Characterizing the continuous degrees

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Prologue

- The *continuous degrees* measure the computability-theoretic content of elements of computable metric spaces.
- They properly extend the Turing degrees and naturally embed into the enumeration degrees.
- In this talk we will see a few characterizations of the continuous degrees inside the enumeration degrees.
- Our main characterization captures the continuous degrees using a simple structural property.
- From this it follows that the continuous degrees are first-order definable in the partial order of the enumeration degrees.

The enumeration degrees

Definition

$A \subseteq \omega$ is *enumeration reducible* to $B \subseteq \omega$ ($A \leq_e B$) if there is a c.e. set Δ such that

$$x \in A \iff (\exists v)(\langle x, v\rangle \in \Delta \ \&\ D_v \subseteq B).$$

The degree structure $\mathcal{D}_e$ induced by $\leq_e$ is called the *enumeration degrees*. It is an upper semi-lattice with a least element (the degree of all c.e. sets).

Proposition

$A \leq_T B \iff A \oplus \overline{A}$ is B-c.e. $\iff A \oplus \overline{A} \leq_e B \oplus \overline{B}$.

The embedding $\iota\colon \mathcal{D}_T \to \mathcal{D}_e$, defined by

$$\iota(d_T(A)) = d_e(A \oplus \overline{A}),$$

preserves the order and the least upper bound.

The *total degrees* are the image of the Turing degrees under this embedding.

Computable metric spaces

Definition

A *computable metric space* is a metric space $\mathcal{M}$ together with a countable dense sequence $Q^{\mathcal{M}} = \{q_n^{\mathcal{M}}\}_{n\in\omega}$ on which the metric is computable (as a function $\omega^2 \to \mathcal{R}$).

Examples:

- $\mathbb{R}$ with $Q^{\mathbb{R}} = \mathbb{Q}$.
- The *Hilbert cube* $[0,1]^\omega$ with the metric
$$d(\alpha,\beta) = \sum_{n\in\omega} |\alpha(n) - \beta(n)|/2^n$$
and $Q^{[0,1]^\omega}$ the sequences of rationals in $[0,1]$ with finite support.
- 2^ω, ω^ω, $\mathcal{C}[0,1]$.

Definition

A *name* of a point $x \in \mathcal{M}$ is a function λ that maps positive rationals $\varepsilon > 0$ to natural numbers so that $d_{\mathcal{M}}(x, q^{\mathcal{M}}_{\lambda(\varepsilon)}) < \varepsilon$.

The continuous degrees

Definition (Miller 2004)

If x and y are points in computable metric spaces, then $x \leq_r y$ if there is a uniform way to compute a name for x from every name for y. This reducibility induces the *continuous degrees*.

Turing degrees correspond to continuous degrees of points in $\mathbb{R}$ (or 2^ω).

Proposition

Every continuous degree contains a point in $\mathcal{C}[0,1]$ and a point in $[0,1]^\omega$.

For $\alpha \in [0,1]^\omega$, let

$$C_\alpha = \bigoplus_{i \in \omega} \{q \in \mathbb{Q} \mid q < \alpha(i)\} \oplus \{q \in \mathbb{Q} \mid q > \alpha(i)\}.$$

Enumerating C_α is as hard as computing a name for α. So $\alpha \mapsto C_\alpha$ induces an embedding of the continuous degrees into the enumeration degrees.

Nontotal continuous degrees

If α has no rational points then
$C_\alpha = \bigoplus_{i\in\omega} \{q \in \mathbb{Q} \mid q < \alpha(i)\} \oplus \{q \in \mathbb{Q} \mid q > \alpha(i)\}$ is total.

Theorem (Miller 2004)

There is a nontotal continuous degree.

Three proofs are known, all essentially topological.

- Miller uses a generalization of Brouwer's fixed point theorem to multivalued functions on $[0,1]^\omega$.
- Levin's neutral measures (1976) have nontotal continuous degree (Day and Miller 2013). Levin uses Sperner's lemma, a combinatorial analogue of the Brouwer fixed point theorem.
- Kihara & Pauly and, independently, Hoyrup gave a short proof using the fact that $[0,1]^\omega$ is not a countable union of zero dimensional subspaces.

The simple structural property

Definition

An enumeration degree $\mathbf{a}$ is *almost total* if whenever $\mathbf{b} \not\leq \mathbf{a}$ is total, $\mathbf{a} \vee \mathbf{b}$ is also total.

Note! The join of any two total degrees is total, so total degrees are almost total.

Are there nontotal almost total degrees?

Continuous degrees are almost total

Proposition (Cai, Lempp, Miller, S. 2014 (unpublished))

Continuous enumeration degrees are almost total.

Proof: Take $\alpha \in [0,1]^\omega$ and $x \in [0,1]$ such that $x \not\leq_r \alpha$. Define

$$\beta \in [0,1]^\omega \text{ by } \beta(n) = (\alpha(n) + x)/2.$$

Note that

- No component of β is rational, so β has total degree.
- $\alpha \oplus x \equiv_r \beta \oplus x$, hence it is also total. □

There are nontotal continuous degrees, so there are nontotal almost total degrees. This is the only way we know how to produce nontotal almost total degrees. (In particular, we have no “direct” construction.)

Codability

Definition

$U \subseteq 2^\omega$ is a *$\Sigma^0_1\langle A\rangle$ class* if there is a set of binary strings $W \leq_e A$, such that

$$U = [W]^\prec = \{X \in 2^\omega \mid (\exists \sigma \in W)\ X \succeq \sigma\}.$$

A *$\Pi^0_1\langle A\rangle$ class* is the complement of a $\Sigma^0_1\langle A\rangle$ class.

Note that a $\Pi^0_1\langle A \oplus \overline{A}\rangle$ class is just a $\Pi^0_1[A]$ class in the usual sense.

Definition

$A \subseteq \omega$ is *(uniformly) codable* if there is a nonempty $\Pi^0_1\langle A\rangle$ class P such that every $X \in P$ (uniformly) enumerates A.

- Codability is equivalent to uniform codability.
- Uniform codability relates to work of Cai, Lempp, Miller, and S. (2014, unpublished) and Kihara and Pauly.

Almost total degrees should not exist

Proposition (AIMS)

Assume that $A \neq \emptyset$ is almost total. There is an enumeration operator Δ such that if X is sufficiently generic, then $\Delta(A \oplus X \oplus \overline{X})$ is the graph of a total function that enumerates A.

- Let $P \subseteq 2^\omega$ be the set of all B such that $A \subseteq B$ and there is no $X \in 2^\omega$ that causes $\Delta(B \oplus X \oplus \overline{X})$ to be a proper multifunction.
- P is a $\Pi^0_1\langle A \rangle$ class. It is nonempty because $A \in P$.
- If $B \in P$, then A is the set of element in the range of $\Delta(B \oplus \sigma \oplus \overline{\sigma})$, as σ ranges over $2^{<\omega}$.

Conclusion (AIMS)

Every almost total degree is uniformly codable.

Exploiting uniform codability

Assume that A is uniformly codable as witnessed by the $\Pi^0_1\langle A\rangle$ class P and the c.e. operator W.

- If $z \in A$, then by compactness, there is a finite set $C \subseteq 2^{<\omega}$ such that $P \subseteq [C]^{\prec}$ and every member of $[C]^{\preceq}$ enumerates z via W.
- If $z \notin A$ and $C \subseteq 2^{<\omega}$ is such a set, then $P \cap [C]^{\prec} = \emptyset$.
- By compactness both facts are $\Sigma^0_1\langle A\rangle$.
- Iterating this observation, we get the notion of *holistic sets*.

Holistic sets

Definition

$S \subseteq \omega^{<\omega}$ is *holistic* if for every $\sigma \in \omega^{<\omega}$,

1. $(\forall n)\ \sigma^\frown(2n)$ and $\sigma^\frown(2n+1)$ are not both in S,
2. If $\sigma \in S$, then $(\exists n)\ \sigma^\frown(2n+1) \in S$.
3. If $\sigma \notin S$, then $(\forall n)\ \sigma^\frown(2n) \in S$,

Think of the n's as indexing potential witnesses that $\sigma \in S$. Either:

- at least one witnesses is verified: $(\exists n)\ \sigma^\frown(2n+1) \in S$,
- or all witnesses are refuted: $(\forall n)\ \sigma^\frown(2n) \in S$.

Conclusion (AIMS)

If $A \subseteq \omega$ is uniformly codable, then there is a holistic set $S \equiv_e A$.

The holistic space

Definition

Let $\mathcal{H} = \{S \subseteq \omega^{<\omega} \colon S \text{ is holistic}\}$. A subbasis for a topology on $\mathcal{H}$ is given by

$$O_\sigma = \{S \in \mathcal{H} \colon \sigma \in S\}$$

for each $\sigma \in \omega^{<\omega}$. The resulting topological space is the *holistic space*.

$\mathcal{H}$ is second countable, Hausdorff, and regular, therefore, satisfies the hypotheses of Urysohn's metrization theorem (1925–1926).

Theorem (Schröder 1998)

Let $\mathcal{X}$ be a computable topological space. If $\mathcal{X}$ is Hausdorff and computably regular, then there is a computable metric on $\mathcal{X}$ that generates the original topology.

The final step in our proof

Lemma (AIMS)

$\mathcal{H}$ satisfies the hypotheses of Schröder's theorem, so it admits a computable metric d.

It is easy to produce a computable dense set of points in $\mathcal{H}$. Therefore:

Lemma (AIMS)

$(\mathcal{H}, d)$ is a computable metric space.

Finally, we can show:

Lemma (AIMS)

If $S \in \mathcal{H}$, then the continuous degree of S as a point in $(\mathcal{H}, d)$ is the same as the enumeration degree of S.

The main theorem

Putting it all together:

Theorem (AIMS)

Let $\mathbf{a}$ be an enumeration degree. The following are equivalent:

1. $\mathbf{a}$ is almost total,
2. the sets in $\mathbf{a}$ are (uniformly) codable,
3. $\mathbf{a}$ contains a holistic set,
4. $\mathbf{a}$ is continuous.

Definability in the enumeration degrees

Cai, Ganchev, Lempp, Miller, and S. (2016) proved that the total degrees and the relation "c.e. in" on total degrees are first order definable in $\mathcal{D}_e$.

Corollary (AIMS)

The continuous degrees are definable in the enumeration degrees.

Miller (2004) proved that if $\mathbf{a}$ and $\mathbf{b}$ are total degrees, then $\mathbf{a}$ is PA above $\mathbf{b}$ iff there is a nontotal continuous degree $\mathbf{c} \in (\mathbf{b}, \mathbf{a})$.

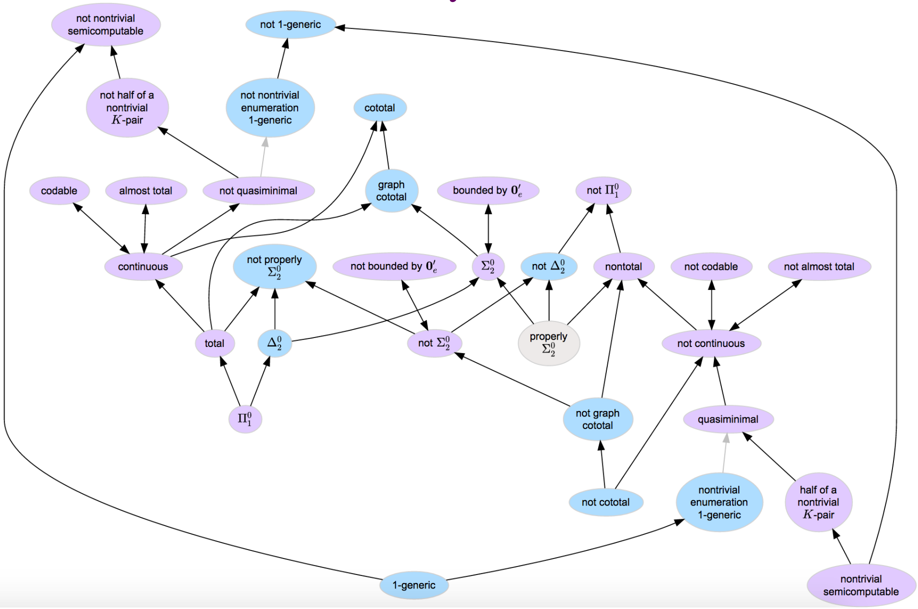
Corollary (AIMS)

The relation "$\mathbf{a}$ is PA above $\mathbf{b}$" (on total degrees) is first order definable in the enumeration degrees.

Question

Can there be a nontrivial automorphism of $\mathcal{D}_T$ that preserves both relations "c.e. in" and "PA above"?

Thank you!



not nontrivial semicomputable
not 1-generic
not half of a nontrivial K-pair
not nontrivial enumeration 1-generic
cototal
codable
almost total
not quasiminimal
graph cotDotal
bounded by $\mathbf{0}'_e$
not Π^0_1
continuous
not properly Σ^0_2
not bounded by $\mathbf{0}'_e$
Σ^0_2
not Δ^0_2
nontotal
not codable
not almost total
total
Δ^0_2
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