The relationship between local and global structure

Mariya I. Soskova
University of Wisconsin–Madison

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Abstract

This talk is about:

- The relationship between degree structures and arithmetic.
- The coding theorem.
- The biinterpretability conjecture in local and global structures.
- The enumeration degrees, where certain first order definability facts lead to interesting local and global structural interactions.

Interpreting the Turing degrees in arithmetic

Second order arithmetic $\mathcal{Z}_2$ is the theory of $(\mathbb{N}, 0, 1, +, \times, \leqslant)$ with an additional sort for sets of natural numbers and a membership relation.

Classical results due to Kleene and Post show that the relations $\leqslant_T$ and $C(X)$: "X is computable" are definable in second order arithmetic.

Thus we can translate statements about the Turing degrees (in the language of partial orderings) to statements about $\mathcal{Z}_2$.

Example

The statement: there exists a minimal Turing degrees

$$(\exists \mathbf{x} \neq \mathbf{0}_T)(\forall \mathbf{y})[\mathbf{y} \leqslant \mathbf{x} \rightarrow \mathbf{y} = \mathbf{0_T} \vee \mathbf{y} = \mathbf{x}]$$

translates to

$$(\exists X)[\neg C(X) \ \& \ (\forall Y)[Y \leqslant_T X \rightarrow C(Y) \vee Y \equiv_T X]].$$

The coding theorem

Theorem (Simpson 77)

The theory of $\mathcal{D}_T$ is computably isomorphic to $\mathcal{Z}_2$.

One way to prove this is using the coding theorem:

Theorem (Slaman, Woodin 1986)

Every countable relation $\mathcal{R} \subseteq \mathcal{D}_T^n$ is uniformly in n definable from finitely many parameters, i.e. for every n there is a fix length $|\vec{\mathbf{p}}|$ and a formula φ_n such that if $\mathcal{R} \subseteq \mathcal{D}_T^n$ then there are parameters $\vec{\mathbf{p}}$ of that length so that

$$\vec{\mathbf{a}} \in \mathcal{R} \leftrightarrow \mathcal{D}_T \models \varphi_n(\mathbf{a}, \vec{\mathbf{p}}).$$

Example

Every countable antichain $\mathcal{A}$ in $\mathcal{D}_T$ can be coded by three parameters $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$:

$\mathbf{x} \in \mathcal{A} \leftrightarrow \mathbf{x} \leqslant \mathbf{a}$ & $\mathbf{x} \neq (\mathbf{b} \vee \mathbf{x}) \wedge (\mathbf{c} \vee \mathbf{x})$ & $\mathbf{x}$ is minimal with these properties.

Interpreting arithmetic in the Turing degrees

We can code a model of arithmetic:

1. Let $\mathcal{N} \subseteq \mathcal{D}_T$ be a countable set. We use parameters $\vec{\mathbf{p}}_1$ code it.
2. If we think of the elements of $\mathcal{N} = \{\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2 \dots\}$ as natural numbers $0, 1, 2, \dots$ then the graphs of $+$, $\times$ and $\leqslant$ are just relations on $\mathcal{D}_T$ coded by parameters $\vec{\mathbf{p}}_2$, $\vec{\mathbf{p}}_3$, and $\vec{\mathbf{p}}_4$.
3. Any set $X \subseteq \mathbb{N}$ corresponds to a countable set $\mathcal{X} \subseteq \mathcal{N}$ and can also be coded by parameters $\vec{\mathbf{p}}_5$.
4. The parameters $\vec{\mathbf{p}} = (\mathbf{a}_0, \mathbf{a}_1, \vec{\mathbf{p}}_1, \vec{\mathbf{p}}_2, \vec{\mathbf{p}}_3, \vec{\mathbf{p}}_4, \vec{\mathbf{p}}_5)$ code a model of arithmetic $\mathbb{N}_{\vec{\mathbf{p}}}$ with a unary predicate X.

It is a definable property of $\vec{\mathbf{p}}$ that they *code* a standard model $\mathbb{N}_{\vec{\mathbf{p}}}$ of arithmetic with a unary predicate X.

We can translate arithmetic statements back into statements about $\mathcal{D}_T$:

Example

The statement $\forall X \exists n[n \in X \ \& \ \forall m[m \geqslant n \vee m \notin X]]$ translates[a] to:

$\forall \vec{\mathbf{p}}$ if $\vec{\mathbf{p}}$ code a standard model $\mathbb{N}_{\vec{\mathbf{p}}}$ with a unary predicate X then there exists
$\mathbf{n} \in \mathbb{N}_{\vec{\mathbf{p}}} \ \& \ X_{\vec{\mathbf{p}}}(\mathbf{n}) \ \&$
$\forall \mathbf{m}[\mathbf{m} \in \mathbb{N}_{\vec{\mathbf{p}}} \ \& \ [\mathbf{m} \geqslant_{\vec{\mathbf{p}}} \mathbf{n} \vee \neg X_{\vec{\mathbf{p}}}(\mathbf{m})]]$.

[a] To make this statement true, we need $X \neq \varnothing$.

The biinterpretability conjecture

Slaman and Woodin conjecture that the relationship between $\mathcal{D}_T$ and $\mathcal{Z}_2$ is even stronger:

Conjecture

The relation $Bi(\vec{\mathbf{p}}, \mathbf{x})$ which is true when $\vec{\mathbf{p}}$ code a model $\mathbb{N}_{\vec{\mathbf{p}}}$ of arithmetic with a unary predicate for a set X and $\deg_T(X) = \mathbf{x}$ is first order definable in $\mathcal{D}_T$.

Theorem (Slaman, Woodin 1990)

The following are equivalent for $\mathcal{D}_T$:

1. The biinterpretability conjecture is true;
2. There are no nontrivial automorphisms of $\mathcal{D}_T$;
3. Every relation on $\mathcal{Z}_2$ that is definable and invariant under Turing reducibility induces a definable relation on Turing degrees.

Slaman and Woodin's automorphism analysis

Using metamathematical methods from set theory Slaman and Wooding came very close to proving the biinterpretability conjecture:

1. There are at most countably many automorphisms of $\mathcal{D}_T$;
2. The relation Bi can be defined using one parameter;
3. Every automorphism on $\mathcal{D}_T$ is the identity on the cone above $\mathbf{0}''_T$.

Definition

An automorphism base is a set $\mathcal{B} \subseteq \mathcal{D}_T$ such that if π is an automorphism of $\mathcal{D}_T$ and for all $\mathbf{x}$ in $\mathcal{B}$ we have that $\pi(\mathbf{x}) = \mathbf{x}$ the π is trivial.

Theorem (Slaman, Woodin)

There exists a single degree $\mathbf{g} \leqslant \mathbf{0}_T^{(5)}$ such that $\{\mathbf{g}\}$ is an automorphism base.

The local structure and first order arithmetic

An effective analysis of the Slaman Woodin theorem yields:

Theorem (The local coding theorem)

Any *uniformly low* sequence of degrees is definable using finitely many parameters in $\mathcal{D}_T(\leqslant \mathbf{0}'_T)$.

Here $\{\mathbf{a}_n\}_{n<\omega}$ is *uniformly low* if it is represented by a sequence $\{A_n\}_{n<\omega}$ such that for all n the set $(\bigoplus_{i<b} A_i)'$ is uniformly computable from $\mathbf{0}'_T$.

Using the local coding theorem we can show:

Theorem (Shore 81)

The theory of $\mathcal{D}_T(\leqslant \mathbf{0}'_T)$ is computably isomorphic to first order arithmetic.

In fact, there is a model of arithmetic that is definable in $\mathcal{D}_T(\leqslant \mathbf{0}'_T)$ without parameters.

The local biinterpretability conjecture

Definition

Fix some $\mathcal{Z} \subseteq \mathcal{D}(\leqslant \mathbf{0}'_T)$ represented by $\{Z_n\}_{n<\omega}$. We say that $\vec{\mathbf{p}}$ define an *indexing* of $\mathcal{Z}$ if $\vec{\mathbf{p}}$ define a standard model of arithmetic $\mathbb{N}_{\vec{\mathbf{p}}}$ along with a bijection $\varphi_{\vec{\mathbf{p}}}$ such that $\varphi_{\vec{\mathbf{p}}}(n_{\vec{\mathbf{p}}}) = \deg_T(Z_n)$.

We can list all Δ^0_2 sets as $\{X_e\}_{e<\omega}$ in some standard way, say $X_e = \Phi_e^{\mathbf{0}'}$, whenever this is well defined and $\varnothing$ otherwise.

Note that if $\vec{\mathbf{p}}$ are Δ^0_2 parameters that code a model of arithmetic with an indexing of Δ^0_2 degrees then $\vec{\mathbf{p}}$ are an automorphism base for $\mathcal{D}_T(\leqslant \mathbf{0}'_T)$.

This is because the degree representing every fixed natural number $e_{\vec{\mathbf{p}}}$ is definable from $\vec{\mathbf{p}}$ and hence so is $\varphi(e_{\vec{\mathbf{p}}}) = \deg_T(X_e)$.

Conjecture (The local biinterpretability conjecture)

There is a definable (without parameters) indexing of the Δ^0_2 Turing degrees.

An indexing of the c.e. Turing degrees

Theorem (Slaman and Woodin 1986)

There are finitely many Δ^0_2 parameters $\vec{\mathbf{p}}$ that code an indexing of the c.e. degrees: a function φ such that $\varphi(e_{\vec{\mathbf{p}}}) = \deg_T(W_e)$.

Consider the set $K = \bigoplus_{e<\omega} W_e$. By Sacks' Splitting theorem there are low disjoint c.e. sets A and B such that $K = A \cup B$.

Represent A and B as $\bigoplus_{e<\omega} A_e$ and $\bigoplus_{e<\omega} B_e$.

The sets $\mathcal{A} = \{d_T(A_e) \mid e < \omega\}$ and $\mathcal{B} = \{d_T(B_e) \mid e < \omega\}$ are uniformly low and hence definable with parameters.

Note that $\deg_T(W_e) = \deg_T(A_e) \vee \deg_T(B_e)$.

The rest of the argument (finding a suitable model of arithmetic and the bijection that defines the indexing) is fairly standard.

Biinterpretability with parameters in the $\mathcal{D}_T(\leqslant \mathbf{0}'_T)$

Theorem (Slaman, Soskova 2015)

There are finitely many Δ^0_2 parameters that code an indexing of the Δ^0_2 degrees.

1. The automorphism group of $\mathcal{D}_T(\leqslant \mathbf{0}'_T)$ is countable.
2. Every relation $\mathcal{S} \subseteq \mathcal{D}_T(\leqslant \mathbf{0}'_T)$ induced by an arithmetically definable degree invariant relation is definable with finitely many Δ^0_2 parameters.
3. $\mathcal{D}_T(\leqslant \mathbf{0}'_T)$ is rigid if and only if $\mathcal{D}_T(\leqslant \mathbf{0}'_T)$ is biinterpretable with first order arithmetic.

Biinterpretability with parameters in the $\mathcal{D}_T(\leqslant \mathbf{0}_T')$

Lemma (Slaman, Soskova 2015)

If $\mathbf{x} \leqslant_T 0'$ then there are low degrees $\mathbf{g}_1$, $\mathbf{g}_2$, $\mathbf{g}_3$, $\mathbf{g}_4$, such that $\mathbf{x} = (\mathbf{g}_1 \vee \mathbf{g}_2) \wedge (\mathbf{g}_3 \vee \mathbf{g}_4)$.

Theorem (Slaman, Soskova 2015)

There exists a uniformly low set of Turing degrees $\mathcal{Z}$, such that every low Turing degree $\mathbf{x}$ is uniquely positioned with respect to the c.e. degrees and the elements of $\mathcal{Z}$.

If $\mathbf{x}, \mathbf{y} \leqslant \mathbf{0}'$, $\mathbf{x}' = \mathbf{0}'$ and $\mathbf{y} \nleqslant \mathbf{x}$ then there are $\mathbf{g}_i \leqslant \mathbf{0}'$, c.e. degrees $\mathbf{a}_i$ and Δ^0_2 degrees $\mathbf{c}_i, \mathbf{b}_i$ for $i = 1, 2$ such that:

1. $\mathbf{b}_i$ and $\mathbf{c}_i$ are elements of $\mathcal{Z}$.
2. $\mathbf{g}_i$ is the least element below $\mathbf{a}_i$ which joins $\mathbf{b}_i$ above $\mathbf{c}_i$.
3. $\mathbf{x} \leqslant \mathbf{g}_1 \vee \mathbf{g}_2$.
4. $\mathbf{y} \nleqslant \mathbf{g}_1 \vee \mathbf{g}_2$.

The enumeratoin degrees

Friedberg and Rogers introduced enumeration reducibility in 1959.

Informally: $A \subseteq \omega$ is *enumeration reducible* to $B \subseteq \omega$ ($A \leqslant_e B$) if there is a uniform way to enumerate A from an enumeration of B.

Definition

$A \leqslant_e B$ if there is a c.e. set W such that

$$A = \{n \colon (\exists e) \langle n, e \rangle \in W \text{ and } D_e \subseteq B\},$$

where D_e is the eth finite set in a canonical enumeration.

The degree structure $\mathcal{D}_e$ induced by $\leqslant_e$ is called the *enumeration degrees*. It is an upper semi-lattice with a least element (the degree of all c.e. sets).

There is a way to define a jump operator in this structure:
$\deg_e(A)' = \deg_e(K_A \oplus \overline{K_A})$, where $K_A = \bigoplus_e \Gamma_e(A)$.

The total enumeration degrees

$A \leqslant_T B \iff A \oplus \overline{A} \leqslant_e B \oplus \overline{B}$.

We can embed $\mathcal{D}_T$ in $\mathcal{D}_e$ by $\iota(d_T(A)) = d_e(A \oplus \overline{A})$.
This embedding preserves the order, the least upper bound, and the jump.

Definition

$A \subseteq \omega$ is *total* if $\overline{A} \leqslant_e A$. A degree is *total* if it contains a total set.

The image of the Turing degrees under the embedding ι is exactly the set of total enumeration degrees.

Nontotal enumeration degrees exist: any generic $A \subseteq \omega$ has nontotal degree.

Theorem (Selman 1971)

$A \leqslant_e B$ if and only if for every set X if $B \leqslant_e X \oplus \overline{X}$ then $A \leqslant_e X \oplus \overline{X}$.

Definability in the enumeration degrees

Theorem (Kalimulli 2003)

The jump operator is first order definable in $\mathcal{D}_e$.

Theorem (Cai, Lempp, Ganchev, Miller, and Soskova 2016)

The total enumeration degree are first order definable in $\mathcal{D}_e$.

Theorem (Cai, Lempp, Ganchev, Miller, and Soskova 2016)

The relation "c.e. in" restricted to total enumeration degrees is first order definable in $\mathcal{D}_e$.

The automorphism group of the enumeration degrees

Corollary

The total (Turing) degrees are a definable automorphism base for $\mathcal{D}_e$.

1. Every automorphism π of $\mathcal{D}_e$ gives rise to an automorphism π^* of $\mathcal{D}_T$.
2. If π is nontrivial then so is π^*.
3. There is a total degree below $\mathbf{0}_e^{(5)}$ which is an automorphism base for $\mathcal{D}_e$.
4. There are at most countably many automorphisms of $\mathcal{D}_e$.

The local structure of the enumeration degrees

The structure $\mathcal{D}_e(\leqslant \mathbf{0}'_e)$ consists of the enumeration degree of Σ^0_2 sets. It has a copy of $\mathcal{D}_T(\leqslant \mathbf{0}'_T)$. The image of $\mathcal{R}$ is exactly the set of enumeration degrees of Π^0_1 sets.

Theorem (Slaman, Woodin 1996)

Every uniformly low set of degrees is codable by parameters in $\mathcal{D}_e(\leqslant \mathbf{0}'_e)$.

Theorem (Ganchev, Soskova 2012)

The theory of $\mathcal{D}_e(\leqslant \mathbf{0}'_e)$ is computably isomorphic to first order arithmetic.

Theorem (Slaman, Soskova 2017)

Let $\mathcal{L}$ be any of the local structures $\mathcal{R}$, $\mathcal{D}_T(\leqslant \mathbf{0}'_T)$, or $\mathcal{D}_e(\leqslant \mathbf{0}'_e)$. The rigidity of $\mathcal{L}$ implies the rigidity of $\mathcal{D}_e$.

A sequence of indexings

Recall, the indexing of the c.e. degrees definable from Δ_2^0 Turing parameters.

Theorem (Slaman Woodin 1986)

There is an indexing of the Π_1^0 enumeration degrees that is definable from finitely many total Δ_2^0 enumeration degrees.

We extend this to:

1. An indexing of all degrees of low 3-c.e. sets;
2. All low Δ_2^0 enumeration degrees;
3. All total Δ_2^0 enumeration degrees;
4. All total degrees that are in the interval $[\mathbf{a}, \mathbf{a}']$ for some total Δ_2^0 enumeration degree.
5. All total Δ_3^0 enumeration degrees;
6. All total Δ_4^0 enumeration degrees;
7. ... All total Δ_n^0 enumeration degrees;

Two sample technical theorems used to produce these

Theorem (Slaman, Soskova 2017)

If Y and W are c.e. sets and A is a low c.e. set such that $W \nleqslant_T A$ and $Y \nleqslant_T A$ then there are sets U and V computable from W such that:

1. $V \leqslant_T Y \oplus U$
2. $V \nleqslant_T A \oplus U$

Relative to X and with $W = \varnothing'$ we get:

Within the class of low and c.e.a. degrees relative to $\mathbf{x}$ which do not compute $\varnothing'$, $\mathbf{y}$ is uniquely positioned with respect to the Δ^0_2 Turing degrees.

Theorem (Slaman, Soskova 2017)

There are high Δ^0_2 degrees $\mathbf{h}_1$ and $\mathbf{h}_2$ such that every 2-generic Δ^0_3 Turing degree $\mathbf{g}$ satisfies $(\mathbf{h}_1 \vee \mathbf{g}) \wedge (\mathbf{h}_2 \vee \mathbf{g}) = \mathbf{g}$.

So every 2-generic below $\mathbf{0}''_T$ can be uniquely determined from degrees in the set $[\mathbf{h}_1, \mathbf{h}'_1] \cup [\mathbf{h}_2, \mathbf{h}'_2]$.

The main theorem and its consequences

Theorem (Slaman, Soskova 2017)

There are finitely many Δ^0_2 total parameters $\vec{\mathbf{p}}$ that define an indexing of the total Δ^0_n enumeration degrees.

- Recall that there is $\mathbf{g} \leqslant \mathbf{0}_T^{(5)}$ such that $\{\mathbf{g}\}$ is an automorphism base for $\mathcal{D}_T$.
- It follows that $\iota(\mathbf{g}) \leqslant \mathbf{0}_e^{(5)}$ and that $\{\iota(\mathbf{g})\}$ is an automorphism base for $\mathcal{D}_e$.
- If $\vec{\mathbf{p}}$ are indexing parameters for the total Δ^0_6 degrees then $\{\iota(\mathbf{g})\}$ is definable from $\vec{\mathbf{p}}$.

Theorem (Slaman, Soskova 2017)

There are finitely many Δ^0_2 total parameters $\vec{\mathbf{p}}$ that form an automorphism base for $\mathcal{D}_e$.

Open questions

Question

Are there any nontrivial automorphisms of any of the mentioned structures: $\mathcal{R}$, $\mathcal{D}_T(\leqslant \mathbf{0}'_T)$, $\mathcal{D}_e(\leqslant \mathbf{0}'_e)$, $\mathcal{D}_T$, $\mathcal{D}_e$?

Question

Is there an indexing of the Σ^0_2 enumeration degrees that is definable (with parameters) in $\mathcal{D}_e(\leqslant \mathbf{0}'_e)$?

Question

Does the rigidity of $\mathcal{D}_T(\leqslant \mathbf{0}'_T)$ or $\mathcal{R}$ imply the rigidity of $\mathcal{D}_T$?

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Thank you!