The Turing universe in the context of enumeration reducibility

Mariya I. Soskova[1]

Sofia University

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- There is an algorithm, which enumerates instances of memberships in A from instances of memberships in B: enumeration reducibility.

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- The halting set relative to B, denoted as $K_B = \{n \mid \varphi_n^B(n) \text{ halts}\}$ is not Turing reducible to B.

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. . .

For imagine we have a procedure called P
that for specified input permits you to see
whether specified source code, with all of its faults,
defines a routine that eventually halts.

. . .

Well, the truth is that P cannot possibly be,
because if you wrote it and gave it to me,
I could use it to set up a logical bind
that would shatter your reason and scramble your mind.

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the p-th program with oracle B
will not in our lifetime enumerate p.

Well, then e too cannot be,
because if you wrote it and gave it to me,
I would make you seem foolish, I will not be kind
with a trick I learned from Cantor and always keep in mind.

The structure of degrees

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- Define a jump operation $'$. For the Turing degrees: $d_T(A)' = d_T(K_A)$. For the e-degrees: $d_e(A)' = d_e(L_A \oplus \overline{L_A})$.

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- $\mathcal{D} = \langle D, \leq, \vee, ' \mathbf{0} \rangle$ is an upper semi-lattice with least element and jump operation.

What connects $\mathcal{D}_T$ and $\mathcal{D}_e$

Proposition

Let A and B be sets of natural numbers. The following are equivalent:

1. $A \leq_T B$;
2. *Both A and $\overline{A}$ can be enumerated by a computable in B function, $A \oplus \overline{A}$ is c.e. in B;*
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$$(\mathcal{D}_T, \leq_T, \vee, ' \mathbf{0}_T) \cong (\mathcal{TOT}, \leq_e, \vee, ', \mathbf{0}_e) \subseteq (\mathcal{D}_e, \leq_e, \vee, ' \mathbf{0}_e)$$

More connections between $\mathcal{D}_T$ and $\mathcal{D}_e$

Theorem (Selman (1971))

Selman's Theorem: $A \leq_e B$ if and only if the set of total enumeration degrees above B is a subset of the set of total enumeration degrees above A.

Corollary

$\mathcal{TOT}$ is an automorphism base for $\mathcal{D}_e$.

Theorem (Soskov's Jump Inversion Theorem (2000))

For every $\mathbf{x} \in \mathcal{D}_e$ there exists a total e-degree $\mathbf{a} \geq \mathbf{x}$, such that $\mathbf{a}' = \mathbf{x}'$.

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Given a set A , characterize the set $\mathcal{S}(A) = \{d_T(Y) \mid A \text{ is c.e. in } Y\}$.

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Theorem (Coles, Downey, Slaman (2000))

For every sets A the set: $\mathcal{S}(A)' = \{d_T(Y)' \mid A \text{ is c.e. in } Y\}$ *has a member of least degree.*

Every torsion free abelian group of rank 1 has a jump degree.

The proof is a forcing construction of this least member.

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- Then $DS_e(\mathcal{A}^+) = \{\iota(\mathbf{a}) \mid \mathbf{a} \in DS_T(\mathcal{A})\}$.
- $\mathcal{A}$ has T-degree $\mathbf{a}$ if and only if $\mathcal{A}^+$ has e-degree $\iota(\mathbf{a})$.

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- If G has an e-degree then this e-degree is precisely $\mathbf{s}_G$.

TFA1 groups in the e-degrees

$$DS_e(G) = \{\mathbf{a} \mid \mathbf{a} \in \mathcal{TOT} \ \& \ \mathbf{s}_G \leq_e \mathbf{a}\}$$

- By Selman's Theorem $\mathbf{s}_G$ is completely determined by the set of total degrees above it.
- G has a degree (both e- and T-) if and only if the type degree $\mathbf{s}_G$ is total.
- If G has an e-degree then this e-degree is precisely $\mathbf{s}_G$.
- By Soskov's Jump inversion Theorem G always has first jump degree (both e- and T-) and it is $\mathbf{s}'_G$.

Global properties of the degree structures

1. Algebraic complexity.
2. Theory strength.
3. First order definability.
4. Characterization of the automorphism group of the structure.

Defining the Turing jump operator

Theorem (Shore, Slaman (1999))

The Turing jump operator is first order definable in $\mathcal{D}_T$.

There is a first order formula φ_J in the language of partial orderings, so that:

$$\mathcal{D}_T \models \varphi_J(\mathbf{x}, \mathbf{y}) \Leftrightarrow \mathbf{x}' = \mathbf{y}.$$

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   Slaman and Woodin's analysis of the automorphisms of the Turing degrees (1996) and *"involves explicit translation of automorphism facts in definability facts via a coding of second order arithmetic".*
2. An additional structural fact, involving a sharp analysis of Kumabe-Slaman forcing.

Semi-computable sets in the enumeration degrees

Definition (Jockusch (1968))

A set of natural numbers A is semi-computable if there is a total computable selector function s_A, such that $s_A(x, y) \in \{x, y\}$ and if $\{x, y\} \cap A \neq \emptyset$ then $s_A(x, y) \in A$.

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$$(\forall \mathbf{x} \in \mathcal{D}_e)((d_e(A) \vee \mathbf{x}) \wedge (d_e(\overline{A}) \vee \mathbf{x}) = \mathbf{x}).$$

$\mathcal{K}$-pairs in the enumeration degrees

Definition (Kalimullin (2003))

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Theorem (Kalimullin)

A pair of sets A, B are a $\mathcal{K}$-pair if and only if their enumeration degrees **a** *and* **b** *satisfy:*

$$\mathcal{K}(\mathbf{a}, \mathbf{b}) \leftrightharpoons (\forall \mathbf{x} \in \mathcal{D}_e)((\mathbf{a} \vee \mathbf{x}) \wedge (\mathbf{b} \vee \mathbf{x}) = \mathbf{x}).$$

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- $\mathcal{K}$-pairs are never total enumeration degrees, the only total degree below either of them is $\mathbf{0}_e$.
- A consequence of the existence of nontrivial $\mathcal{K}$-pairs in $\mathcal{D}_e$ is that the second ingredient from the Slaman-Shore definition of the Turing jump fails in $\mathcal{D}_e$.
- There are no $\mathcal{K}$-pairs in the structure of the Turing degrees.

$\mathcal{K}$-pairs and the definability of the enumeration jump

Theorem (Kalimullin (2003))

$\mathbf{0}'_e$ *is the largest degree which can be represented as the least upper bound of a triple* $\mathbf{a}, \mathbf{b}, \mathbf{c}$*, such that* $\mathcal{K}(\mathbf{a}, \mathbf{b})$*,* $\mathcal{K}(\mathbf{b}, \mathbf{c})$ *and* $\mathcal{K}(\mathbf{c}, \mathbf{a})$*.*

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The enumeration jump is first order definable in $\mathcal{D}_e$.

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Theorem (Ganchev, S (2012))

For every nonzero enumeration degree $\mathbf{u} \in \mathcal{D}_e$, $\mathbf{u}'$ is the largest among all least upper bounds $\mathbf{a} \vee \mathbf{b}$ of nontrivial $\mathcal{K}$-pairs $\{\mathbf{a}, \mathbf{b}\}$, such that $\mathbf{a} \leq_e \mathbf{u}$.

Zooming in: the local structures

- The local structure of the Turing degrees $\mathcal{D}_T(\leq \mathbf{0}'_T)$ consists of all Turing degrees reducible to $\mathbf{0}'_T$.

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$$\mathcal{D}_T(\leq \mathbf{0}'_T) \cong \mathcal{TOT}(\leq \mathbf{0}'_e) \subseteq \mathcal{D}_e(\leq \mathbf{0}'_e)$$

Definability in the local structures

Definition

1. A degree $\mathbf{a}$ is low if $\mathbf{a}' = \mathbf{0}'$.
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It is not known if the low degrees are first order definable in $\mathcal{D}_T(\leq \mathbf{0}'_T)$.

Definability in the local structure of the enumeration degrees

Initial motivation: Prove that the theory of first order arithmetic is interpretable in $\mathcal{D}_e(\leq \mathbf{0}'_e)$.

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Theorem (Slaman, Woodin (1997))

The theory of $\mathcal{D}_e(\leq \mathbf{0}_e')$ is undecidable.

Idea: Use $\mathcal{K}$-pairs to extend this result.

An obstacle

$\mathcal{K}(\mathbf{a},\mathbf{b}) \leftrightharpoons (\forall \mathbf{x})((\mathbf{a} \vee \mathbf{x}) \wedge (\mathbf{b} \vee \mathbf{x}) = \mathbf{x})$

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Theorem (Ganchev, S (2012))

There is a first order formula $\mathcal{LK}(x, y)$, which defines the $\mathcal{K}$-pairs in $\mathcal{D}_e(\leq \mathbf{0}'_e)$.

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Theorem (Ganchev, S)

The first order theory of $\mathcal{D}_e(\leq \mathbf{0}_e')$ is computably isomorphic to the first order theory of true arithmetic.

A surprising consequence

Extending a result of Giorgi, Sorbi and Yang:

Theorem (Ganchev, S)

An enumeration degree $\mathbf{a}$ *is low if and only if every degree* $\mathbf{b} \leq_e \mathbf{a}$ *bounds a* $\mathcal{K}$*-pair.*

The class L_1 *is first order definable in* $\mathcal{D}_e(\leq \mathbf{0}'_e)$*.*

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Extending Jouckusch's theorem:

Theorem (Ganchev, S)

A degree $\mathbf{a} \leq_e \mathbf{0}'_e$ *is total if and only if it is the least upper bound of a maximal* $\mathcal{K}$*-pair.*

The class of total degrees is first order definable in $\mathcal{D}_e(\leq \mathbf{0}'_e)$.

Open question

We know that:

- $\mathcal{TOT} \cap \mathcal{D}_e(\geq \mathbf{0}_e')$ is first order definable.

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Is $\mathcal{TOT}$ first order definable in $\mathcal{D}_e$?

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A positive answer would connect the problems of the existence of a non-trivial automorphism in both structures.

One step further in the dream world

Theorem (Ganchev,S)

For every nonzero enumeration degree $\mathbf{u} \in \mathcal{D}_e$,

$$\mathbf{u}' = max\,\{\mathbf{a} \vee \mathbf{b} \mid \mathcal{K}(\mathbf{a},\mathbf{b})\ \&\ \mathbf{a} \leq_e \mathbf{u}\}\,.$$

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- Suppose that a degree is total if and only if it is the least upper bound of a maximal $\mathcal{K}$-pair.
- The relation **x** is c.e. in **u** would also be definable for total degrees by :

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- Then for total **u**, our definition of the jump would read $\mathbf{u}'$ is the largest total degree, which is c.e. in **u**.

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4. Definability with parameters: Every relation on $\mathcal{D}$ induced by a degree invariant relation on sets of natural numbers, which is definable in second order arithmetic, is definable in $\mathcal{D}$ with parameters.

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Corollary (S (2013))

$\mathcal{TOT}$ is definable with parameters in $\mathcal{D}_e$.

The end

Thank you!