The enumeration degrees zoo

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Abstract

Enumeration reducibility captures a natural relationship between sets of natural numbers in which positive information about the first set is used to produce positive information about the second set.

By identifying sets that are reducible to each other we obtain an algebraic representation of this reducibility as a partial order: the structure of the enumeration degrees $\mathcal{D}_e$.

Motivation for the interest in this area comes from its nontrivial connections to the study of the Turing degrees and computable mathematics.

We will outline a series of interactions between the structure of the enumeration degrees and notions stemming from topology and descriptive set theory that give rise to a zoo of subclasses within the enumeration degrees.

Enumeration reducibility

Definition (Friedberg and Rogers 1959)

$A \leqslant_e B$ if there is a program that transforms an enumeration of B (i.e. a function on the natural numbers with range B) to an enumeration of A.

The program is a c.e. table of axioms of the sort:

$$\textit{If } \{x_1, x_2, \ldots, x_k\} \subseteq B \textit{ then } x \in A.$$

Compare this to Turing reducibility, which can be defined as follows:
$A \leqslant_T B$ if there is a c.e. table of axioms of two sorts:

$$\textit{If } \{x_1, x_2, \ldots, x_k\} \subseteq B \textit{ and } \{y_1, y_2, \ldots, y_n\} \subseteq B^c \textit{ then } x \in A.$$
$$\textit{If } \{x_1, x_2, \ldots, x_k\} \subseteq B \textit{ and } \{y_1, y_2, \ldots, y_n\} \subseteq B^c \textit{ then } x \in A^c.$$

Of course this c.e. table must not give contradictory instructions.

Proposition. $A \leqslant_T B$ if and only if $A \oplus A^c \leqslant_e B \oplus B^c$.

Degree structures

Definition

1. $A \equiv B$ if and only if $A \leqslant B$ and $B \leqslant A$.
2. $\mathbf{d}(A) = \{B \mid A \equiv B\}$.
3. $\mathbf{d}(A) \leqslant \mathbf{d}(B)$ if an only if $A \leqslant B$.
4. Let $A \oplus B = \{2n \mid n \in A\} \cup \{2n+1 \mid n \in B\}$. Then $\mathbf{d}(A \oplus B) = \mathbf{d}(A) \vee \mathbf{d}(B)$.

And so we have two partial orders with least upper bound:

1. The Turing degrees $\mathcal{D}_T$ with least element $\mathbf{0}_T$ consisting of all computable sets.
2. The enumeration degrees $\mathcal{D}_e$ with least element $\mathbf{0}_e$ consisting of all computably enumerable sets..

The total enumeration degrees

Proposition. $A \leqslant_T B$ iff $A \oplus A^c \leqslant_e B \oplus B^c$.

This suggests a natural embedding of the Turing degrees into the enumeration degrees.

Proposition. The function $\iota\colon \mathcal{D}_T \to \mathcal{D}_e$, where

$$\iota(d_T(A)) = d_e(A \oplus A^c),$$

is an embedding of $\mathcal{D}_T$ into $\mathcal{D}_e$.

Definition

A set A is *total* if $A \geqslant_e A^c$ (or equivalently if $A \equiv_e A \oplus A^c$). An enumeration degree is *total* if it contains a total set.

The image of the Turing degrees under the embedding ι is exactly the set of total enumeration degrees.

Theorem (Medvedev 1955). There are *quasiminimal* degrees—degrees that are nontotal and do not bound any nonzero total degree.

Definability of the total enumeration degrees

Definition (Kalimullin 2003)

A pair of sets $\{A, B\}$ is a *$\mathcal{K}$-pair relative to U* if and only if there is a set $W \leqslant_e U$ such that $A \times B \subseteq W$ and $A^c \times B^c \subseteq W^c$.

Example. If A is a left cut in a linear ordering L on ω then $\{A, A^c\}$ is a $\mathcal{K}$-pair relative to L as witnessed by the set $\{(n, m) \mid n \leqslant_L m\}$.

Theorem(Kalimullin 2003). $\{A, B\}$ is a $\mathcal{K}$-pair relative to U if and only if the degrees $\deg_e(A) = \mathbf{a}$, $\deg_e(B) = \mathbf{b}$ and $\deg_e(U) = \mathbf{u}$ satisfy:

$$(\forall \mathbf{x} \geqslant \mathbf{u})[(\mathbf{a} \vee \mathbf{x}) \wedge (\mathbf{b} \vee \mathbf{x}) = \mathbf{x}.]$$

Theorem(Cai, Ganchev, Lempp, Miller, S 2016). A nonzero degree $\mathbf{x}$ is total if and only if $\mathbf{x} = \mathbf{a} \vee \mathbf{b}$ for a *maximal* $\mathcal{K}$-pair $\{\mathbf{a}, \mathbf{b}\}$.

The complexity of points in computable metric spaces

Definition (Lacombe 1957)

A *computable metric space* is a metric space $\mathcal{M}$ together with a countable dense sequence $Q^{\mathcal{M}} = \{q_n^{\mathcal{M}}\}_{n\in\omega}$ on which the metric is computable.

Examples. 2^ω, ω^ω, $\mathbb{R}$, $\mathcal{C}[0,1]$, and *Hilbert cube* $[0,1]^\omega$.

Definition

$\lambda\colon \mathbb{Q}^+ \to \omega$ is a *name* of a point $x \in \mathcal{M}$ if for all rationals $\varepsilon > 0$ we have $d_{\mathcal{M}}(x, q^{\mathcal{M}}_{\lambda(\varepsilon)}) < \varepsilon$.

Note that points in $2^\omega, \omega^\omega$ and $\mathbb{R}$ have a name of least Turing degree. For example, if $r \in \mathbb{R}$ then consider the degree of $\{q \in \mathbb{Q} \mid q < r\} \oplus \{q \in \mathbb{Q} \mid q > r\}$.

Question (Pour-El, Lempp). Does every point in a computable metric space have a name of least Turing degree?

The continuous degrees

Definition (Miller 2004)

If x and y are members of (possibly different) computable metric spaces, then $x \leqslant_r y$ if there is a uniform way to compute a name for x from a name for y.

This reducibility induces the *continuous degrees* $\mathcal{D}_r$.

Theorem (Miller 2004). Every continuous degree contains a point from $[0,1]^\omega$ and a point from $C[0,1]$.

Nontotal continuous degrees exist

For $\alpha \in [0,1]^\omega$, let

$$C_\alpha = \bigoplus_{i\in\omega} \{q \in \mathbb{Q} \mid q < \alpha(i)\} \oplus \{q \in \mathbb{Q} \mid q > \alpha(i)\}.$$

Observation. Enumerating C_α is exactly as hard as computing a name for α. So $\alpha \mapsto C_\alpha$ induces an embedding of the continuous degrees into the enumeration degrees.

Elements of 2^ω, ω^ω, and $\mathbb{R}$ are mapped onto the *total* degree of their least Turing degree name. And so we have

$$\mathcal{D}_T \hookrightarrow \mathcal{D}_r \hookrightarrow \mathcal{D}_e.$$

Theorem (Miller 2004). There is a nontotal continuous degree.

Every known proof of this result uses nontrivial topological facts: Brouwer's fixed point theorem for multivalued functions on an infinite dimensional space, or Sperner's lemma, or results from topological dimension theory: $[0,1]^\omega$ is strongly infinite dimensional.

Topology realized as a structural property

Theorem (Andrews, Igusa, Miller, S 2018). The continuous degrees are definable in $\mathcal{D}_e$: they are the degrees that join every total degree not below them to a total degree—*the almost total degrees*.

Uses the effective version of Urysohn's metrization theorem proved by Schröder (1998).

Theorem (Ganchev, Kalimullin, Miller, S 2019). An enumeration degree is continuous if and only if it is not half of a nontrivial $\mathcal{K}$-pair relative to any degree $\mathbf{x}$.

It follows that there is a structural dichotomy: an e-degree is either continuous or quasiminimal relative to some total degree.

Theorem (Miller 2018). The existence of nontotal almost total degrees in every cone implies that $[0,1]^\omega$ is strongly infinite dimensional.

The enumeration degree zoo

Definition (Kihara, Pauly 2018)

A *represented space* is a pair of a second countable T_0 space X and a listing of an open basis $B^X = \{B_i\}_{i<\omega}$.

A *name* for a point $x \in X$ is an enumeration of the set $N_x = \{i \mid x \in B_i\}$.

For $x, y \in X$, say that $x \leqslant y$ if every name for y (uniformly) computes a name for x.

Thus a represented space X gives rise to a class of e-degrees $\mathcal{D}_X \subseteq \mathcal{D}_e$.

Examples:.

- $\mathcal{D}_{2^\omega} = \mathcal{D}_{\mathbb{R}}$ is the total enumeration degrees.
- $\mathcal{D}_{[0,1]^\omega}$ is the continuous degrees.
- $\mathcal{D}_{S^\omega} = \mathcal{D}_e$, where S is the Sierpinski topology $\{\varnothing, \{1\}, \{0,1\}\}$.

Kihara, Ng, and Pauly (2019) investigate $\mathcal{D}_X$, where X is the ω-power of the: cofinite topology on ω, telophase space, double origin space, quasi-Polish Roy space, irregular lattice space.

The semicomputable sets

Definition (Jockusch 1968)

A set A is *semicomputable* if and only if there is a computable selector function $s_A : \omega^2 \to \omega$ such that $s_A(x, y) \in \{x, y\}$ and if $\{x, y\} \cap A \neq \varnothing$ then $s_A(x, y) \in A$.

Equivalently, A is a left cut in a computable linear ordering.

Theorem (Kihara, Pauly 2018). $\mathcal{D}_{\mathbb{R}^<}$, where $\mathbb{R}^<$ is the real line with topology generated by $\{(q, \infty)\}_{q \in \mathbb{Q}}$, is exactly the semicomputable degrees.

Theorem (Cai, Ganchev, Lempp, Miller, S 2016). The semicomputable degrees are first order definable as the halves of maximal nontrivial $\mathcal{K}$-pairs.

Question. Can $\mathcal{K}$-pairs be captured through a topological definition?

PA relative to an enumeration oracle

Definition

A set $P \subseteq 2^\omega$ is a Π^0_1 *class relative to the enumeration oracle* A or a $\Pi^0_1\langle A\rangle$*-class* if there is a set $W \subseteq 2^{<\omega}$ such that $W \leqslant_e A$ and $P = \{\alpha \in 2^\omega \mid \forall\sigma(\sigma \in W \rightarrow \sigma \not\preccurlyeq \alpha)\}$.

So a Π^0_1 class relative to a Turing oracle A is just a Π^0_1 class relative to the enumeration oracle $A \oplus A^c$.

Recall that B is PA relative to A if and only if B computes a member of any nonzero $\Pi^0_1(A)$.

Definition

We will say that $\langle B\rangle$ *is PA relative to* $\langle A\rangle$ if B enumerates (the set of initial segments of) a member of any nonzero $\Pi^0_1\langle A\rangle$.

Question. Which oracles behave "well" with respect to the relation PA?

Good oracles and bad oracles

If X has continuous degree then there is a nonempty $\Pi^0_1\langle X\rangle$ class whose members uniformly enumerate X.

Theorem(Ganchev, Kalimullin, Miller, S 2019). If X has continuous degree:

1. If $\langle Y\rangle$ is PA relative to $\langle X\rangle$ then $X \leqslant_e Y$—we will say that X is *PA-bounded*.
2. There is a $\Pi^0_1\langle X\rangle$ class P such that for every nonempty $\Pi^0_1\langle X\rangle$ class Q every path in P uniformly enumerates a path in Q—the oracle X *has a universal class*.

Theorem(Franklin, Lempp, Miller, Schweber, and S 2019). The continuous degrees are exactly the PA bounded enumeration degrees.

Theorem(Miller, S 2014). There are enumeration oracles X such that $\langle X\rangle$ is PA relative to $\langle X\rangle$. We call these *⟨self⟩-PA oracles*.

If X is ⟨self⟩-PA then X cannot have a universal class P:otherwise, let $Y \leqslant_e X$ represent a member of P. $Y \equiv_e Y \oplus Y^c$ so every $\Pi^0_1(Y)$-class is a $\Pi^0_1\langle X\rangle$ class and Y computes a member of it.

Other ways to have a universal class

Definition

An enumeration oracle $\langle A \rangle$ is *low for PA* if every set $X \oplus X^c$ that is PA (in the Turing sense) is PA relative to $\langle A \rangle$.

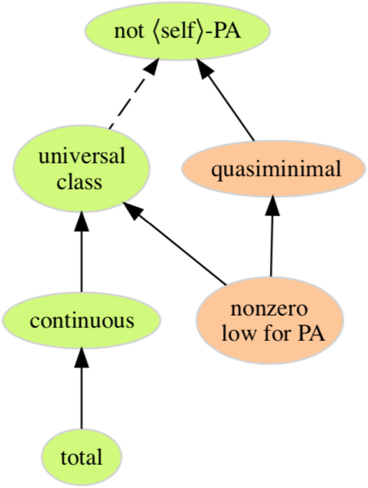
Low for PA oracles have a universal class (e.g. DNC_2) and are quasiminimal.

Theorem(Goh, Kalimullin, Miller, S). The following classes of enumeration oracles are low for PA.

1. The 1-generic degrees.
2. Halves of nontrivial $\mathcal{K}$-pairs (and hence degrees of semi-computable sets).

Note that these two classes are disjoint from the continuous degrees as they consist of quasiminimal degrees.

The picture so far



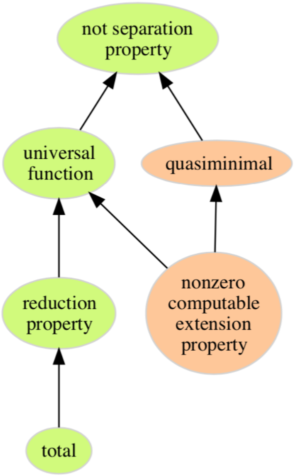
Notions from descriptive set theory

Definition (Kalimullin, Puzarenko 2005)

Let X be an enumeration oracle.

1. X has the *reduction property* if for all pairs of set $A, B \leqslant_e X$ there are sets $A_0, B_0 \leqslant_e X$ such that $A_0 \subseteq A$, $B_0 \subseteq B$, $A_0 \cap B_0 = \varnothing$, and $A_0 \cup B_0 = A \cup B$;
2. X has the *uniformization property* if whenever $R \leqslant_e X$ is a binary relation there is a function f with graph $G_f \leqslant_e X$ such that $dom(f) = dom(R)$.
3. X has the *separation property* if for every pair of disjoint sets $A, B \leqslant_e X$ there is a separator C such that $A \subseteq C$, $B \subseteq C^c$, and $C \oplus C^c \leqslant_e X$.
4. X has the *computable extension* property if every partial function φ with $G_\varphi \leqslant_e X$ has a (partial) computable extension $\psi \subseteq \varphi$.
5. X has a *universal function* if there is a partial function U with $G_U \leqslant_e X$ such that if φ is a partial function with $G_\varphi \leqslant_e X$ then for some e we have that $\varphi = \lambda x.U(e, x)$

Kalimullin and Puzarenko's theorem



The reduction property

X has the *reduction property* if whenever $A, B \leqslant_e X$ there are disjoint $A_0, B_0 \leqslant_e X$ with $A_0 \subseteq A$, $B_0 \subseteq B$, and $A_0 \cup B_0 = A \cup B$;

We want to construct a $\Pi^0_1\langle X\rangle$ class U such that if $P_e = 2^\omega \smallsetminus [\Gamma_e(X)]$ then the e-th column in any member of U uniformly codes a member of P_e.

If X were total we would fix enumerations of $\Gamma_e(X)$ relative to X and let U be the class of separators for

1. The set A of all $\langle e, \sigma\rangle$ such that all extensions of $\sigma 0$ are enumerated in $\Gamma_e(X)$ *first*.
2. The set B of all $\langle e, \sigma\rangle$ such that all extensions of $\sigma 1$ are enumerated in $\Gamma_e(X)$ *first*.

If X is not total then we cannot fix the order in which $\Gamma_e(X)$ is enumerated and so must not use the word *first*!

But then for σ with no extension in P_e we will have $\langle e, \sigma\rangle \in A \cap B$.
The reduction property lets us solve exactly this problem!

Theorem(Goh, Kalimullin, Miller, S). The reduction property implies having a universal class.

Separation classes

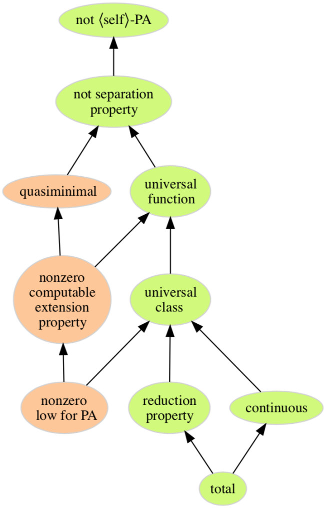
Definition

A $\Pi^0_1\langle X\rangle$ class P is a *separation class* if $P = \{C \mid A \subseteq C \;\&B \subseteq C^c\}$ for some disjoint $A, B \leqslant_e X$. Call such classes *Sep*$\langle X\rangle$ for short.

Theorem(Goh, Kalimullin, Miller, S). Let X be an enumeration oracle.

1. X has the separation property if and only if X is $\langle$self$\rangle$-Sep$\langle X\rangle$, i.e. X enumerates a member in any Sep$\langle X\rangle$-class.
2. X has the computable extension property if and only if X is low for Sep$\langle X\rangle$-classes.
3. X has a universal function if and only if X has a universal class for Sep$\langle X\rangle$-classes, i.e., there is a (separation) class U such that for every separation class P_e there is a uniform way to compute a path in P_e from any path in U.

A summary of the results by Goh, Kalimullin, Miller, and Soskova



A challenge

Question. Can we find a uniform topological classification of the new members of the zoo?

Thank you!