Fragments of the first order theory of the partial order of the enumeration degrees

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The theory of a degree structure

Let $\mathcal{D}$ be a degree structure.

Question

- Is the theory of the structure in the language of partial orders decidable?
- How complicated is the theory?
- How many quantifiers does it take to break decidability?

Degree structure	Complexity of $Th(\mathcal{D})$	$\exists\forall\exists$-$Th(\mathcal{D})$	$\forall\exists$-$Th(\mathcal{D})$
$\mathcal{D}_T$	Simpson 77	Lerman-Schmerl 83	Shore 78; Lerman 83
$\mathcal{D}_T(\leqslant \mathbf{0})$	Shore 81	Lerman-Schmerl 83	Lerman-Shore 88
$\mathcal{R}$	Slaman-Harrington 80s	Lempp-Nies-Slaman 98	Open
$\mathcal{D}_e$	Slaman-Woodin 97	Open	Open
$\mathcal{D}_e(\leqslant \mathbf{0}')$	Ganchev-Soskova 12	Kent 06	Open

Related problems

- To understand what existential sentences are true $\mathcal{D}$ we need to understand what finite partial orders can be embedded into $\mathcal{D}$;

- At the next level of complexity is the *extension of embeddings problem*:

Problem

We are given a finite upper-semilattice P and a partial order $Q \supseteq P$. Does every embedding of P extend to an embedding of Q?

- To understand what $\forall\exists$-sentences are true in $\mathcal{D}$ we need to solve a slightly more complicated problem:

Problem

We are given a finite upper-semilattice P and a partial orders $Q_0, \dots Q_n \supseteq P$. Does every embedding of P extend to an embedding of one of the Q_i?

The Turing degrees and initial segment embeddings

Theorem (Lerman 71)

Every finite lattice can be embedded into $\mathcal{D}_T$ as an initial segment.

- Suppose that P is a finite lattice and $Q \supseteq P$ is a partial order extending P.
- The initial segment embedding of P can be extended to an embedding of Q only if no element in $Q \setminus P$ is below any element of P.
- Q also needs to respect least upper bounds if $x \in Q \setminus P$ and $u, v \in P$ and $x \geqslant u, v$ then $x \geqslant u \vee v$.

Theorem (Shore 78; Lerman 83)

That is the only obstacle.

Theorem (Schmerl 83)

If all finite lattices can be embedded into $\mathcal{D}$ as intervals then $\exists\forall\exists$-$Th(\mathcal{D})$ is undecidable.

By Nies Transfer Lemma embedding finite distributive lattices is sufficient.

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The enumeration degrees

Theorem (Gutteridge 71)

The enumeration degrees are downwards dense.

A degree $\mathbf{b}$ is a *minimal cover* of a degree $\mathbf{a}$ if $\mathbf{a} < \mathbf{b}$ and the interval $(\mathbf{a}, \mathbf{b})$ is empty.

Theorem (Slaman, Calhoun 96)

There are Π^0_2 enumeration degrees $\mathbf{a} < \mathbf{b}$ such that $\mathbf{b}$ is a minimal cover of $\mathbf{a}$

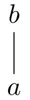
A degree $\mathbf{b}$ is a *strong minimal cover* of a degree $\mathbf{a}$ if $\mathbf{a} < \mathbf{b}$ and for every degree $\mathbf{x} < \mathbf{b}$ we have that $\mathbf{x} \leqslant \mathbf{a}$.

Theorem (Kent, Lewis-Pye, Sorbi 12)

There is a Δ^0_3 degree $\mathbf{a}$ and Π^0_2 enumeration degree $\mathbf{b}$ such that $\mathbf{b}$ is a strong minimal cover of $\mathbf{a}$

The simplest lattice

Consider the lattice $\mathcal{L} = \{a < b\}$. What properties should possible extensions $Q_0, Q_1 \dots Q_n$ have so that every embedding of $\mathcal{L}$ extends to Q_i for some i:



1. We can embed this lattice as degrees $\mathbf{a} < \mathbf{b}$ such that $\mathbf{b}$ is a strong minimal cover of $\mathbf{a}$. Thus we need at least one Q_i where all new x satisfy: if $x < b$ then $x < a$.
2. We can embed this lattice as degrees $\mathbf{0}_e < \mathbf{b}$. Thus we need at least one Q_i where all new x satisfy: if $x < b$ then $x > a$.

Theorem (Slaman, Sorbi 14)

Every countable partial order can be embedded below any nonzero enumeration degree.

So these two conditions suffice.

A wild conjecture

Conjecture (Lempp, Slaman, Soskova)

Every finite lattice can be embedded into $\mathcal{D}_e$ as an interval of Π^0_2 enumeration degrees $[\mathbf{a}, \mathbf{b}]$ so that if $\mathbf{x} \leqslant \mathbf{b}$ then $\mathbf{x} \in [\mathbf{a}, \mathbf{b}]$ or $\mathbf{x} < \mathbf{a}$.

- Note! This would only solve the extension of embeddings problem: Every embedding of P would extend to an embedding of Q if Q satisfies the same two properties: have no new degree below any member of P and respect least upper bounds.
- If we allow more than one Q then we need a wilder conjecture: Q_1 could place new elements below the least element in P, Q_2 could place new elements below some minimal element in P and we can't rule both out simultaneously.

Step 1

Slightly extend the Kent, Lewis-Pye, Sorbi result:

Theorem

There are Π^0_2 degrees $\mathbf{a} < \mathbf{b}$ such that $\mathbf{b}$ is a strong minimal cover of $\mathbf{a}$.

Proof.

Construct Π^0_2 sets A and B so that:

- $\mathcal{M}_e$: $\Psi_e(A, B) = \Gamma(B)$ or $A, B \leqslant \Psi_e(A, B)$;
- $\mathcal{T}_e$: $A \neq \Phi_e(B)$.

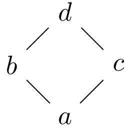
A number is in A or B if and only if it is in A_s or B_s at infinitely many stages. $\mathcal{M}_e$-strategies promise to add numbers to B if certain numbers enter A, B.

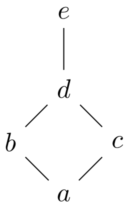
Attempts at diagonalization of $\mathcal{T}$ may fail: a witness $x \in A$ if and only if $y \in \Psi(A, B)$ influencing what a higher priority $\mathcal{M}$-strategy wants in B. Instead we produce a stream of elements $x_0, x_1, \ldots$ whose membership in A is reflected in membership in $\Psi(A, B)$. We code B using x_{2i}.

$A \leqslant_e \Psi(A, B)$ because A consists of (1) elements enumerated by higher priority strategies, (2) elements in the stream, (3) elements enumerated in A to code B at higher priority strategies.

Step 2, 3, 4

Generalize the previous construction to show that each of the following lattices can be embedded in a *strong minimal cover* way.





A small victory

Theorem (Lempp, Slaman, Soskova)

Every finite distributive lattice can be embedded as an interval $[\mathbf{a}, \mathbf{b}]$ so that if $\mathbf{x} \leqslant \mathbf{b}$ then $\mathbf{x} \in [\mathbf{a}, \mathbf{b}]$ or $\mathbf{x} \leqslant \mathbf{b}$.

Proof.

Fix a finite distributive lattice $\mathcal{L}$ with join irreducible elements $a_0, a_1, \dots a_n$. Every other element of the lattice has a unique representation as $a_F = \bigvee_{i \in F} a_i$, where F is downwards closed.

We build Π^0_2 sets $X_0, \dots X_n$ so that $A_F = \bigoplus_{j \in F} X_j$ represents a_F.

- $\mathcal{T}^i_e$: $X_i \neq \Phi_e(A_{F_i})$, where $F_i = \{j \mid a_i \nleqslant_{\mathcal{L}} a_j\}$;
- $\mathcal{M}^{G,F}_e$: Fix $F \subseteq G$ such that a_G is minimal above a_F. Note that $G = F \cup \{i\}$ for some fixed number i. Denote by $G \setminus F = \{j \in G \mid a_j \leqslant_{\mathcal{L}} a_i\}$. The requirement asks that $\Psi_i(A_G) = \Gamma(A_F)$ or else $A_{G \setminus F} \leqslant_e \Psi(A_G)$.

□

A small victory

Corollary

The $\exists\forall\exists$-theory of $\mathcal{D}_e$ is undecidable.

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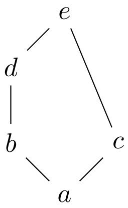
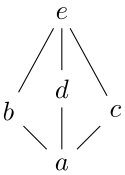
Questions

Question

Can we embed all finite lattices in $\mathcal{D}_e$ as strong intervals?

We are currently working on N_5 and M_3:





Question

What would be a plausible algorithm for deciding the $\forall\exists$-theory of $\mathcal{D}_e$?

Question

Can we embed all countable (distributive) lattices into $\mathcal{D}_e$ as strong intervals?

Thank you!