Cototality and the skip operator

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Fall Central Sectional Meeting, Minneapolis, Minnesota
Effective Mathematics in Discrete and Continuous Worlds

[1]Supported by the “Women in Science” program and Sofia University Science Fund.

Motivation from symbolic dynamics

Definition

- A *subshift* is a closed set $X \subseteq 2^\omega$ such that if $a\alpha \in X$ then $\alpha \in X$.
- X is *minimal* if there is no $Y \subset X$, such that Y is a subshift.

Given a minimal subshift X, we would like to characterize the set of Turing degrees of members of X.

Definition

The *language* of subshift X is the set
$L_X = \{\sigma \in 2^{<\omega} \mid \exists \alpha \in X(\sigma \text{ is a subword of } \alpha)\}$.

1. If X is minimal and $\sigma \in L_X$ then for every $\alpha \in X$, σ is a subword of α. So every element of X can enumerate the set L_X.
2. If we can enumerate L_X then we can compute a member of X.

The enumeration degrees and cototal sets

Definition

$A \leq_e B$ if every enumeration of B can compute an enumeration of A.

The enumeration degree of L_X characterizes the set of Turing degrees of members of X.

(Jaendel:) If we can enumerate the set of *forbidden words* $\overline{L_X}$ then we can enumerate L_X.

So $L_X \leq_e \overline{L_X}$.

Definition

A set A is *cototal* if $A \leq_e \overline{A}$.

Four classes of degrees

1. A set A is *total* if $\overline{A} \leq_e A$. A degree $\mathbf{a}$ is *total* if it contains a total set.

   Equivalently, $\mathbf{a}$ contains the graph of a total function G_f or even the graph of a $\{0, 1\}$-valued total function.
2. (Solon:) A degree $\mathbf{a}$ is *graph-cototal* if it contains the complement of the graph of a total function.
3. A degree $\mathbf{a}$ is *cototal* if it contains a cototal set.
4. (Solon:) A degree $\mathbf{a}$ is *Solon cototal* if it contains a set A, such that $\overline{A}$ is of total degree.

$$\text{total} \Rightarrow \text{graph-cototal} \Rightarrow \text{cototal} \Rightarrow \text{Solon cototal.}$$

Σ_2^0 enumeration degrees are graph-cototal

The degrees that contain Σ_2^0 sets are called Σ_2^0 *enumeration degrees*.

Proposition

Every Σ_2^0 e-degree is graph-cototal.

Proof.

Fix a Σ_2^0 set A and an approximation $\{A_s\}_{s<\omega}$.

$$f(a) = \begin{cases} 0, & \text{if } a \notin A; \\ \text{the least stage } s \text{ such that } a \in A_t \text{ for all } t \geq s-1, & \text{otherwise.} \end{cases}$$

□

Graph-cototal does not imply total.

Universal examples of cototal degrees

Definition

Let $G = (\omega, E)$ be a graph. A set $M \subseteq \omega$ is *independent*, if no two members of M are edge related. M is a *maximal independent* set, if it has no independent proper superset.

Any enumeration of M computes an enumeration of $\overline{M}$.

Theorem

Every cototal enumeration degree contains the complement of a maximal independent set for the graph $\omega^{<\omega}$.

Theorem

There is a maximal independent set S for $\omega^{<\omega}$, such that $\overline{S}$ does not have graph-cototal degree.

Cototal does not imply graph-cototal.

Universal examples of cototal degrees

Theorem (McCarthy)

Every cototal enumeration degree contains:

1. The complement of a maximal antichain in $\omega^{<\omega}$.
2. A perfect uniformly enumeration pointed tree.

A tree $T \subseteq 2^{<\omega}$ is *uniformly enumeration pointed* if there is a single algorithm that allows us to enumerate T given any infinite path in T.

Theorem (McCarthy)

Every cototal enumeration degree is the degree of the language of a minimal subshift.

The skip operator

Definition

$X \le_e Y$ if an only if there is a c.e. set Γ such that
$X = \Gamma(Y) = \{x \mid \exists D(\langle x, D\rangle \in \Gamma \ \wedge \ D \subseteq Y)\}$.

Let $K_A = \bigoplus_e \Gamma_e(A)$. Then $K_A \equiv_e A$.

The *enumeration jump* of A is defined by $A' = K_A \oplus \overline{K_A}$.

1. If A is cototal then $K_A \le_e A \le_e \overline{A} \le_e \overline{K_A}$.
2. If $A \le_e B$ then $\overline{K_A} \le_1 \overline{K_B}$.

Definition

The skip of A is the set $A^\Diamond = \overline{K_A}$. The skip of a degree $\mathbf{a}$ is $\mathbf{a}^\Diamond = \mathbf{d}_e(A^\Diamond)$.

Proposition

A degree $\mathbf{a}$ is cototal if and only if $\mathbf{a} \le \mathbf{a}^\Diamond$ (if and only if $\mathbf{a}^\Diamond = \mathbf{a}'$).

Skip inversion

Theorem

Let $S \geq_e \emptyset'$. There is a set A such that $A^\Diamond \equiv_e S$.

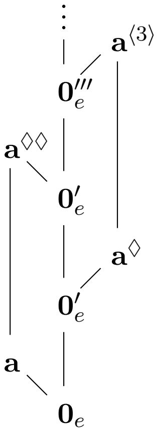
We further ensure that if S is not total then A is not of cototal degree.

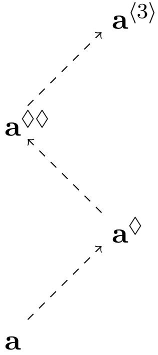
Corollary

Solon cototal does not imply *cototal*.

Start with S that is not total, but of total degree. Skip-invert to A. Then the degree of A is not cototal, but it is *Solon cototal*, because the complement of K_A is of total degree.

Iterated skips





Two properties of skips:

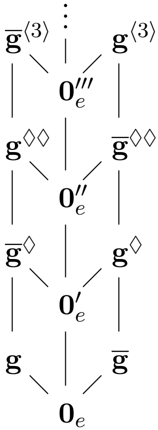
1. If $\mathbf{a} \leq \mathbf{b}$ then $\mathbf{a}^{\Diamond} \leq \mathbf{b}^{\Diamond}$;
2. $\mathbf{a} \leq \mathbf{a}^{\Diamond\Diamond}$.

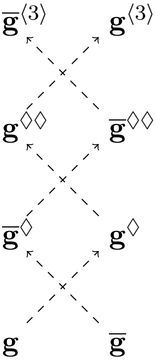
Examples of skips

Proposition

If G is generic relative to a total set X then $(G \oplus X)^{\Diamond} \equiv_e \overline{G} \oplus X'$.

If G is arithmetically generic then the skips of G and $\overline{G}$ form a double zigzag.





A skip two-cycle

Proposition

There are sets A and B such that $B = A^{\Diamond}$ and $A = B^{\Diamond}$. The sets A and B are above all hyperarithmetical sets.

The double skip operator is monotone. Apply Knaster-Tarski's fixed point theorem.

The end

Thank you!