Introduction to Harrington's Nonsplitting Theorem
Part Two

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Review

Theorem

There exists a c.e. degree a such that $0'$ can not be split over a.

- $N_\Psi : E \neq \Psi^A$ - hence A is not complete
- $P_{\Theta,U,V} : E = \Theta^{U,V} \Rightarrow (\exists \Gamma, \Lambda)[K = \Gamma^{U,A} \vee K = \Lambda^{V,A}]$

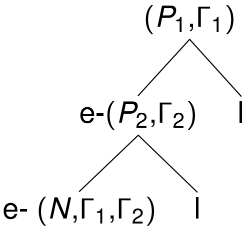
Review

- The (P, Γ) - strategy - builds a c.e. set of axioms, so that $\Gamma^{U,A} = K$
- The (N, Γ) - strategy
  1. Initialization - choose threshold and witness
  2. Honestifictaion - insure $\theta(x) < u(d)$
  3. Attack - If $\Psi^A(x) = 0$, enumerate x in E
  4. Successful attack - A U-change - move the γ-markers above $\psi(x)$ to preserve the computation $\Psi^A(x) = 0$.
  5. Unsuccessful attack - A V-change - cancel witness, enumerate $\gamma(d)$ in A and start over, (P, Λ) and (N, Λ) start work.

Important Details

- Working intervals - secure working space for successive strategies
- The Functionals must be total - the markers have to come to rest for each element after a certain stage

Two P- requirements above one N requirement



- P_1 building Γ_1
- P_2 building Γ_2
- (N, Γ_1, Γ_2)

Two P- requirements above one N requirement

- N needs to clear its computation of both γ_1 and γ_2 markers.
- Initialization - two thresholds
- Honestification - both Γ_1 and Γ_2 have to be honest hence two outcome h_1 and h_2
- Attack - one witness enumerated in E will force both (U_1, V_1) changes and (U_2, V_2) changes

Synchronization Problems

- One P requirement - a single U change insures success
- Two P requirements - we need a U_1 and U_2 change simultaneously.
- Worst case scenario:
  $(U_1, V_2) \Rightarrow (V_1, U_2) \Rightarrow (U_1, V_2) \Rightarrow \dots$
- We need two outcomes g_1 and g_2
- Outcomes g and h destroy the functional Γ. Now all outcomes h_1, h_2, g_1, and g_2 can destroy both functionals.

Solutions

- A change in the strategy of a higher priority requirement can afford to restart all lower priority strategies:
    1. Below infinite Honestification for Γ_1 a new (P_2, Γ_2) node
    2. Below Unsuccessful attack for Γ_1 a new (P_2, Γ_2) node
- Hence a change in P_1's strategy can afford to destroy P_2's work and insure a safe working space for the strategies below
- A change in strategy in lower priority requirements has to preserve higher priority strategies - more work on insuring safe working interval

Honestification

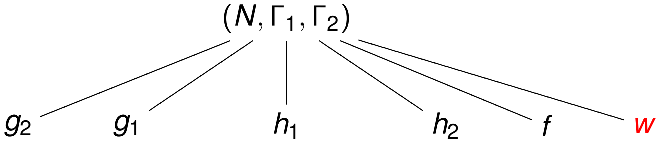
- True outcome is h_2, then P_1 must be preserved.
- Hence $h_1 <_L h_2$.
- Outcome h_1: both Γ_1 and Γ_2 destroyed, P_1 satisfied, P_2 postponed
- Outcome h_2: Γ_1 uninfluenced, P_1 remains intact. P_2 satisfied.

Unsuccessful Attacks

- Unsuccessful attacks - Cancel current witness, Honestification with new witness but same threshold at following stages. If the true outcome is infinitely many unsuccessful attacks - $\gamma(d)$ is unbounded, hence (P, Γ) destroyed.
- Infinitely many P_2-unsuccessful attacks destroy (P_1, Γ_1).
- Only solution - moving thresholds: Define $d_2 < d_1$, unsuccessful attack for P_2 $(o = g_2)$ - cancel current P_1 - threshold d_1.
- Hence $g_2 <_L g_1$.
- Dangerous possibility: true outcome is g_2, but g_1 is visited infinitely often solved!

The detailed $(N_\Psi, \Gamma_1, \Gamma_2)$ strategy

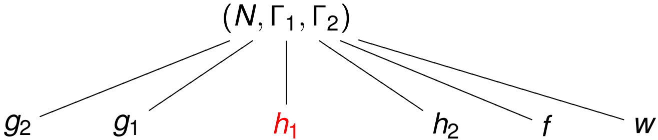
Initialization



1. Define new thresholds: $d_2 < d_1$, if undefined
2. Define a witness $x > d_1$, $x \notin E$
3. Wait for $x < l(E, \Theta_1^{U_1,V_1})$ and $x < l(E, \Theta_2^{U_2,V_2})$.
   $(o = w)$

The detailed $(N_\Psi, \Gamma_1, \Gamma_2)$ strategy

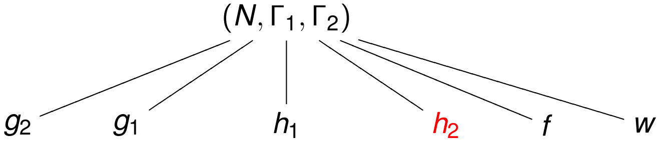
Honestification



- Perform Honestification(Γ_1): Check if $\theta_1(x)$ has grown since the last stage (Result = h), change markers as appropriate. Otherwise (Result = w).
- If (Result = h) then enumerate $\gamma(d_2)$ in A, $(o = h_1)$ working within boundaries $(x, \gamma(d_1))$.
- If (Result = w), perform Honestification(Γ_2).
- If (Result = h) then $(o = h_2)$ working within boundaries $(L = \max(x, \gamma(d_1)), R = \gamma(d_2))$.
- If (Result = w), then wait for $\Psi^A(x) = 0$, with $\psi(x) < R$. (o = w), working within boundaries $(L = \max(\gamma_1(d_1), \gamma_2(d_2)), R)$.

The detailed $(N_\Psi, \Gamma_1, \Gamma_2)$ strategy

Honestification



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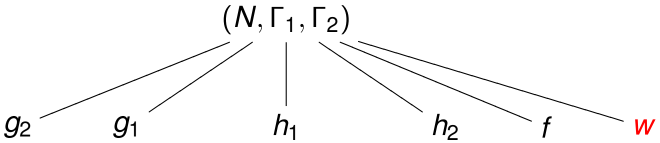
Honestification



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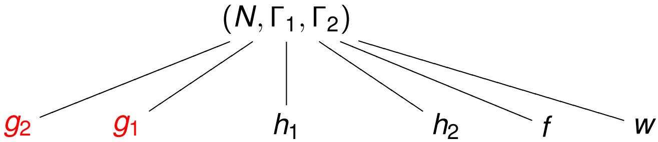
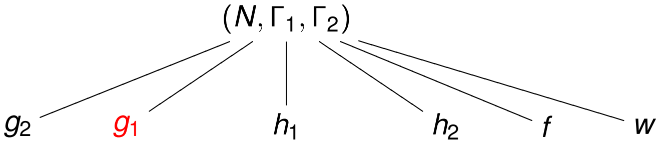
The detailed $(N_\Psi, \Gamma_1, \Gamma_2)$ strategy
Attack



- Enumerate x in E and restrain A on $\psi(x)$. The outcome $(o = g_i)$, where g_i is the most recently visited g-outcome
- Wait for the next expansionary stage

The detailed $(N_\Psi, \Gamma_1, \Gamma_2)$ strategy

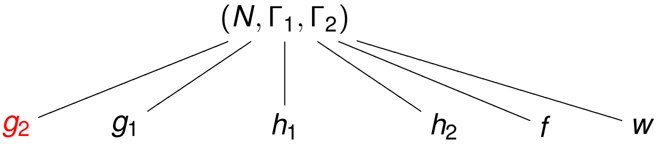
Attack



- Unsuccessful Γ_1-attack. Enumerate $\gamma_1(d_1)$ and $\gamma_2(d_2)$ in A. Remove the restraint on A. Cancel the current witness x. Return to Initialization at the next stage $(o = g_1)$ working within $(L = d_1, R = x)$.
- Unsuccessful Γ_2-attack. Enumerate $\gamma_2(d_2)$ in A and cancel d_1. Remove the restraint on A. Cancel the current witness x. Return to Initialization at the next stage $(o = g_2)$ working within $(L = d_2, R = x)$.
- Successful attack - $(o = f)$, working within $(L = \max(\gamma_1(d_1), \gamma_2(d_2)), R)$.

The detailed $(N_\Psi, \Gamma_1, \Gamma_2)$ strategy

Attack



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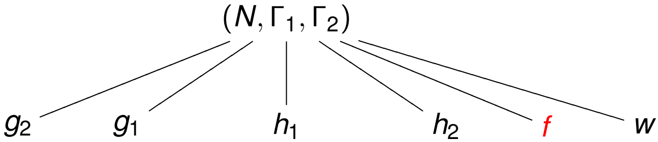
The detailed $(N_\Psi, \Gamma_1, \Gamma_2)$ strategy

Attack



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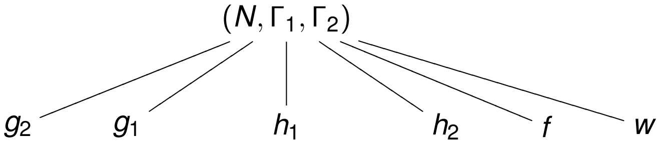
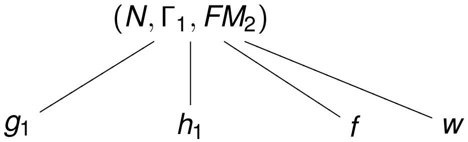
Working Intervals



- g_2 - $(L = d_2, R = x)$. d_1 is cancelled and redefined bigger than x.
- g_1 - $(L = d_1, R = x)$. $\gamma(d_1)$ and $\gamma(d_2)$ are cancelled, redefined bigger than x.
- h_1 - $(L = x, R = \gamma(d_1))$. $\gamma(d_2)$ is cancelled and redefined bigger than $\gamma(d_1)$.
- h_2 -$(L = \gamma(d_1), R = \gamma(d_2))$
- w, f - $(L = \gamma(d_2), R)$

Working below Honestification outcomes

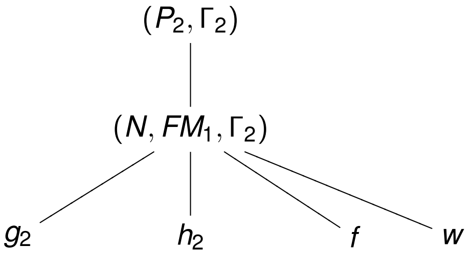
Below h_2



- h_1 is not visited - $\theta_1(x)$ remains constant, N does not enumerate any γ_1 markers in A.
- $\theta_2(x)$ - unbounded, hence P_2 is satisfied.
- (N, Γ_1, FM_2) working only with P_1 as described previously - within the working interval

Working below Honestification outcomes

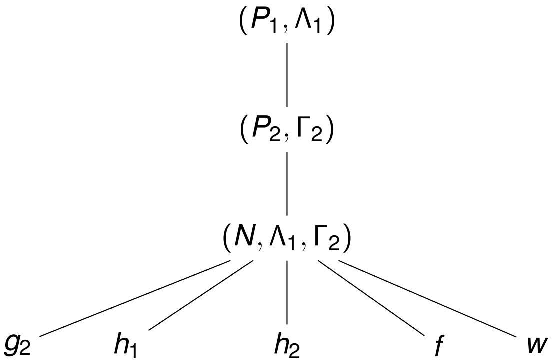
Below h_1



- $\theta_1(x)$ unbounded, hence P_1 satisfied.
- $\gamma_2(d_2)$ also grows unbounded - Γ_2 is destroyed and works outside the working interval.
- P_2 restarted on the next node
- (N, FM_1, Γ_2) working only with the new P_2, within the right working interval.

Working below Unsuccessful outcomes outcomes

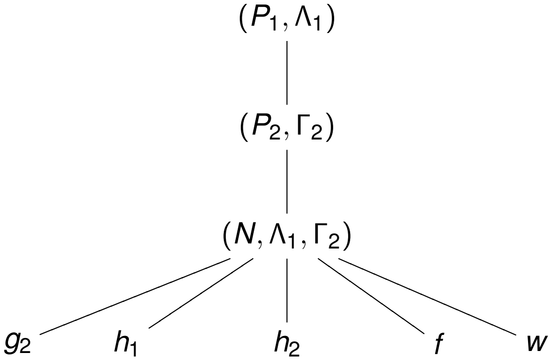
Below g_1



- Cofinitely many Γ_1- Unsuccessful attacks, P_1 switches to a Λ_1 strategy.
- $\gamma_2(d_2)$ also grows unbounded - Γ_2 is destroyed and works outside the working interval. P_2 restarted on the next node

Working below Unsuccessful Attack outcomes

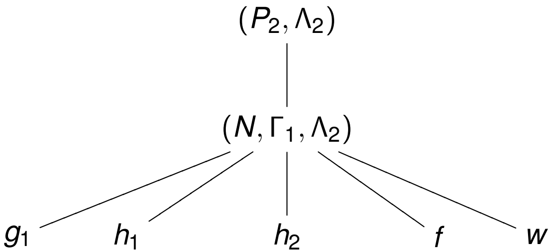
Below g_1



- (N, Λ_1, Γ_2) working with the new P_2, within the right working interval.
- Every attack synchronized with (N, Γ_1, Γ_2) - hence Λ_1 successful. No outcome g_1, only g_2

Working below Unsuccessful Attack outcomes

Below g_2



- Infinitely many Γ_2- Unsuccessful attacks, P_2 switches to a Λ_2 strategy.
- d_1 grows unboundedly - Γ_1 is preserved.
- (N, Γ_1, Λ_2) working with thresholds and witness $d_2 < \widehat{d_2} < \widehat{d_1} < \widehat{x} < d_1 < x$.
- Every attack synchronized with (N, Γ_1, Γ_2) - hence Λ_2 successful. No outcome g_2, only g_1.

Main tricks

Initialization

- A change in K below a threshold d can injure the computation $\Psi^A(x) = 0$
- True outcomes w, f, h_1, h_2, g_1 - finitely many times initialization
- True outcome g_2 - first threshold d_1 changes infinitely often - infinitely many times initialization.
- Solution - when a $K \upharpoonright d$ change occurs, initialize only nodes below outcomes that assume d remains constant.
- $K \upharpoonright d_1$-change $\Rightarrow$ initialize all nodes except below outcome g_2.

Main tricks

Initialization

- The threshold $\widehat{d_1}$ for (N, Γ_1, Λ_2) remains constant.
- Attack $\Rightarrow$ Success $\Rightarrow$ Change in $K \upharpoonright d_1$, above $\widehat{d_1}$.
- (N, Γ_1, Λ_2) expects a V_2- change - that it obviously did not get.
- Solution: There was a $(U_1, V_1) \upharpoonright \theta_1(\widehat{x})$ - change.
    1. $U_1 \upharpoonright \theta_1(\widehat{x})$-change $\Rightarrow$ the markers for all elements $n > \widehat{d_1}$ have been moved above $\psi(x)$ and the computation will be preserved. Do not initialize in this case.
    2. $V_1 \upharpoonright \theta_1(\widehat{x})$-change. Then (N, Γ_1, Λ_2) would have outcome g_1 at next true stage.

Main tricks

Unsuccessful attacks - Worst case scenario

- Both outcomes g_2 and g_1 are visited infinitely often.
- An attack synchronized with g_2, may be followed by outcome g_1. Then when g_2 is true again, the synchronization between the attacks is lost!
- We can not be sure that there is a $V_2 \upharpoonright \theta_2(\widehat{x})$ - change
- We can be sure that in this case there is a $V_1 \upharpoonright \theta_1(\widehat{x})$- change - hence (N, Γ_1, Λ_2) will have outcome g_1 followed by a new P_2 strategy and a (N, Λ_1, Γ_2) - node.