The three quantifier theory of the partial order of the enumeration degrees

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The theory of a degree structure

Let $\mathcal{D}$ be a degree structure.

Question

- Is the theory of the structure in the language of partial orders decidable?
- How complicated is the theory?
- How many quantifiers does it take to break decidability?

Degree structure	Complexity of $Th(\mathcal{D})$	$\exists\forall\exists$-$Th(\mathcal{D})$	$\forall\exists$-$Th(\mathcal{D})$
$\mathcal{D}_T$	Simpson 77	Lerman-Schmerl 83	Shore 78; Lerman 83
$\mathcal{D}_T(\leqslant \mathbf{0})$	Shore 81	Lerman-Schmerl 83	Lerman-Shore 88
$\mathcal{R}$	Slaman-Harrington 80s	Lempp-Nies-Slaman 98	Open
$\mathcal{D}_e$	Slaman-Woodin 97	Open	Open
$\mathcal{D}_e(\leqslant \mathbf{0}')$	Ganchev-Soskova 12	Kent 06	Open

Related problems

- To understand what existential sentences are true $\mathcal{D}$ we need to understand what finite partial orders can be embedded into $\mathcal{D}$;
- At the next level of complexity is the *extension of embeddings problem*:

Problem

We are given a finite partial order P and a finite partial order $Q \supseteq P$. Does every embedding of P extend to an embedding of Q?

- To understand what $\forall\exists$-sentences are true in $\mathcal{D}$ we need to solve a slightly more complicated problem:

Problem

We are given a finite partial order P and finite partial orders $Q_0, \dots Q_n \supseteq P$. Does every embedding of P extend to an embedding of one of the Q_i?

The Turing degrees and initial segment embeddings

Theorem (Lerman 71)

Every finite lattice can be embedded into $\mathcal{D}_T$ as an initial segment.

- Suppose that P is a finite partial order and $Q \supseteq P$ is a finite partial order extending P.
- We can extend P to a lattice by adding extra points for joins when necessary.
- The initial segment embedding of the lattice P can be extended to an embedding of Q only if new elements in $Q \setminus P$ are compatible with joins in P:
  1. If $q \in Q \setminus P$ is bounded by some element in P then q is one of the added joins.
  2. If $x \in Q \setminus P$ and $u, v \in P$ and $x \geqslant u, v$ then $x \geqslant u \vee v$.

Theorem (Shore 78; Lerman 83)

That is the only obstacle.

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The enumeration degrees

Theorem (Gutteridge 71)

The enumeration degrees are downwards dense.

A degree $\mathbf{b}$ is a *minimal cover* of a degree $\mathbf{a}$ if $\mathbf{a} < \mathbf{b}$ and the interval $(\mathbf{a}, \mathbf{b})$ is empty.

Theorem (Slaman, Calhoun 96)

There are Π^0_2 enumeration degrees $\mathbf{a} < \mathbf{b}$ such that $\mathbf{b}$ is a minimal cover of $\mathbf{a}$

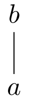
A degree $\mathbf{b}$ is a *strong minimal cover* of a degree $\mathbf{a}$ if $\mathbf{a} < \mathbf{b}$ and for every degree $\mathbf{x} < \mathbf{b}$ we have that $\mathbf{x} \leqslant \mathbf{a}$.

Theorem (Kent, Lewis-Pye, Sorbi 12)

There is a Δ^0_3 degree $\mathbf{a}$ and Π^0_2 enumeration degree $\mathbf{b}$ such that $\mathbf{b}$ is a strong minimal cover of $\mathbf{a}$

The simplest lattice

Consider the lattice $\mathcal{L} = \{a < b\}$. What properties should possible extensions $Q_0, Q_1 \dots Q_n$ have so that every embedding of $\mathcal{L}$ extends to Q_i for some i:



1. We can embed this lattice as degrees $\mathbf{a} < \mathbf{b}$ such that $\mathbf{b}$ is a strong minimal cover of $\mathbf{a}$. Thus we need at least one Q_i where all new x satisfy: if $x < b$ then $x < a$.
2. We can embed this lattice as degrees $\mathbf{0}_e < \mathbf{b}$. Thus we need at least one Q_i where all new x satisfy: if $x < b$ then $x > a$.

Theorem (Slaman, Sorbi 14)

Every countable partial order can be embedded below any nonzero enumeration degree.

So these two conditions suffice.

A wild conjecture

Conjecture (Lempp, Slaman, Soskova)

Every finite lattice can be embedded into $\mathcal{D}_e$ as an interval of Π^0_2 enumeration degrees $[\mathbf{a}, \mathbf{b}]$ so that if $\mathbf{x} \leqslant \mathbf{b}$ then $\mathbf{x} \in [\mathbf{a}, \mathbf{b}]$ or $\mathbf{x} < \mathbf{a}$.

- Note! This would only solve the extension of embeddings problem: Every embedding of P would extend to an embedding of Q if Q satisfies the same two properties: have no new degree below any member of P and respect least upper bounds.
- If we allow more than one Q then we need a wilder conjecture, e.g.:

Conjecture

Every finite lattice can be embedded into $\mathcal{D}_e$ so that:

1. If $\mathbf{x} \leqslant \mathbf{b}$, where $\mathbf{b}$ is the image of the largest element then $\mathbf{x}$ is the image of an element from the lattice or bounded by an atom of the lattice.
2. Incomparable atoms and co-atoms form minimal pairs.

This implies the existence of strong minimal pairs.

Our results: Step 1

Slightly extend the Kent, Lewis-Pye, Sorbi result:

Theorem

There are Π^0_2 degrees $\mathbf{b} < \mathbf{a}$ such that $\mathbf{a}$ is a strong minimal cover of $\mathbf{b}$.

Proof.

Construct Π^0_2 sets A and B so that:

- $\mathcal{M}_e$: $\Psi_e(A, B) = \Gamma(B)$ or $A, B \leqslant_e \Psi_e(A, B)$;
- $\mathcal{T}_e$: $A \neq \Phi_e(B)$.

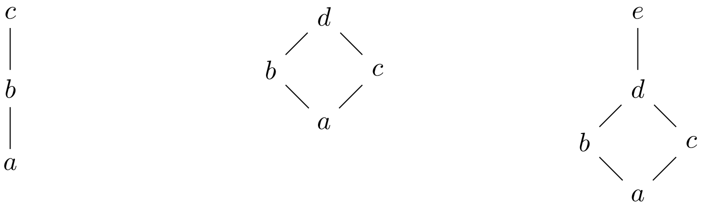
A number is in A or B if and only if it is in A_s or B_s at infinitely many stages. $\mathcal{M}_e$-strategies promise to add numbers to B if certain numbers enter A, B.

Attempts at diagonalization of $\mathcal{T}$ may fail: a witness $x \in A$ if and only if $y \in \Psi(A, B)$ influencing what a higher priority $\mathcal{M}$-strategy wants in B. Instead we produce a stream of elements $x_0, x_1, \ldots$ whose membership in A is reflected in membership in $\Psi(A, B)$. We code B using x_{2i}.

$A \leqslant_e \Psi(A, B)$ because A consists of (1) elements enumerated by higher priority strategies, (2) elements in the stream, (3) elements enumerated in A to code B at higher priority strategies.

Step 2, 3, 4

Generalize the previous construction to show that each of the following lattices can be embedded in a *strong minimal cover* way.



A small victory

Theorem (Lempp, Slaman, Soskova)

Every finite distributive lattice can be embedded as an interval $[\mathbf{a}, \mathbf{b}]$ so that if $\mathbf{x} \leqslant \mathbf{b}$ then $\mathbf{x} \in [\mathbf{a}, \mathbf{b}]$ or $\mathbf{x} \leqslant \mathbf{b}$.

Corollary

The $\exists\forall\exists$-theory of $\mathcal{D}_e$ is undecidable.

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An additional application

Theorem (Lempp, Slaman, Soskova)

The extension of embeddings problem in $\mathcal{D}_e$ is decidable.

Proof sketch:

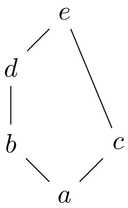
- Fix partial orders $P \subseteq Q$.
- If $q \in Q \setminus P$ is a point that violates the conditions of the usual algorithm (the one for $\mathcal{D}_T$) then we build a specific embedding that blocks q.
- We extend P to P^* by carefully adding points to make $B(q) = \{p \in P^* \mid p < q\}$ a distributive lattice and embed that strongly.
- We use generic extensions for the rest of P to make $\bigwedge A(q) = \bigvee B(q)$, where $A(q) = \{p \in P^* \mid q < p\}$.
- This leaves $\bigvee B(q)$ as the only possible position for q.

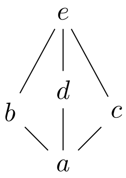
Questions

Question

Can we embed all finite lattices in $\mathcal{D}_e$ as strong intervals?

Important test cases are N_5 and M_3:





Question

Are there strong minimal pairs in $\mathcal{D}_e$: minimal pairs $\mathbf{a}$ and $\mathbf{b}$ such that all nonzero $\mathbf{x} \leqslant \mathbf{a}$ we have that $\mathbf{x} \vee \mathbf{b} \geqslant \mathbf{a}$?

Question

Can we embed all countable (distributive) lattices into $\mathcal{D}_e$ as strong intervals?

Thank you!