A Locally Definable Set of Low Enumeration Degrees
Work in progress

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Outline

1. $\mathcal{K}$-pairs: definition and properties
2. The main theorem and the definition of a set of low degrees
3. Basic ideas in the construction

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$\mathcal{K}$-pairs: Definition

Iskander Kalimullin: Definability of the jump operator in the enumeration degrees
Journal of Mathematical Logic (2003)

Definition

Let A, B and U be sets of a ntural numbers. The pair (A, B) is a $\mathcal{K}$-pair over U (U-e-ideal) if there exists a set $W \leq_e U$ such that $A \times B \subseteq W$ and $\overline{A} \times \overline{B} \subseteq \overline{W}$.

If $U = \emptyset$ then (A, B) is called a K-pair.

$\mathcal{K}$-pairs: A trivial example

Example

Let V be a c.e set. Then (V, A) is a $\mathcal{K}$-pair for any set of natural numbers A.

Let $W = V \times \mathbb{N}$. Then $V \times A \subseteq W$ and $\overline{V} \times \overline{A} \subseteq \overline{W}$.

$\mathcal{K}$-pairs: A more interesting example

Definition (Jockusch 1968)

A set of natural numbers A is semi-recursive if there is a computable function s_A such that for every pair of natural numbers (x, y):

1. $s_A(x, y) \in \{x, y\}$.
2. If $x \in A$ or $y \in A$ then $s_A(x, y) \in A$.

Example

Let A be a semi-recursive set. Then $(A, \overline{A})$ is a $\mathcal{K}$-pair.

Let s_A be the selector function for A.
Set $W = \{(s_A(x, y), \overline{s_A(x, y)}) \mid x, y \in \mathbb{N}\}$,
where $\overline{s_A(x, y)} = x$ if $s_A(x, y) = y$ and $\overline{s_A(x, y)} = y$ if $s_A(x, y) = x$.

An order theoretic characterization of $\mathcal{K}$-pairs

Theorem (Kalimullin)

(A, B) is a $\mathcal{K}$-pair if and only if the degrees $\mathbf{a} = d_e(A)$ and $\mathbf{b} = d_e(B)$ have the following property:

$$(\forall \mathbf{c} \in \mathcal{D}_\mathbf{e})((\mathbf{a} \vee \mathbf{c}) \wedge (\mathbf{b} \vee \mathbf{c}) = \mathbf{c})$$

1. A pair of degrees $(\mathbf{a}, \mathbf{b})$ is a $\mathcal{K}$-pair
   $\Leftrightarrow$ There is a $\mathcal{K}$-pair of sets (A, B), such that $A \in \mathbf{a}$ and $B \in \mathbf{b}$.
   $\Leftrightarrow$ For all $A \in \mathbf{a}$ and $B \in \mathbf{b}$ the pair of sets (A, B) is a $\mathcal{K}$-pair.
2. If $(\mathbf{a}, \mathbf{b})$ is a $\mathcal{K}$-pair then $(\mathbf{a}, \mathbf{b})$ is a minimal pair:

$$\underbrace{(\mathbf{a} \vee \mathbf{0}_e)}_{\mathbf{a}} \wedge \underbrace{(\mathbf{b} \vee \mathbf{0}_e)}_{\mathbf{b}} = \mathbf{0}_e$$

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$\mathcal{K}$-pairs in the local structure $\mathcal{G}_e$

Lemma (Kalimullin)

Let A and B be Σ^0_2 sets with Σ^0_2 approximation $\{A^{\{s\}}\}_{s<\omega}$ and $\{B^{\{s\}}\}_{s<\omega}$ such that:

$$(\forall s)(A^{\{s\}} \subseteq A \vee B^{\{s\}} \subseteq B).$$

Then (A, B) is a $\mathcal{K}$-pair.

Proof:
Let $W = \bigcup_{s<\omega} A^{\{s\}} \times B^{\{s\}}$.
If $(x, y) \in A \times B$ then there is a stage s such that $(x, y) \in A^{\{s\}} \times B^{\{s\}}$.
If $(x, y) \in \overline{A} \times \overline{B}$ then for all stages s, $A^{\{s\}} \subseteq A$ or $B^{\{s\}} \subseteq B$, hence $x \notin A^{\{s\}}$ or $y \notin B^{\{s\}}$, so $(x, y) \notin W$.

Constructing a nontrivial $\mathcal{K}$-pair

We want to construct Σ_2^0 approximations $\{A^{\{s\}}\}$ and $\{B^{\{s\}}\}$ to sets A and B such that for all e:

$$\mathcal{N}_e^A : W_e \neq A$$

$$\mathcal{N}_e^B : W_e \neq B$$

and for alll s

$$\mathcal{K}_s : A^{\{s\}} \subseteq A \vee B^{\{s\}} \subseteq B.$$

Strategy for satisfying $\mathcal{N}$-requirements

To satisfy $\mathcal{N}_e^A$ at every stage s do the following:

1. If a witness x_e is not yet selected then select a witness x_e as a fresh number and enumerate it in $A^{\{s\}}$.
2. If $x_e \in W_e^{\{s\}}$ then extract x_e from $A^{\{s\}}$.

To satisfy $\mathcal{N}_e^B$ at every stage s do the following:

1. If a witness y_e is not yet selected then select a witness y_e as a fresh number and enumerate it in $B^{\{s\}}$.
2. If $y_e \in W_e^{\{s\}}$ then extract y_e from $B^{\{s\}}$.

Strategy for satisfying $\mathcal{N}$-requirements

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Conflict and solution

Conflict: If x_e is enumerated in $A^{\{s_{start}\}}$ and extracted from $A^{\{s_{end}\}}$ then for all t such that $s_{start} \leq t < s_{end}$, $A^{\{t\}} \not\subseteq B$. We must ensure that $B^{\{t\}} \subseteq B$.

Solution: We order the $\mathcal{N}$ requirements linearly by priority:

$$\mathcal{N}_0^A < \mathcal{N}_0^B < \mathcal{N}_1^A < \mathcal{N}_1^B < \dots$$

and add injury:
If an $\mathcal{N}$-strategy is injured then the value of its witness is cancelled and the witness is enumerated back in the corresponding set A or B.

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Final construction

At every stage s activate the first s $\mathcal{N}$-strategies in order of their priority:

If the stratey $\mathcal{N}_e^A$ is activated at stage s the:

1. If a witness x_e is not yet selected then select a witness x_e as a fresh number and enumerate it in $A^{\{s\}}$.
2. If $x_e \in W_e^{\{s\}}$ then extract x_e from $A^{\{s\}}$ and injure all lower priority strategies.

If the stratey $\mathcal{N}_e^B$ is activated at stage s the:

1. If a witness y_e is not yet selected then select a witness y_e as a fresh number and enumerate it in $B^{\{s\}}$.
2. If $y_e \in W_e^{\{s\}}$ then extract y_e from $B^{\{s\}}$ and injure all lower priority strategies.

Properties of Σ^0_2 $\mathcal{K}$-pairs

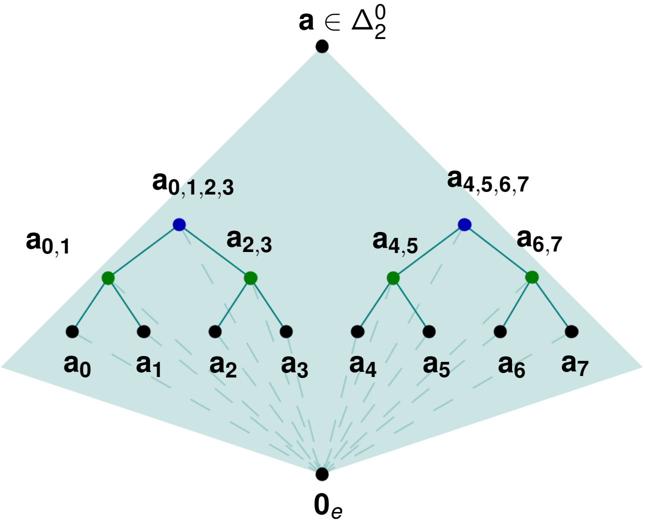
Theorem (Kallimulin)

1. *If $(\mathbf{a}, \mathbf{b})$ are a Σ^0_2 $\mathcal{K}$-pair then* **a** *and* **b** *are low and quasi-minimal.*
2. *Every Δ^0_2 enumeration degree bounds a $\mathcal{K}$-pair.*
3. *There is a Δ^0_2 $\mathcal{K}$- pair $(\mathbf{a}, \mathbf{b})$ which splits $\mathbf{0}'_e$.*
4. *The set of degrees* **c** *which form a $\mathcal{K}$-pair with a fixed degree* **a** *is an ideal.*

Theorem (Ganchev, S)

Every Δ^0_2 enumeration degree bounds a countable $\mathcal{K}$-system, i.e. a sequence $(\mathbf{a_0}, \mathbf{a_1}, \dots)$ of Δ^0_2 enumeration degrees such that for every $i \neq j$ the pair $(\mathbf{a_i}, \mathbf{a_j})$ is a $\mathcal{K}$-pair.

Properties of Σ^0_2 $\mathcal{K}$-pairs



Definability of $\mathcal{K}$-pairs

$\mathcal{K}(\mathbf{a},\mathbf{b})$= " $(\mathbf{a},\mathbf{b})$ is a $\mathcal{K}$-pair" is definable in $\mathcal{D}_e$:

$$\mathcal{K}(\mathbf{a},\mathbf{b}) \Leftrightarrow (\forall \mathbf{c})((\mathbf{a} \vee \mathbf{c}) \wedge (\mathbf{b} \vee \mathbf{c}) = \mathbf{c}).$$

Theorem (Kalimullin)

If $(d_e(A), d_e(B))$ is not a $\mathcal{K}$-pair then there is a witness C computable from $A \oplus B \oplus K$ such that:

$$(d_e(A) \vee d_e(C)) \wedge (d_e(B) \vee d_e(C))) \neq d_e(C))$$

- If A and B are Δ^0_2 then C is also Δ^0_2 and hence in the structure of the Δ^0_2 enumeration degrees the property $\mathcal{K}(\mathbf{a},\mathbf{b})$ is definable and $\{\mathbf{a} \mid (\exists \mathbf{b})\mathcal{K}(\mathbf{a},\mathbf{b})\}$ is a definable set of low degrees.
- If A and B are Σ^0_2 then C is Δ^0_3.

Question

Is the property $\mathcal{K}(\mathbf{a},\mathbf{b})$ first order definable in $\mathcal{G}_e$?

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Is the property $\mathcal{K}(\mathbf{a}, \mathbf{b})$ first order definable in $\mathcal{G}_e$?

The main theorem

Theorem (S,Wu)

Every nonzero Δ^0_2 enumeration degree $\mathbf{a}$ is low cuppable, i.e. there is a low $\mathbf{b}$ such that $\mathbf{a} \vee \mathbf{b} = \mathbf{0}'_e$.

Theorem (Cooper, Sorbi, Yi)

There are non-cuppable Σ^0_2 enumeration degrees.

Theorem (The main theorem)

If $\mathbf{u}$ and $\mathbf{v}$ are Σ^0_2 enumeration degrees such that $\mathbf{u} \vee \mathbf{v} = \mathbf{0}'_e$ then $\mathbf{u}$ is low cuppable or $\mathbf{v}$ is low cuppable.

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A definition of a low set

Corollary

The formula $\mathcal{L}(\mathbf{a}) \Leftrightarrow (\exists \mathbf{b})(\mathcal{K}(\mathbf{a}, \mathbf{b}) \wedge \mathbf{a} \vee \mathbf{b} = \mathbf{0}'_e)$ *defines in* $\mathcal{G}_e$ *a nonempty set of low enumeration degrees.*

Proof:
There is a Δ^0_2 $\mathcal{K}$-pair which splits $\mathbf{0}'_e$ so the set $\{\mathbf{a} \mid \mathcal{G}_e \vdash \mathcal{L}(\mathbf{a})\} \neq \emptyset$.

Let $\mathbf{a}$ be a Σ^0_2 degree such that $\mathcal{G}_e \vdash \mathcal{L}(\mathbf{a})$. Let $\mathbf{b}$ a witness such that $\mathcal{K}(\mathbf{a}, \mathbf{b}) \wedge \mathbf{a} \vee \mathbf{b} = \mathbf{0}'_e$.

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Proof:

Case 1: $\mathbf{b}$ is low cuppable. Let $\mathbf{c}$ be a low Δ^0_2 e-degree which cups $\mathbf{b}$.

$$(\mathbf{a} \vee \mathbf{c}) \wedge \underbrace{(\mathbf{b} \vee \mathbf{c})} = \mathbf{c}$$

$$(\mathbf{a} \vee \mathbf{c}) \wedge \quad \mathbf{0}'_e \quad = \mathbf{c}$$

$$(\mathbf{a} \vee \mathbf{c}) \qquad = \mathbf{c}$$

Hence $\mathbf{a} \leq \mathbf{c}$ and $\mathbf{a}' \leq \mathbf{c}' = \mathbf{0}'_e$ and the degree $\mathbf{a}$ is low.

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The formula $\mathcal{L}(\mathbf{a}) \Leftrightarrow (\exists \mathbf{b})(\mathcal{K}(\mathbf{a}, \mathbf{b}) \wedge \mathbf{a} \vee \mathbf{b} = \mathbf{0}_e')$ *defines in* $\mathcal{G}_e$ *a nonempty set of low enumeration degrees.*

Proof:

Case 1: **b** is low cuppable. Let **c** be a low Δ_2^0 e-degree which cups **b**.

$$(\mathbf{a} \vee \mathbf{c}) \wedge \underbrace{(\mathbf{b} \vee \mathbf{c})} = \mathbf{c}$$

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Case 2: **a** is low cuppable. Let **c** be a low Δ^0_2 e-degree which cups **a**.

$$\underbrace{(\mathbf{a} \vee \mathbf{c})} \wedge (\mathbf{b} \vee \mathbf{c}) = \mathbf{c}$$

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So $\mathbf{b} \leq \mathbf{c}$ and hence **b** is low.

Every low degree is Δ^0_2 hence low cuppable.

So **b** is low cuppable and we are back to Case 1.

A definition of a low set

Corollary

The formula $\mathcal{L}(\mathbf{a}) \Leftrightarrow (\exists \mathbf{b})(\mathcal{K}(\mathbf{a},\mathbf{b}) \land \mathbf{a} \lor \mathbf{b} = \mathbf{0}'_e)$ *defines in* $\mathcal{G}_e$ *a nonempty set of low enumeration degrees.*

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Main theorem

Theorem (The main theorem)

If $\mathbf{u}$ *and* $\mathbf{v}$ *are* Σ^0_2 *enumeration degrees such that* $\mathbf{u} \vee \mathbf{v} = \mathbf{0}'_e$ *then* $\mathbf{u}$ *is low cuppable or* $\mathbf{v}$ *is low cuppable.*

Proof:(Sketch)

Uses a construction very similar to the construction of a non-splitting enumeration degree.

Non-splitting as a strong form of cupping

Theorem (S)

There is an incomplete Σ^0_2 e-degree $\mathbf{a}$ such that there is no splitting of $\mathbf{0}'_e$ in the e-degrees above $\mathbf{a}$.

Proof idea: Construct a Π^0_1 set E and a Σ^0_2 set A such that the following requirements are satified:

$$\mathcal{N}_\Psi : \Psi_e(A) \neq E.$$

$$\mathcal{R}_{\Theta,U,V} : \Theta(U,V) = E \Rightarrow \exists \Gamma, \Lambda(\Gamma(U,A) = \overline{K} \vee \Lambda(V,A) = \overline{K}).$$

So the constructed degree $\mathbf{a} = d_e(A)$ has the following property:

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Level 1: gaining control over the give sets

- Let U and V be Σ^0_2 degrees such that $U \oplus V \equiv_e \overline{K}$. Fix good Σ^0_2 approximations to U, V and $U \oplus V$.
- We will construct a Π^0_1 set E, which we use to force changes in the approximations to U and V.
- As $U \oplus V$ has complete e-degree there is an enumeration operator Θ such that $\Theta(U, V) = E$.
- The length of agreement function $l(\Theta(U, V)^{\{s\}}, E^{\{s\}}, s)$ is unbounded in s.
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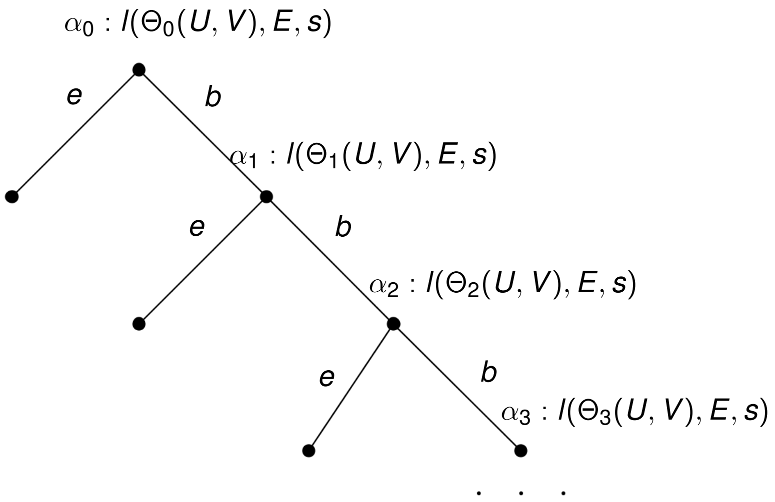

Level 2: Working below outcome e

Suppose α is monitoring $I(\Theta(U, V), E, s)$ and see infinitely many expansionary stages.

The strategy α starts the construction of a set A and an operator Γ so that $\Gamma(U, A) = \overline{K}$.

On every visit it rectifies Γ on all $n \leq s$.

1. If $n \in \overline{K}^{\{s\}}$ enumerate an axiom in Γ of the form $\langle n, U^{\{s\}} \upharpoonright u(n) + 1, A^{\{s\}} \upharpoonright a(n) + 1 \rangle$
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Making A low

Definition

A set A is 1-generic if for every c.e. set W viewed as a set of finite binary strings

$$(\exists\tau \subseteq A)(\tau \in W \vee (\forall\mu \supseteq \tau)\mu \notin W)).$$

Theorem (Copestake)

Everey Δ^0_2 1-generic set is low.

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Naive strategy for β

1. Select a threshold $d \in \overline{K}$.
2. Wait until α is done correctign the operator Γ on elements $n \leq d$ and set $\tau = A \upharpoonright a(d) + 1$.
3. If there is no extension $\mu \supseteq \tau$ such that $\mu \in W^{\{s\}}$ then the outcome is w.
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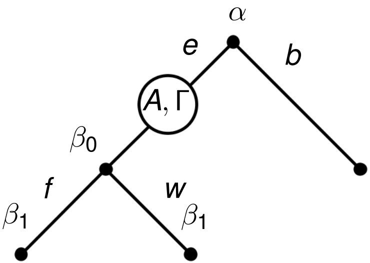

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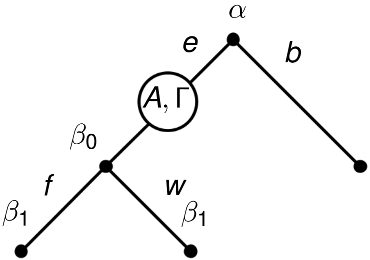

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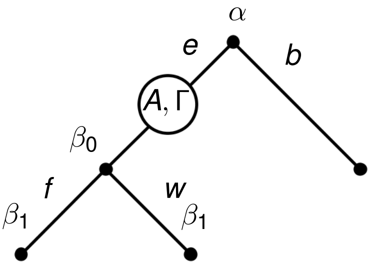

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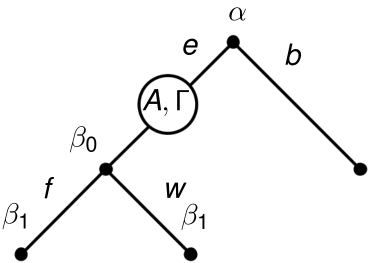


Including the set E to force changes

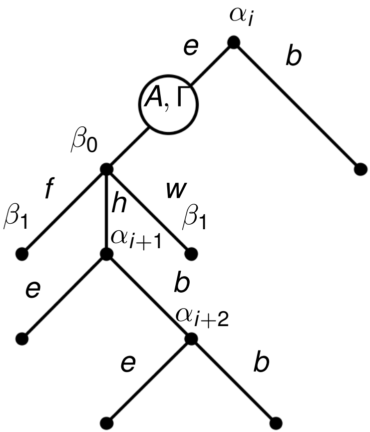
Solution: β will use an agitator.

1. Select an agitator $e \in E$.
2. As β is working under the assumption that $\Theta(U, V) = E$ so $e \in \Theta(U, V)$ with axiom $\langle e, U_e, V_e \rangle$.
3. Extracting e from the set E forces e out of $\Theta(U, V)$ and $\langle e, U_e, V_e \rangle$ is not valid at further stages. This is called an attack.
4. Arrange things so that $U_e \not\subseteq U$ invalidates all axioms in Γ for elements $n \geq d$.
5. This moves α's influence on A on elements larger than $|\mu|$ and we can preserve $\mu \subseteq A$.

Honestification before an attack

- Make sure that $u(d) > \max U_e$ so that $U_e \not\subseteq U$ ensures $\langle d, U \upharpoonright u(n)+1, A \upharpoonright a(d)+1 \rangle$ invalid.
- This must be repeated every time the valid axiom for e in Θ changes before the attack.
- If this happens infinitely many times then there are no valid axioms for e in Θ and $\Theta(U, V) \neq E$.
- A new outcome h is added to signify this.

Honestification



Transferring results from the Turing degrees

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