Embedding Partial Orderings in Degree Structures

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Embeddability in the Turing degrees

Definition (Kleene, Post 1954)

A sequence of sets $\{A_i\}_{i<\omega}$ is called computably independent if for every i:

$$A_i \not\leq_T \bigoplus_{j \neq i} A_j.$$

- Mostowski 1938: There exists a computable partial ordering $\mathcal{R} = \langle \mathbb{N}, \leq \rangle$ in which every countable partial ordering can be embedded.
- Sacks 1963: The existence of a computably independent sequence of sets gives an embedding of any computable partial ordering.

Embeddability in the Turing degrees

Localizing independent sequences of sets:

Corollary

Every countable partial ordering can be embedded

1. *Kleene and Post 1954: in the Turing degrees, even in the* Δ^0_2 *Turing degrees.*
2. *Muchnik 1958: in the c.e. Turing degrees.*
3. *Robinson 1971: densely in the c.e. Turing degrees, i.e. in any nonempty interval of c.e. Turing degrees.*

The enumeration degrees

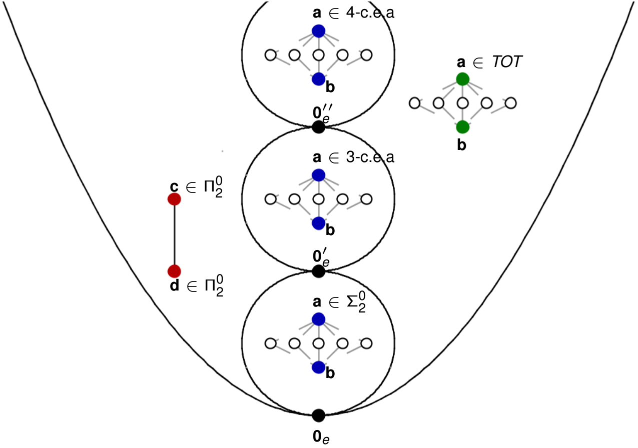
- Case 1971: Any countable partial ordering can be embedded in the e-degrees below the degree of any generic function.
- Copestake 1988: below any 1-generic enumeration degree.
- Lageman 1972: below any nonzero Δ^0_2 e-degree.
- Bianchini 2000: densely in the Σ^0_2 enumeration degrees.

Theorem

Let $\mathbf{b} < \mathbf{a}$ be enumeration degrees such that $\mathbf{a}$ contains a member with a good approximation. Then every countable partial ordering can be embedded in the interval $[\mathbf{b}, \mathbf{a}]$.

Method: *e*-independent sequences of sets.

The general picture



The ω e-degrees: Basic definitions

Let $\mathcal{S}$ be the set of all sequences of sets of natural numbers.

Definition

Let $\mathcal{A} = \{A_n\}_{n<\omega}$ be a sequence of sets natural numbers and V be an e-operator. The result of applying the enumeration operator V to the sequence $\mathcal{A}$, denoted by $V(\mathcal{A})$, is the sequence $\{V[n](A_n)\}_{n<\omega}$. We say that $V(\mathcal{A})$ is enumeration reducible ($\leq_e$) to the sequence $\mathcal{A}$.

So $\mathcal{A} \leq_e \mathcal{B}$ is a combination of two notions:

- Enumeration reducibility: for every n we have that $A_n \leq_e B_n$ via, say, Γ_n.
- Uniformity: the sequence $\{\Gamma_n\}_{n<\omega}$ is uniform.

Basic definitions

With every member $\mathcal{A} \in \mathcal{S}$ we connect a *jump sequence* $P(\mathcal{A})$.

Definition

The *jump sequence* of the sequence $\mathcal{A}$, denoted by $P(\mathcal{A})$ is the sequence $\{P_n(\mathcal{A})\}_{n<\omega}$ defined inductively as follows:

- $P_0(\mathcal{A}) = A_0$.
- $P_{n+1}(\mathcal{A}) = A_{n+1} \oplus P'_n(\mathcal{A})$, where $P'_n(\mathcal{A})$ denotes the enumeration jump of the set $P_n(\mathcal{A})$.

The ω-enumeration degrees

Let $\mathcal{A}, \mathcal{B} \in \mathcal{S}$.

Definition

- ω-enumeration reducibility: $\mathcal{A} \leq_\omega \mathcal{B}$, if $\mathcal{A} \leq_e P(\mathcal{B})$.
- $\mathcal{A} \equiv_\omega \mathcal{B}$ iff $\mathcal{A} \leq_\omega \mathcal{B}$ and $\mathcal{B} \leq_\omega \mathcal{A}$.
- $d_\omega(\mathcal{A}) = \{\mathcal{B} \mid \mathcal{A} \equiv_\omega \mathcal{B}\}$
- $\mathcal{D}_\omega$ is an upper semi-lattice with jump operation and least element $\mathbf{0}_\omega = d_\omega((\emptyset, \emptyset, \dots))$.

The e-degrees as a substructure

$\langle \mathcal{D}_e, \leq_e, \vee, ' \rangle$ can be embedded in $\langle \mathcal{D}_\omega, \leq_\omega, \vee, ' \rangle$ via the embedding κ defined as follows:

$$\kappa(d_e(A)) = d_\omega((A, \emptyset, \emptyset, \dots)) = d_\omega((A, A', A'', \dots)).$$

Theorem (Soskov, Ganchev)

- *The structure $\mathcal{D}_1 = \kappa(\mathcal{D}_e)$ is first order definable in $\mathcal{D}_\omega$.*
- *The structures $\mathcal{D}_e$ and $\mathcal{D}_\omega$ with jump operation have isomorphic automorphism groups.*

The embeddability question

Consider the structure $\mathcal{L}_\omega$ consisting of all degrees reducible to $0'_\omega = d_\omega((\emptyset', \emptyset'', \emptyset''', \dots))$ also called the Σ^0_2 ω-enumeration degrees.

Theorem (Soskov)

The structure $\mathcal{L}_\omega$ is dense.

Theorem

Let $\mathbf{b} <_\omega \mathbf{a} \leq_\omega \mathbf{0}'_\omega$. Every countable partial ordering can be embedded in the interval $[\mathbf{b}, \mathbf{a}]$.

Proof techniques: Independent sequences of sequences sets, embeddability results in the enumeration degrees, good approximations for sequences, recursion theorem.

The c.e. degrees modulo iterated jump

Definition (Jockusch, Lerman, Soare and Solovay)

Let **a** and **b** be c.e. Turing degrees. $\mathbf{a} \sim_\infty \mathbf{b}$ iff there exists a natural number n such that $\mathbf{a}^n = \mathbf{b}^n$.

- Induced degree structure $\mathcal{R}/\sim_\infty$ with $[\mathbf{a}]_{\sim_\infty} \leq [\mathbf{b}]_{\sim_\infty}$ if and only if there exists a natural number n such that $\mathbf{a}^n \leq_T \mathbf{b}^n$.
- Least element $L = \bigcup_{n<\omega} L_n$.
- Greatest element $H = \bigcup_{n<\omega} H_n$.
- $\mathcal{R}/\sim_\infty$ is a dense structure.
- Lempp : There is a splitting of the highest ∞-degree and a minimal pair of ∞-degrees.

Starting with other classes of degree

- $\mathcal{L}_T/\sim_\infty$: the Δ_2^0 Turing degrees modulo iterated jump. Shoenfield, Sacks: The range of the jump operator restricted to the c.e. Turing degrees coincides with the range of the jump operator restricted to the Δ_2^0 Turing degrees. It is namely the set of all Turing degrees c.e. in and above $\mathbf{0}'$. Hence:

$$\mathcal{L}_T/\sim_\infty \simeq \mathcal{R}/\sim_\infty .$$

- $\mathcal{L}_e/\sim_\infty$: the Σ_2^0 e-degrees modulo iterated jump. McEvoy: The range of the enumeration jump operator restricted to the Σ_2^0-enumeration degrees coincides with the range of the enumeration jump operator restricted to the Π_1^0 enumeration degrees. Hence:

$$\mathcal{R}/\sim_\infty \simeq (\Pi_1^0 \text{ e-degrees})/\sim_\infty \simeq \mathcal{L}_e/\sim_\infty .$$

The ω-enumeration degrees modulo iterated jump

Consider $\mathcal{L}_\omega / \sim_\infty$.
$\mathcal{R} / \sim_\infty$ embeds in $\mathcal{L}_\omega / \sim_\infty$.

$$\mathcal{R} \subseteq \mathcal{L}_T \hookrightarrow \iota(\mathcal{L}_T) = \mathit{Tot} \subseteq \mathcal{L}_e \hookrightarrow \kappa(\mathcal{L}_e) = \mathcal{D}_1 \subseteq \mathcal{L}_\omega$$

Lemma

Let $\mathbf{a}$ *and* $\mathbf{b}$ *be two* Σ^0_2 ω*-enumeration degrees.*

1. *If* $\mathbf{a} \leq_\omega \mathbf{b}$ *then* $[\mathbf{a}]_{\sim_\infty} \leq [\mathbf{b}]_{\sim_\infty}$*.*
   Proof idea: Monotonicity of the jump.
2. *If* $[\mathbf{a}]_{\sim_\infty} \leq [\mathbf{b}]_{\sim_\infty}$ *then there is a representative* $\mathbf{c} \in [\mathbf{a}]_{\sim_\infty}$ *such that* $\mathbf{c} \leq_\omega \mathbf{b}$*.*
   Proof idea: Existence of least jump inverts.

The almost degrees

Definition

Let $\mathcal{A} = \{A_n\}_{n<\omega}$ be a sequence of sets of natural numbers. We shall say that the sequence $\mathcal{B} = \{B_n\}_{n<\omega}$ is *almost-$\mathcal{A}$* if for every n we have that $P_n(\mathcal{A}) \equiv_e P_n(\mathcal{B})$.
If $\mathcal{A}$ is almost-$\mathcal{B}$ then we shall say that $d_\omega(\mathcal{A})$ is almost-$d_\omega(\mathcal{B})$.

Properties:

- If $\mathbf{a} <_\omega \mathbf{b}$ and $\mathbf{a} <_\infty \mathbf{b}$ then there exists an almost-$\mathbf{a}$ degree $\mathbf{z}$ such that $\mathbf{a} <_\omega \mathbf{z} \leq_\omega \mathbf{b}$.
- ω-reducibility and ∞-reducibility coincide when restricted to the almost $\mathbf{a}$-degrees.

The final result

Theorem

1. *$\mathcal{L}_\omega / \sim_\infty$ properly extends $\mathcal{R} / \sim_\infty$.*
2. *Every countable partial ordering can be embedded densely in $\mathcal{L}_\omega / \sim_\infty$.*

The final result

Theorem

1. *$\mathcal{L}_\omega / \sim_\infty$ properly extends $\mathcal{R} / \sim_\infty$.*
2. *Every countable partial ordering can be embedded densely in $\mathcal{L}_\omega / \sim_\infty$.*

Thank you!