The hyper enumeration degrees

University of Wisconsin–Madison
Dagstuhl
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Abstract

My work heavily focuses on enumeration reducibility and the induced structure of the enumeration degrees $\mathcal{D}_e$.

Motivation for my interest in this area comes from

- its nontrivial connections to the study of the Turing degrees, specifically in relation to the automorphism problem;
- and from many interesting interactions between the structure of the enumeration degrees and notions stemming from topology and descriptive set theory.

Today I plan to tell you about a higher-order analog of enumeration reducibility that I find interesting and about which I know very little: hyper-enumeration reducibility.

The enumeration degrees: definition

Let A and B be sets of natural numbers.

Definition (Friedberg and Rogers 1959)

$A \leqslant_e B$ if there is a uniform way to enumerate A given any enumeration of B.

We ask that there is a c.e. set W such that

$$A = \{x \mid (\exists v)[\langle x, v\rangle \in W \ \& \ D_v \subseteq B]\},$$

where D_v denotes the finite set with canonical code v.

By identifying sets that are reducible to each other we get the partial order of the *enumeration degrees* $\mathcal{D}_e$:

- $A \equiv_e B$ if $A \leqslant_e B$ and $B \leqslant_e A$;
- The enumeration degree of A is $\deg_e(A) = [A]_{\equiv_e}$.
- $\deg_e(A) \leqslant \deg_e(B)$ iff $A \leqslant_e B$.

The Turing degrees within the enumeration degrees

Proposition Let A and B be sets of natural numbers.

- A is c.e. in B if and only if $A \leqslant_e B \oplus B^c$.
- $A \leqslant_T B$ if and only if $A \oplus A^c \leqslant_e B \oplus B^c$.

And so the Turing degrees embed into the enumeration degrees by $\iota : \mathcal{D}_T \to \mathcal{D}_e$ defined by

$$\iota(\deg_T(A)) = \deg_e(A \oplus A^c).$$

The image of the Turing degrees is the set of *total enumeration degrees*.

A degree is total if and only if it contains the graph of a total function.

Proposition There are nontotal enumeration degrees.

Proof We can build a non-c.e. set A such that if φ is a partial function such that $G_\varphi \leqslant_e A$ then φ has a computable extension.

The connection between $\mathcal{D}_e$ and $\mathcal{D}_T$

Theorem (Selman)

$A \leqslant_e B$ if an only if every total degree above $\deg_e(B)$ is also above $\deg_e(A)$.

Theorem (Cai, Ganchev, Lempp, Miller, S.)

The total enumeration degrees are first order definable in $\mathcal{D}_e$.

Hyperenumeration reducibility

Let A and B be sets of natural numbers.

Definition (Sanchis)

$A \leqslant_{he} B$ (A is *hyperenumeration reducible* to B) if and only if there is a c.e. set W such that

$$A = \{x \mid \forall f \in \omega^\omega \exists n \exists v [\langle f \upharpoonright n, x, v \rangle \in W \ \& \ D_v \subseteq B]\}.$$

We can think of W as defining a labelled tree $T_x \subseteq \omega^{<\omega}$ for every x and B as cutting branches whenever it sees a label D_v such that $D_v \subseteq B$. We have that $x \in A$ if and only if B reduces T_x to a well-founded tree.

Known properties of hyper-enumeration reducibility

Sanchis proved that hyperenumeration reducibility has some natural properties:

1. We can replace the c.e. set W by a Π^1_1 set and we will have the same notion.
2. Hyperenumeration reducibility is a pre-order on $\mathcal{P}(\omega)$, and so it induces $\mathcal{D}_{he}$ the hyper enumeration degrees.
3. It extends enumeration reducibility: furthermore, if $A \leqslant_e B$ then $A \leqslant_{he} B$ and $A^c \leqslant_{he} B^c$.
4. A is Π^1_1 in B if and only if $A \leqslant_{he} B \oplus B^c$.
5. $A \leqslant_h B$ if and only if $A \oplus A^c \leqslant_{he} B \oplus B^c$.

Let's call the hyperenumeration degrees of sets of the form $A \oplus A^c$ *hypertotal*.

Theorem (Sanchis)

There are non-hypertotal degrees: There is a set A that is not Π^1_1 and such that if φ is a partial function such that $G_\varphi \leqslant_{he} A$ then φ has a Π^1_1 extension.

Questions

Question

Is the hyperenumeration degree of a set A characterized by the hypertotal degrees above it?

In $\mathcal{D}_e$ assuming $A \not\leqslant_e B$ we built a sufficiently generic enumeration of B that is not above A.

Here we can't use enumerations to distinguish between hyperenumeration degrees: if f is an enumeration of A then $A \leqslant_e G_f$ and so $A \oplus A^c \leqslant_{he} G_f$.

Questions

Question

Is there a way to stratify hyperenumeration reducibility?

Recall that $A \leqslant_h B$ if there is some B-computable ordinal α such that $A \leqslant_T B^{(\alpha)}$.

In the e-degrees we have an analog of the jump operator, but it is always a total set. So if $A \leqslant_e B^{(\alpha)}$ then $A \oplus A^c \leqslant_e B^{(\alpha+1)}$, i.e. $A \leqslant_h B$.

However, we (Andrews et al) recently introduced the skip operator: an operator in the enumeration degrees that in some ways behaves more naturally. Perhaps it can be used.