Characterizing the continuous degrees

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Prologue

- The *continuous degrees* measure the computability-theoretic content of elements of computable metric spaces.
- They properly extend the Turing degrees and naturally embed into the enumeration degrees.
- In this talk we will see a few characterizations of the continuous degrees inside the enumeration degrees.
- Our main characterization captures the continuous degrees using a simple structural property.
- From this it follows that the continuous degrees are first-order definable in the partial order of the enumeration degrees.

The enumeration degrees

A is *enumeration reducible to* B if every enumeration of B can compute an enumeration of A.

Formally, for every function $f \in \omega^\omega$ with $\mathrm{range}(f) = B$ there is a function $g \in \omega^\omega$ with $\mathrm{range}(g) = A$ such that $g \leqslant_T f$.

Selman proved that it is equivalent to demand that f uniformly computes g.

Definition

$A \subseteq \omega$ is *enumeration reducible* to $B \subseteq \omega$ ($A \leqslant_e B$) if there is a c.e. set Δ such that

$$x \in A \iff (\exists v)(\langle x, v\rangle \in \Delta \ \&\ D_v \subseteq B).$$

The degree structure $\mathcal{D}_e$ induced by $\leqslant_e$ is called the *enumeration degrees*. It is an upper semi-lattice with a least element (the degree of all c.e. sets).

The total enumeration degrees

Proposition

$A \leqslant_T B \iff A \oplus \overline{A}$ is B-c.e. $\iff A \oplus \overline{A} \leqslant_e B \oplus \overline{B}$.

The embedding $\iota\colon \mathcal{D}_T \to \mathcal{D}_e$, defined by $\iota(d_T(A)) = d_e(A \oplus \overline{A})$, preserves the order and the least upper bound.

The *total degrees* are the image of the Turing degrees under this embedding.

Note! Consider the set of enumerations of a total set $A \oplus \overline{A}$. It has an element of least Turing degrees: $d_T(A)$.
On the other hand if the set of enumerations of a set X has an element of least Turing degree $d_T(A)$ then $X \equiv_e A \oplus \overline{A}$.

It is easy to see that nontotal degrees exist: every sufficiently generic/random set must have nontotal degree.

The simple structural property

Definition

An enumeration degree $\mathbf{a}$ is *almost total* if whenever $\mathbf{b} \nleqslant \mathbf{a}$ is total, $\mathbf{a} \vee \mathbf{b}$ is also total.

Note! Total degrees are almost total.

Are there nontotal almost total degrees?

Proposition (Cai, Lempp, Miller, S. 2014 (unpublished))

Continuous enumeration degrees are almost total.

There are nontotal continuous degrees, so there are nontotal almost total degrees. We need some background from computable analysis.

Computable analysis: the real numbers

Definition

$\lambda\colon \mathcal{Q}^+ \to \mathcal{Q}$ is a *name* of a real $x \in \mathbb{R}$ if for all rationals $\varepsilon > 0$ we have $|\lambda(\varepsilon) - x| < \varepsilon$.

Names can be easily coded as binary sequences, allowing us to transfer computability-theoretic notions. For example:

Definition

A function $f : \mathbb{R} \to \mathbb{R}$ is *computable* if there is a Turing functional that takes a name for any real $x \in \mathcal{R}$ to a name for $f(x)$.

- The binary expansion of a real x is computable from every name. (But this is nonuniform because of the dyadic rationals!)
- The binary expansion of x computes a name for x.
- This is the least Turing degree name for x; it is natural to take this as the *Turing degree* of x.

Computable metric spaces

Definition

A *computable metric space* is a metric space $\mathcal{M}$ together with a countable dense sequence $Q^{\mathcal{M}} = \{q_n^{\mathcal{M}}\}_{n\in\omega}$ on which the metric is computable (as a function $\omega^2 \to \mathbb{R}$).

Examples:

- $\mathbb{R}$ with $Q^{\mathbb{R}} = \mathbb{Q}$.
- The *Hilbert cube* $[0,1]^\omega$ with the metric
$$d(\alpha,\beta) = \sum_{n\in\omega} |\alpha(n) - \beta(n)|/2^n$$
and $Q^{[0,1]^\omega}$ the sequences of rationals in $[0,1]$ with finite support.
- $2^\omega, \omega^\omega, \mathcal{C}[0,1]$.

Definition

A *name* of a point $x \in \mathcal{M}$ is a function λ that maps positive rationals $\varepsilon > 0$ to natural numbers so that $d_{\mathcal{M}}(x, q^{\mathcal{M}}_{\lambda(\varepsilon)}) < \varepsilon$.

The continuous degrees

Recall, that real numbers $x \in \mathbb{R}$ always have a name of least Turing degree (the degree of their binary expansion).

Question (Pour-El, Lempp)

Does every point in a computable metric space have a name of least Turing degrees?

Definition (Miller 2004)

If x and y are points in computable metric spaces, then $x \leqslant_r y$ if there is a uniform way to compute a name for x from every name for y.

This reducibility induces the *continuous degrees*.

The continuous degrees within known structures

The Turing degrees can be embedded in the continuous degrees: they correspond to the continuous degrees of points in $\mathbb{R}$ (or 2^ω, or even $[0,1]$).

Proposition

Every continuous degree contains a point in $\mathcal{C}[0,1]$ and a point in $[0,1]^\omega$.

For $\alpha \in [0,1]^\omega$, let

$$C_\alpha = \bigoplus_{i \in \omega} \{q \in \mathbb{Q} \mid q < \alpha(i)\} \oplus \{q \in \mathbb{Q} \mid q > \alpha(i)\}.$$

Enumerating C_α is as hard as computing a name for α. So $\alpha \mapsto C_\alpha$ induces an embedding of the continuous degrees into the enumeration degrees.

Question (Pour El, Lempp)

Is there a *continuous enumeration degree* that is nontotal?

Nontotal continuous degrees

If α has no rational points then
$C_\alpha = \bigoplus_{i\in\omega} \{q \in \mathbb{Q} \mid q < \alpha(i)\} \oplus \{q \in \mathbb{Q} \mid q > \alpha(i)\}$ is total.

Theorem (Miller 2004)

There is a nontotal continuous degree.

Three proofs are known, all essentially topological.

- Miller uses a generalization of Brouwer's fixed point theorem to multivalued functions on $[0,1]^\omega$.
- Levin's neutral measures (1976) have nontotal continuous degree (Day and Miller 2013). Levin uses Sperner's lemma, a combinatorial analogue of the Brouwer fixed point theorem.
- Kihara & Pauly and, independently, Hoyrup gave a short proof using the fact that $[0,1]^\omega$ is not a countable union of zero dimensional subspaces.

Nontotal continuous degrees

Theorem (Miller 2004)

There is a nontotal continuous degree.

Proof

- If $x \in [0,1]^\omega$ has total degree, then there is a $y \in 2^\omega$ and Turing functionals Γ, Ψ that map (names of) x to (names of) y and back.
- The subspaces on which the functions induced by Γ and Ψ are inverses are homeomorphic (because computable functionals induce continuous functions).
- Subspaces of 2^ω are zero dimensional, so if $x \in [0,1]^\omega$ has total degree, then it is in one of *countably many* zero dimensional "patches".
- The Hilbert cube $[0,1]^\omega$ is strongly infinite dimensional, hence not a countable union of zero dimensional subspaces.
- So some $x \in [0,1]^\omega$ is not covered by one of these patches. □

Continuous degrees are almost total

An enumeration degree $\mathbf{a}$ is *almost total* if whenever $\mathbf{b} \nleqslant \mathbf{a}$ is total, $\mathbf{a} \vee \mathbf{b}$ is also total.

Proposition (Cai, Lempp, Miller, S. 2014 (unpublished))

Continuous enumeration degrees are almost total.

Proof: Take $\alpha \in [0,1]^\omega$ and $x \in [0,1]$ such that $x \nleqslant_r \alpha$. Define

$$\beta \in [0,1]^\omega \text{ by } \beta(n) = (\alpha(n) + x)/2.$$

Note that

- No component of β is rational, so β has total degree.
- $\alpha \oplus x \equiv_r \beta \oplus x$, hence it is also total. □

Metalogue

What do we know so far?

- Every continuous degrees is almost total.
- There are nontotal continuous degrees, so there are nontotal almost total degrees.
- This is the only way we know how to produce nontotal almost total degrees. (In particular, we have no "direct" construction.)

This suggests that the almost total degrees might be exactly the continuous degrees, but we originally thought that this was too surprising to be true.

Codability

Definition

$U \subseteq 2^\omega$ is a *$\Sigma^0_1\langle A\rangle$ class* if there is a set of binary strings $W \leqslant_e A$, such that

$$U = [W]^{\prec} = \{X \in 2^\omega \mid (\exists \sigma \in W)\ X \succeq \sigma\}\,.$$

A *$\Pi^0_1\langle A\rangle$ class* is the complement of a $\Sigma^0_1\langle A\rangle$ class.

Note that a $\Pi^0_1\langle A \oplus \overline{A}\rangle$ class is just a $\Pi^0_1[A]$ class in the usual sense.

Definition (AIMS)

$A \subseteq \omega$ is *(uniformly) codable* if there is a nonempty $\Pi^0_1\langle A\rangle$ class P such that every $X \in P$ (uniformly) enumerates A.

- Codability is equivalent to uniform codability.
- Uniform codability relates to work of Cai, Lempp, Miller, and S. (2014, unpublished) and Kihara and Pauly.

Almost total degrees should not exist

Goal: Given a set $A \subseteq \omega$, we try to build a sufficiently generic set $X \in 2^\omega$ so that $X \oplus \overline{X}$ witnesses that A is not cototal.

We build X by initial segments as $\bigcup_s \sigma_s$, satisfying for all e:

$$\mathcal{R}_e : \Gamma_e(A) \neq X \oplus \overline{X}.$$

$$\mathcal{P}_e : \Delta_e(A \oplus X \oplus \overline{X}) \text{ is not an enumeration of } A.$$

Suppose we have built σ and we want to extend and satisfy a requirement.

We can never fail to satisfy a requirement $\mathcal{R}_e$: Check if $2|\sigma| \in \Gamma(A)$ and if so extend by $\sigma\hat{}0$ otherwise extend by $\sigma\hat{}1$.

We can satisfy $\mathcal{P}_e$ if we can extend σ to ρ that forces $\Delta_e(A \oplus X \oplus \overline{X})$ to be:

1. A multifunction.
2. To have range outside of A.
3. To not be a total function.
4. To not enumerate all of A.

A failed forcing argument

Proposition (AIMS)

Assume that A is almost total. There is an enumeration operator Δ such that if X is sufficiently generic, then $\Delta(A \oplus X \oplus \overline{X})$ is the graph of a total function that enumerates A.

- Let $P \subseteq 2^\omega$ be the set of all B such that $A \subseteq B$ and there is no $X \in 2^\omega$ that causes $\Delta(B \oplus X \oplus \overline{X})$ to be a proper multifunction.
- P is a $\Pi^0_1\langle A\rangle$ class. It is nonempty because $A \in P$.
- If $B \in P$, then A is the set of elements in the range of $\Delta(B \oplus \sigma \oplus \overline{\sigma})$, as σ ranges over $2^{<\omega}$.

Conclusion (AIMS)

Every almost total degree is uniformly codable.

Exploiting uniform codability

Assume that A is uniformly codable as witnessed by the $\Pi^0_1\langle A\rangle$ class P and the c.e. operator W.

- If $z \in A$, then by compactness, there is a finite set $C \subseteq 2^{<\omega}$ such that $P \subseteq [C]^{\prec}$ and every member of $[C]^{\preceq}$ enumerates z via W.
- If $z \notin A$ and $C \subseteq 2^{<\omega}$ is such a set, then $P \cap [C]^{\prec} = \varnothing$.
- By compactness both facts are $\Sigma^0_1\langle A\rangle$.
- We think of finite sets $C \subseteq 2^{<\omega}$ such that $(\forall X \in [C]^{\prec})\ z \in W^X$ as potential witnesses that $z \in A$.
- If $z \in A$, then *at least one* witness is verified (positively from an enumeration of A). If $z \notin A$, then *all* witnesses are refuted $(\dots)$.
- Iterating this observation, we get the notion of *holistic sets*.

Holistic sets

Definition

$S \subseteq \omega^{<\omega}$ is *holistic* if for every $\sigma \in \omega^{<\omega}$,

1. $(\forall n)\ \sigma^\frown(2n)$ and $\sigma^\frown(2n+1)$ are not both in S,
2. If $\sigma \in S$, then $(\exists n)\ \sigma^\frown(2n+1) \in S$.
3. If $\sigma \notin S$, then $(\forall n)\ \sigma^\frown(2n) \in S$,

Think of the n's as indexing potential witnesses that $\sigma \in S$. Either:

- at least one witnesses is verified: $(\exists n)\ \sigma^\frown(2n+1) \in S$,
- or all witnesses are refuted: $(\forall n)\ \sigma^\frown(2n) \in S$.

Conclusion (AIMS)

If $A \subseteq \omega$ is uniformly codable, then there is a holistic set $S \equiv_e A$.

The holistic space

Definition

Let $\mathcal{H} = \{S \subseteq \omega^{<\omega} \colon S \text{ is holistic}\}$. A subbasis for a topology on $\mathcal{H}$ is given by

$$O_\sigma = \{S \in \mathcal{H} \colon \sigma \in S\}$$

for each $\sigma \in \omega^{<\omega}$. The resulting topological space is the *holistic space*.

Proposition (AIMS)

$\mathcal{H}$ is second countable, Hausdorff, and regular.

Therefore, $\mathcal{H}$ satisfies the hypotheses of Urysohn's metrization theorem (1925–1926):

Proposition (AIMS)

$\mathcal{H}$ is metrizable.

Effective Urysohn's theorem

Theorem (Schröder 1998)

Let $\mathcal{X}$ be a computable topological space. If $\mathcal{X}$ is Hausdorff and computably regular, then there is a computable metric on $\mathcal{X}$ that generates the original topology.

Lemma (AIMS)

$\mathcal{H}$ satisfies the hypotheses of Schröder's theorem, so it admits a computable metric d.

This metric is computable in the sense we need, i.e., if $S, T \in \mathcal{H}$, then from enumerations of S and T we can compute $d(S, T)$.

Final step

It is easy to produce a computable dense set of points in $\mathcal{H}$. Therefore:

Lemma (AIMS)

$(\mathcal{H}, d)$ is a computable metric space.

Finally, we can show:

Lemma (AIMS)

If $S \in \mathcal{H}$, then the continuous degree of S as a point in $(\mathcal{H}, d)$ is the same as the enumeration degree of S.

The main theorem

Putting it all together:

Theorem (AIMS)

Let $\mathbf{a}$ be an enumeration degree. The following are equivalent:

1. $\mathbf{a}$ is almost total,
2. the sets in $\mathbf{a}$ are (uniformly) codable,
3. $\mathbf{a}$ contains a holistic set,
4. $\mathbf{a}$ is continuous.

Definability in the enumeration degrees

Theorem (Cai, Ganchev, Lempp, Miller, and S. (2016))

The set of total enumeration degrees and the relation “c.e. in” on total degrees are first order definable in $\mathcal{D}_e$.

Corollary (AIMS)

The almost total, and hence the continuous degrees are definable in the enumeration degrees.

The relation “PA above”

Definition

Let $\mathbf{a}$ and $\mathbf{b}$ be total enumeration degrees. We say that $\mathbf{a}$ is PA above $\mathbf{b}$ if the pre-image of $\mathbf{a}$ is PA above the pre-image of $\mathbf{b}$.

Theorem (Miller (2004))

If $\mathbf{a}$ and $\mathbf{b}$ are total degrees, then $\mathbf{a}$ is PA above $\mathbf{b}$ iff there is a nontotal continuous degree $\mathbf{c} \in (\mathbf{b}, \mathbf{a})$.

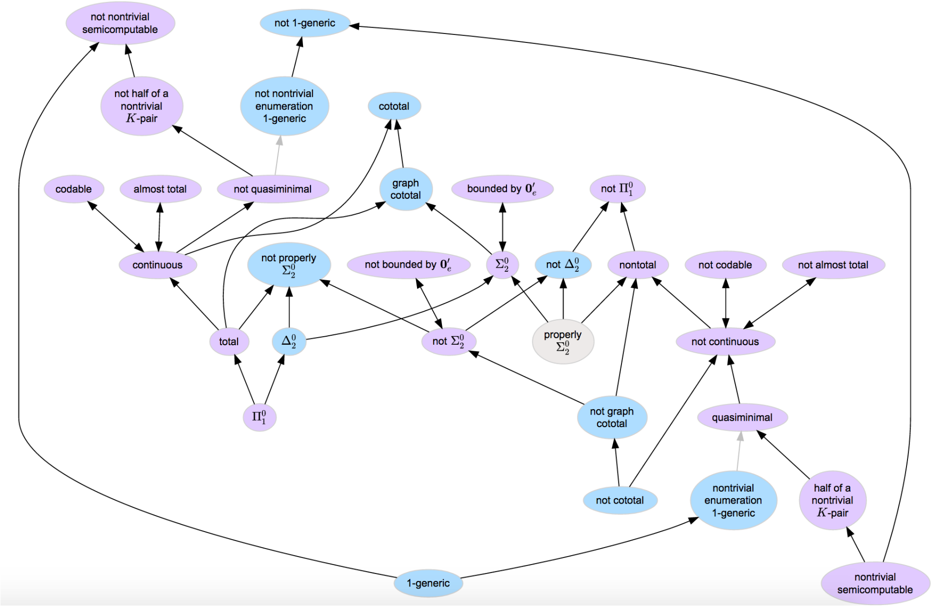
Corollary (AIMS)

The relation “$\mathbf{a}$ is PA above $\mathbf{b}$” (on total degrees) is first order definable in the enumeration degrees.

Question

Can there be a nontrivial automorphism of $\mathcal{D}_T$ that preserves both relations “c.e. in” and “PA above”?

http://ludovicpatey.com/zooviewer/



Epilogue

- We have discussed various characterizations of the continuous degrees inside the enumeration degrees.
- The main one, almost totality, is a natural structural property.
- We do not know how to directly build a nontotal almost total degree.
- All known constructions of nontotal continuous degrees involve a nontrivial topological component.

Question

Why would a structural property in the enumeration degrees reflect a topological obstruction?