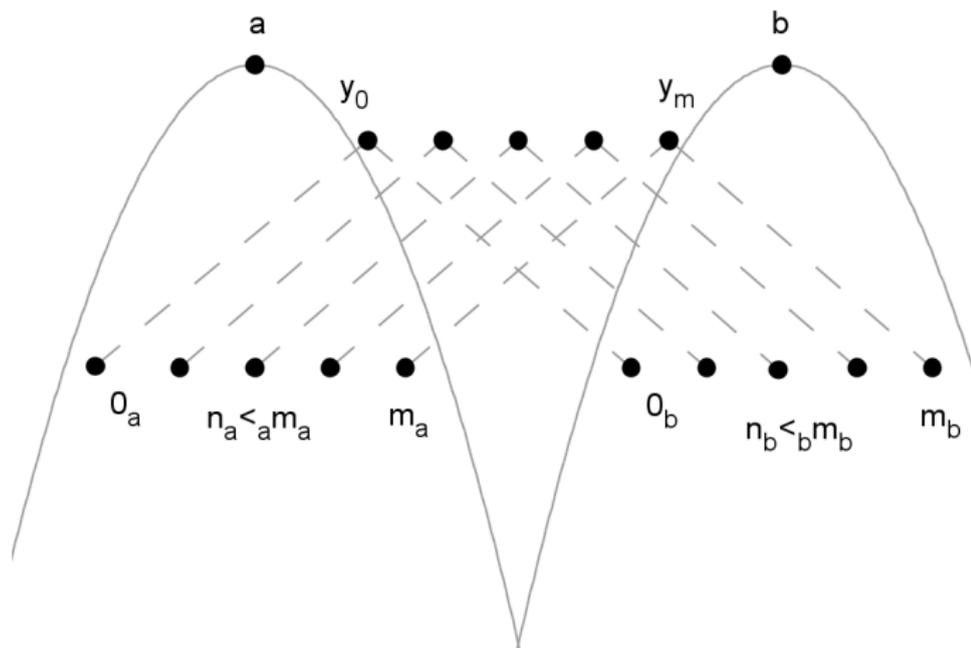
Comparison maps

If $\mathfrak{A}(\mathbf{a}, \overline{\mathbf{p}}, \overline{\mathbf{q}})$ is an SMA then for every $\mathfrak{A}(\mathbf{b}, \overline{\mathbf{p}}', \overline{\mathbf{q}}')$ this is true.

Application II

If **a** bounds a nonzero Δ_2^0 degree then it bounds a nontrivial $\mathcal{K}$-pair.

If **a** is a downwards properly Σ_2^0 degree, then it bounds no $\mathcal{K}$-pair.

Theorem

*A degree **a** is downwards properly Σ_2^0 if and only if:*

$$\mathcal{G}_e \models \forall \mathbf{b}, \mathbf{c}[(\mathbf{b} \leq \mathbf{a} \ \& \ \mathbf{c} \leq \mathbf{a}) \Rightarrow \neg \mathcal{LK}(\mathbf{b}, \mathbf{c})].$$

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Definition

x is upwards properly Σ_2^0 every **y** $\in [\mathbf{x}, \mathbf{0}'_e)$ is properly Σ_2^0.

Example

1. If **a** is a non-splitting degree then it is upwards properly Σ_2^0.
2. (Bereznyuk, Coles, Sorbi) For every enumeration degree $\mathbf{a} < \mathbf{0}'_e$ there exists an upwards properly Σ_2^0 degree $\mathbf{c} \geq \mathbf{a}$.

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Theorem (Jockusch)

For every noncomputable set B there is a semi recursive set $A \equiv_T B$ such that both A and $\overline{A}$ are not c.e.

Corollary

Every nonzero total enumeration degree can be represented as the least upper bound of a nontrivial $\mathcal{K}$-pair.

Theorem (Arslanov, Cooper, Kalimullin)

For every Δ^0_2 enumeration degree $\mathbf{a} < \mathbf{0}'_e$ there is a total enumeration degree $\mathbf{b}$ such that $\mathbf{a} \leq \mathbf{b} < \mathbf{0}'_e$.

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So a degree **a** is upwards properly Σ_2^0 if and only if no element above it other than $\mathbf{0}'_e$ can be represented as the least upper bound of a nontrivial $\mathcal{K}$-pair.

Theorem

A degree **a** *is upwards properly* Σ_2^0 *if and only if:*

$$\mathcal{G}_e \models \forall \mathbf{c}, \mathbf{d}(\mathcal{LK}(\mathbf{c}, \mathbf{d}) \,\&\, \mathbf{a} \leq \mathbf{c} \vee \mathbf{d} \Rightarrow \mathbf{c} \vee \mathbf{d} = \mathbf{0}'_e).$$

Dynamic characterizations

Lemma (McEvoy)

C is low if and only if it has a Δ_2^0 approximation $\{C^{\{s\}}\}_{s<\omega}$ such that for all enumeration operators Θ, $\{\Theta^{\{s\}}(C^{\{s\}})\}_{s<\omega}$ is a Δ_2^0 approximation to $\Theta(C)$.

Lemma (Kalimullin)

A pair of Δ_2^0 sets A and B form a $\mathcal{K}$-pair if and only if there are Δ_2^0 approximations $\{A^{\{s\}}\}_{s<\omega}$ and $\{B^{\{s\}}\}_{s<\omega}$ to A and B respectively such that for all s:

$$A^{\{s\}} \subseteq A \vee B^{\{s\}} \subseteq B.$$

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Proposition

If (**a**, **b**) *is a nontrivial $\mathcal{K}$-pair in $\mathcal{G}_e$ then* **a** $\vee$ **b** *is either low or total.*

Proof:
Suppose that A and B are a Δ_2^0 nontrivial $\mathcal{K}$-pair such that $A \oplus B$ is not low. Let $\{A^{\{s\}}\}_{s<\omega}$ and $\{B^{\{s\}}\}_{s<\omega}$ be their $\mathcal{K}$-approximations.
Then $\{A^{\{s\}} \oplus B^{\{s\}}\}_{s<\omega}$ is not a low approximation.

Let Θ and x be such that at infinitely many stages s:
$x \in \Theta^{\{s\}}(A^{\{s\}} \oplus B^{\{s\}}) \setminus \Theta^{\{s+1\}}(A^{\{s+1\}} \oplus B^{\{s+1\}})$.

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We have an infinite computable sequence of bad stages for
$\{A^{\{s\}} \oplus B^{\{s\}}\}_{s<\omega}$:

$$s_0 < s_1 \cdots < s_n \ldots$$

Let $G_A = \{n \mid A^{\{s_n\}} \subseteq A\}$ and $G_B = \{n \mid B^{\{s_n\}} \subseteq B\}$.

- Then $G_B = \overline{G_A}$.
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Theorem (Giorgi, Sorbi, Yang)

Every non-low total degree bounds a downwards properly Σ_2^0 enumeration degree.

Corollary

The class of the non-low total Σ_2^0 enumeration degrees is first order definable in $\mathcal{G}_e$

Proof:

A Σ_2^0 e-degree **a** is non-low and total is and only if

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