A gentle introduction to Harrington non-splitting and beyond

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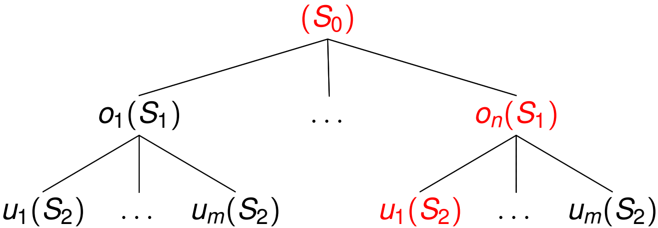
The Priority Method

- 1958 Post's Problem and Friedberg and Muchnik's solution
- 1963 Sacks Jump Theorem and 1964 Density theorem
- 1975 Lachlan's Monster Theorem and the priority tree method

The priority tree method

- Construct a set A satisfying R_0, $R_1, \dots$.
- Priority ordering $R_0 < R_1 \dots$.
- Strategies and outcomes: S_0 with outcomes $o_1, \dots o_n$, S_1 with outcomes $u_1, u_2, \dots u_m$.
- The tree of strategies is a computable tree of all possible ways that the construction might go.

The construction

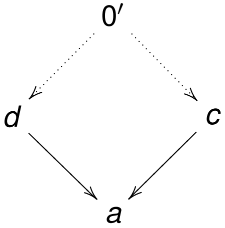


- At stage s we construct a finite path through the tree, approximating the true path.
- We *injure* strategies to the right.
- If a strategy is not injured infinitely many times and is visited on infinitely many stages it satisfies its requirement.
- The most left infinite path each initial segment of which is visited infinitely many times is the true path.

Harrington's Nonsplitting theorem

Theorem

There exists a c.e. degree $a < 0'$ such that $0'$ can not be split over a.



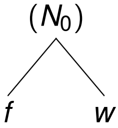
The Requirements

We will construct the c.e. sets A and E

- $N_\Psi : E \neq \Psi^A$ - hence A is not complete
- $P_{\Theta,U,V} : E = \Theta^{U,V} \Rightarrow (\exists \Gamma, \Lambda)[K = \Gamma^{U,A} \vee K = \Lambda^{V,A}]$

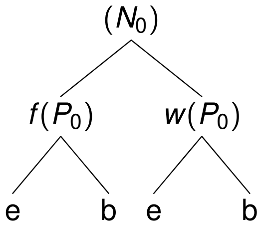
Priority: $N_0 < P_0 < N_1 \ldots$
We start off with $A = E = \emptyset$.

The first N-strategy



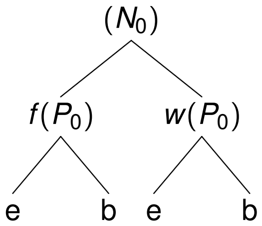
- $N_0 : E \neq \Psi_0^A$
- Select a witness x for N_0
- Wait for $\Psi_0^{A_s}(x) = 0$
- Enumerate x in E and restrain $A \upharpoonright \psi(x)$.

The first P strategy



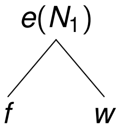
- $P_{\Theta,U,V} : E = \Theta^{U,V} \Rightarrow (\exists \Gamma, \Lambda)[K = \Gamma^{U,A} \vee K = \Lambda^{V,A}]$
- Monitor the length of agreement $l(s) = l(E_s, \Theta^{U,V}[s])$.
- If the length of agreement is bounded, then $E \neq \Theta^{U,V}$.
- If $E = \Theta^{U,V}$ then we have infinitely many expansionary stages.

The first P strategy



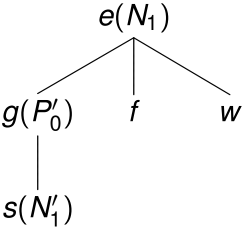
- $P_{\Theta,U,V} : (\exists \Gamma, \Lambda)[K = \Gamma^{U,A} \vee K = \Lambda^{V,A}]$
- At expansionary stages construct a Turing operator Γ, so that $\Gamma^{U,A} = K$.
- Γ is a c.e. set of axioms of the form $\Gamma^{\tau_1,\tau_2}(z) = v$.
- For each $z < l$: axiom $\Gamma^{U_s \upharpoonright (u(z)+1), A_s \upharpoonright (\gamma(z)+1)}(z) = K_s(z)$.
- We are allowed to enumerate new axioms only if the previous ones are not valid anymore.
- If $K(z)$ changes, enumerate $\gamma(z)$ in A and rectify Γ.

The second N - strategy



- A-restraint by N_1 conflicts with the need to rectify Γ by P_0.
- Expansionary stages: $\Theta^{U,V}(x) = E(x) = 0$
- Enumerating the witness x in E ensures a change in the set $U \oplus V \upharpoonright \theta(x)$.
- A U-change enables us to move the markers $\gamma(n)$ above $\psi(x)$ without changing A.

The second N - strategy



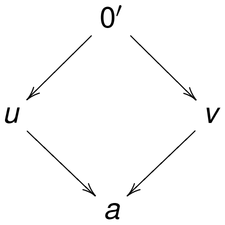
- A V - change is not useful at all - we have to try again with a new witness.
- It would be useful if we were constructing $\Lambda^{V,A} = K$.
- A backup strategy P'_0 will work only when the attack ends with a V-change.
- A copy of N_1 - will now be able to satisfy its requirement.

Further generalizations

- The requirement that both degrees above a are c.e is strong.
- Even mildly weakening it is not possible: Arslanov's Splitting Theorem

Theorem

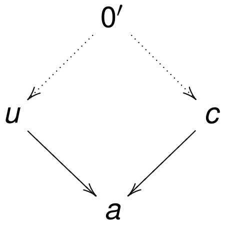
There is a d.c.e. splitting of $0'$ above each c.e. degree $a < 0'$.



The strongest non-splitting theorem

Theorem

There exists a c.e. degree $a < 0'$ such that there exists no nontrivial splitting of $0'$ into a c.e. and a Δ_2 degree above a.



Embedding the Turing degrees into the enumeration degrees

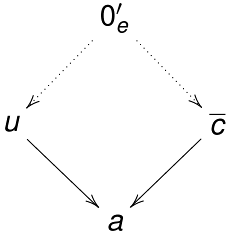
There exists an order theoretic embedding $\iota : D_T \to D_e$ with following properties.

1. ι preserves least element, joins and jump operators
2. The c.e. Turing degrees embed exactly onto the Π_1 enumeration degrees
3. There are partial Δ_2 degrees.

A theorem in the e-degrees

Theorem

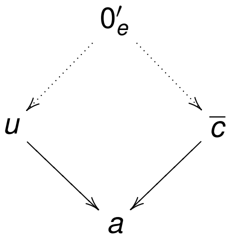
There exists a Π_1 e-degree $a < 0'_e$ such that there exist no nontrivial splittings of $0'_e$ into a Π_1 e-degree and Δ_2 e-degree above a.



A more general theorem in the e-degrees

Theorem

There exists a Π_1 e-degree $a < 0'_e$ such that there exist no nontrivial splittings of $0'_e$ into a Π_1 e-degree and a Σ_2 e-degree above a.



The Requirements

We will construct the Π_1 sets A and E

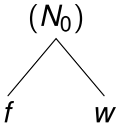
- For all enumeration operators Ψ:

$$N_\Psi : E \neq \Psi^A$$

- For each pair of a Σ_2 set U and a Π_1 set $\overline{W}$ and each enumeration operator Θ:

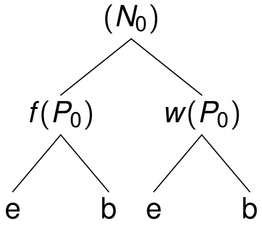
$$P_{\Theta,U,W} : E = \Theta^{U,\overline{W}} \Rightarrow (\exists \Gamma, \Lambda)[\overline{K} = \Gamma^{U,A} \vee \overline{K} = \Lambda^{\overline{W},A}]$$

The first N strategy



- Select a witness $x \in E$ for N_Ψ
- Wait for $x \in \Psi^A[s]$, i.e. for an axiom $\langle x, A_x \rangle \in \Psi_s$ with $A_x \subset A_s$.
- Extract x from E and restrain each $y \in A \upharpoonright \psi(x)$.

The first P-strategy



- Monitor the length of agreement $l(s) = l(E_s, \Theta^{U,\overline{W}}[s])$.
- At expansionary stages construct a enumeration operator Γ, so that $\Gamma^{U,A}[s] = \overline{K}[s]$
- For each $n < l$ such that $n \in \overline{K}_s$: axiom $\langle n, U_s \upharpoonright (u(n)+1), A_s \upharpoonright (\gamma(n)+1)\rangle \in \Gamma$.
- For each $n < l$ such that $n \notin \overline{K}_s$: make all previously defined axioms invalid by extracting $\gamma(n)$ from A.

Complications

- The set U is now Σ_2.
- A Σ_2-approximation U_s gives us:
  1. If $a \in U$ then $a \in U_s$ for almost all s.
  2. If $a \notin U$ then $a \notin U_s$ for infinitely many s
- It could happen that $l(E, \Theta^{U,\overline{W}})$ is bounded but $E = \Theta^{U,\overline{W}}$.
- We may never see the right approximation to $U \upharpoonright u(n)$.

Good approximations

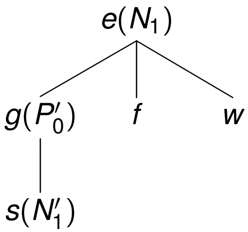
Definition

An approximation U_s to a set U is good if it has the following properties:

1. For all n there exists a good stage s such that $U \upharpoonright n \subset U_s \subset U$.
2. For all n there exists a stage s such that if $t > s$ is a good stage then $U \upharpoonright n \subset U_t$

We define a good Σ_2 approximation to U and $U \oplus \overline{W}$ with infinitely many common good stages.

The second N-strategy



- Choose a witness x and try to achieve $\gamma(n) > \psi(x)$ before the imposition of the restraint.
- Extract x from E. Return of $l(E, \Theta^{U,\overline{W}})$ forces U or $\overline{W}$ to change below $\theta(x)$.
- Only trust $\overline{W}$-changes - start over with new witness, implement the backup strategy which insures $\Lambda^{\overline{W},A} = \overline{K}$
- Otherwise: lift the gamma markers and preserve the restraint, but keep an eye on $\overline{W}$, for further changes.

The overall picture for the enumeration degrees

- The strongest non-splitting theorem
- Theorem(Sorbi, Arslanov) There is a Δ_2 splitting of $0'_e$ above each Δ_2 degree.
- What about splitting/non-splitting above a Σ_2 degree?

Motivation

- Non-cuppable degrees:
  1. There is non-cuppable c.e. degree in the Turing degrees. Cooper, Yates
  2. Every Δ_2 e-degree is Δ_2 cuppable. Cooper, Sorbi, Xiaoding Yi.
  3. There is a Σ_2 non-cuppable e-degree. Cooper, Sorbi, Xiaoding Yi.
- Non-bounding degree
  1. There is non-bounding c.e. degree in the Turing degrees. Lachlan
  2. Every Δ_2 e-degree bounds a minimal pair. Cooper, Li, Sorbi, Yang
  3. There is a Σ_2 non-bounding e-degree. Cooper, Li, Sorbi, Yang
- Non-splitting
  1. Harrington's Theorem
  2. Arslanov and Sorbi's Theorem

Non-splitting in the enumeration degrees

We will construct a Σ_2 set A and a Π_1 set E

- For all enumeration operators Ψ:

$$N_\Psi : E \neq \Psi^A$$

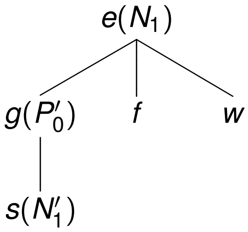
- For each pair of a Σ_2 set U and V and each enumeration operator Θ:

$$P_{\Theta,U,V} : E = \Theta^{U,V} \Rightarrow (\exists \Gamma, \Lambda)[\overline{K} = \Gamma^{U,A} \vee \overline{K} = \Lambda^{V,A}]$$

Differences

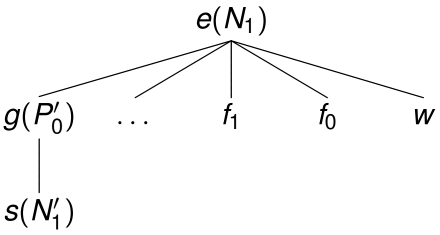
- Now we need to deal with two Σ_2 sets.
- We use good approximation again.
- We loose the stability that using the Π_1 set $\overline{W}$ gave us.

The Problem in Detail



- Lets Look at the second N-strategy
- When we attack with a witness x, we get a change in $U \oplus V$.
- The pair of Σ_2 sets can now trick us - giving us false information: a V-change that is later on corrected.
- The N-requirement that counted on this V-change will be injured.

Longer memory



- We keep a record of all previous attempts - detailed information about each witness $x_0 < x_1 < \dots$.
- Every time we attack - first take a look at what happened to previous witnesses.
- Only when we have all changes in V_0, in $V_1, \dots$ do we let the backup strategy work.
- Delayed successfulness of previous attacks.

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