Finite automorphism bases for degree structures

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Automorphism bases

Definition

Let $\mathcal{A}$ be a structure with domain A. A set $B \subseteq A$ is an automorphism base for $\mathcal{A}$ if whenever f and g are automorphisms of $\mathcal{A}$, such that $(\forall x \in B)(f(x) = g(x))$, then $f = g$.

Equivalently if f is an automorphism of $\mathcal{A}$ and $(\forall x \in B)(f(x) = x)$ then f is the identity.

Theorem (Slaman and Woodin)

There is an element $\mathbf{g} \leq \mathbf{0}^{(5)}$ *such that* $\{\mathbf{g}\}$ *is an automorphism base for the structure of the Turing degrees* $\mathcal{D}_T$.

$Aut(\mathcal{D}_T)$ *is countable and every member has an arithmetically definable presentation.*

Part I: The local structure of the Turing degrees

Definition

A set of degrees $\mathcal{Z}$ contained in $\mathcal{D}_T(\leq \mathbf{0}')$ is *uniformly low* if it is bounded by a low degree and there is a sequence $\{Z_i\}_{i<\omega}$, representing the degrees in $\mathcal{Z}$, and a computable function f such that $\{f(i)\}^{\emptyset'}$ is the Turing jump of $\bigoplus_{j<i} Z_j$.

Example: If $\bigoplus_{i<\omega} A_i$ is low then $\mathcal{A} = \{d_T(A_i) \mid i < \omega\}$ is uniformly low.

Theorem (Slaman and Woodin)

If $\mathcal{Z}$ is a uniformly low subset of $\mathcal{D}_T(\leq \mathbf{0}')$ then $\mathcal{Z}$ is definable from parameters in $\mathcal{D}_T(\leq \mathbf{0}')$.

Applications of the coding theorem

1. Using parameters we can code a model of arithmetic $\mathcal{M} = (\mathbb{N}^{\mathcal{M}}, 0^{\mathcal{M}}, +^{\mathcal{M}}, \times^{\mathcal{M}}, \leq^{\mathcal{M}})$.
2. If $\mathcal{Z} \subseteq \mathcal{D}_T(\leq \mathbf{0}')$ is uniformly low and represented by the sequence $\{Z_i\}_{i<\omega}$ then there are Δ^0_2 parameters that code a model of arithmetic $\mathcal{M}$ and a function $\varphi : \mathbb{N}^{\mathcal{M}} \to \mathcal{D}_T(\leq \mathbf{0}')$ such that $\varphi(i^{\mathcal{M}}) = d_T(Z_i)$.

   We call such a function *an indexing* of $\mathcal{Z}$.

Applications of the coding theorem

3. Using parameters we can define the set of c.e. degrees: Consider the set $K = \bigoplus_{e<\omega} W_e$. By Sacks' Splitting theorem there are low disjoint c.e. sets A and B such that $K = A \cup B$.

   Represent A and B as $\bigoplus_{e<\omega} A_e$ and $\bigoplus_{e<\omega} B_e$. Note that W_e is the disjoint union of A_e and B_e.

   The set $\mathcal{A} = \{d_T(A_e) \mid e < \omega\}$ and $\mathcal{B} = \{d_T(B_e) \mid e < \omega\}$ are uniformly low and hence definable with parameters.

   A degree $\mathbf{x}$ is c.e. if it is the join of an element from $\mathcal{A}$ and an element from $\mathcal{B}$.

The goal

Theorem (Slaman and Woodin)

There are finitely many Δ^0_2 parameters which code a model of arithmetic $\mathcal{M}$ and an indexing of the c.e. degrees: a function $\psi : \mathbb{N}^{\mathcal{M}} \to \mathcal{D}_T(\leq \mathbf{0}')$ such that $\psi(e^{\mathcal{M}}) = d_T(W_e)$.

Note that if we have an automorphism π of $\mathcal{D}_T(\leq \mathbf{0}')$ which fixes these parameters then π fixes every c.e degree.

The Goal

Extend this result to find finitely many Δ^0_2 parameters that code a model of arithmetic $\mathcal{M}$ and an indexing φ of the Δ^0_2 Turing degrees.

We will call e an index for a Δ^0_2 set X if $\{e\}^{\emptyset'}$ is the characteristic function of X.

Step 1: Reducing to low sets

Lemma

If $\mathbf{x} \leq_T 0'$ *then there are low degrees* $\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3, \mathbf{g}_4$*, such that* $\mathbf{x} = (\mathbf{g}_1 \vee \mathbf{g}_2) \wedge (\mathbf{g}_3 \vee \mathbf{g}_4)$.

- Suppose that we know how to map an index $e^{\mathcal{M}}$ of a low Δ^0_2 set G to the degree $\varphi(e^{\mathcal{M}}) = d_T(G)$.
- If in $\mathcal{M}$ "e is an index of a non-low Δ^0_2 set X" then we search in $\mathcal{M}$ for indices e_1, e_2, e_3, e_4 of low Δ^0_2 sets which define the degree of X.
- We map $e^{\mathcal{M}}$ to $(\varphi(e_1^{\mathcal{M}}) \vee \varphi(e_2^{\mathcal{M}})) \wedge (\varphi(e_3^{\mathcal{M}}) \vee \varphi(e_4^{\mathcal{M}}))$.

Step 2: Distinguishing between low Δ^0_2 sets

Theorem

There exists a uniformly low set of Turing degrees $\mathcal{Z}$, such that every low Turing degree $\mathbf{x}$ is uniquely positioned with respect to the c.e. degrees and the elements of $\mathcal{Z}$.

If $\mathbf{x}, \mathbf{y} \leq \mathbf{0}'$, $\mathbf{x}' = \mathbf{0}'$ and $\mathbf{y} \not\leq \mathbf{x}$ then there are $\mathbf{g}_i \leq \mathbf{0}'$, c.e. degrees $\mathbf{a}_i$ and Δ^0_2 degrees $\mathbf{c}_i, \mathbf{b}_i$ for $i = 1, 2$ such that:

1. *$\mathbf{b}_i$ and $\mathbf{c}_i$ are elements of $\mathcal{Z}$.*
2. *$\mathbf{g}_i$ is the least element below $\mathbf{a}_i$ which joins $\mathbf{b}_i$ above $\mathbf{c}_i$.*
3. *$\mathbf{x} \leq \mathbf{g}_1 \vee \mathbf{g}_2$.*
4. *$\mathbf{y} \not\leq \mathbf{g}_1 \vee \mathbf{g}_2$.*

Applications

Theorem (Biinterpretability with parameters)

There are finitely many Δ^0_2 parameters that code a model of arithmetic $\mathcal{M}$ and an indexing of the Δ^0_2 degrees.

1. The automorphism group of $\mathcal{D}_T(\leq \mathbf{0}')$ is countable.
2. Every automorphism π of $\mathcal{D}_T(\leq \mathbf{0}')$ has an arithmetic presentation.
3. Every relation $\mathcal{R} \subseteq \mathcal{D}_T(\leq \mathbf{0}')$ induced by an arithmetically definable degree invariant relation is definable with finitely many Δ^0_2 parameters. If $\mathcal{R}$ is invariant under automorphisms then it is definable.
4. $\mathcal{D}_T(\leq \mathbf{0}')$ is rigid if and only if $\mathcal{D}_T(\leq \mathbf{0}')$ is biinterpretable with first order arithmetic.

Part II: The structure of the enumeration degrees

Definition

$A \leq_e B$ if there is a c.e. set W, such that

$$A = W(B) = \{x \mid \exists D(\langle x, D\rangle \in W \;\&\; D \subseteq B)\}.$$

- $A \equiv_e B$ if $A \leq_e B$ and $B \leq_e A$.
- The enumeration degree of a set A is $d_e(A) = \{B \mid A \equiv_e B\}$.
- $d_e(A) \leq d_e(B)$ iff $A \leq_e B$.
- The least element: $\mathbf{0_e} = d_e(\emptyset)$, the set of all c.e. sets.
- The least upper bound: $d_e(A) \vee d_e(B) = d_e(A \oplus B)$.
- The enumeration jump: $d_e(A)' = d_e(K_A \oplus \overline{K_A})$, where $K_A = \{\langle e, x\rangle \mid x \in W_e(A)\}$.

What connects $\mathcal{D}_T$ and $\mathcal{D}_e$

Proposition

$$A \leq_T B \Leftrightarrow A \oplus \overline{A} \leq_e B \oplus \overline{B}.$$

A set A is *total* if $A \equiv_e A \oplus \overline{A}$. An enumeration degree is *total* if it contains a total set. The set of total degrees is denoted by $\mathcal{TOT}$.

The embedding $\iota : \mathcal{D}_T \to \mathcal{D}_e$, defined by $\iota(d_T(A)) = d_e(A \oplus \overline{A})$, preserves the order, the least upper bound and the jump operation.

$$(\mathcal{D}_T, \leq_T, \vee, ', \mathbf{0}_T) \cong (\mathcal{TOT}, \leq_e, \vee, ', \mathbf{0}_e) \subseteq (\mathcal{D}_e, \leq_e, \vee, ', \mathbf{0}_e)$$

If $\mathbf{x} \in \mathcal{D}_T$ then we will call $\iota(\mathbf{x})$ *the image of* $\mathbf{x}$.

Definability in the enumeration degrees

Theorem (Kalimullin)

The enumeration jump is first order definable in $\mathcal{D}_e$.

Theorem (Cai, Ganchev, Lempp, Miller, S)

The set of total enumeration degrees is first order definable in the enumeration degrees.

Definition

A Turing degree $\mathbf{a}$ is *c.e. in* a Turing degree $\mathbf{x}$ if some $A \in \mathbf{a}$ is c.e. in some $X \in \mathbf{x}$.

Theorem (Cai, Ganchev, Lempp, Miller, S)

The image of the relation " c.e. in " in the enumeration degrees is first order definable in $\mathcal{D}_e$.

The total degrees as an automorphism base

Theorem (Selman)

A is enumeration reducible to B if and only if
$\{\mathbf{x} \in \mathcal{TOT} \mid d_e(A) \leq \mathbf{x}\} \supseteq \{\mathbf{x} \in \mathcal{TOT} \mid d_e(B) \leq \mathbf{x}\}$.

Corollary

The total enumeration degrees form a definable automorphism basis of the enumeration degrees.

- *If $\mathcal{D}_T$ is rigid then $\mathcal{D}_e$ is rigid.*
- *The automorphism analysis for the enumeration degrees follows.*
- *The total degrees below $\mathbf{0}_e^{(5)}$ are an automorphism base of $\mathcal{D}_e$.*

Towards a better automorphism base of $\mathcal{D}_e$

Theorem

There are total Δ^0_2 parameters that code a model of arithmetic $\mathcal{M}$ and an indexing of the total Δ^0_2 enumeration degrees.

1. The parameters $\vec{p}$ code an indexing of the image of the c.e. Turing degrees.
2. The parameters $\vec{p}$ code an indexing of the image of a uniformly low set $\mathcal{Z}$.
3. Every low total Δ^0_2 enumeration degree is uniquely positioned with respect to the image of the c.e. degrees and the image of $\mathcal{Z}$.
4. Every total Δ^0_2 enumeration degree is uniquely positioned with respect to the low total Δ^0_2 enumeration degrees.

An improvement

Theorem

1. *Every low Δ^0_2 enumeration degree is uniquely positioned with respect to the image of the c.e. Turing degrees and the low 3-c.e. enumeration degrees.*
2. *Every low 3-c.e. enumeration degree is uniquely positioned with respect to the image of the c.e. Turing degrees.*

If $\vec{p}$ defines a model of arithmetic $\mathcal{M}$ and an indexing of the images of the c.e. Turing degrees then $\vec{p}$ defines an indexing of the total Δ^0_2 enumeration degrees.

Stepping outside the local structure

New Goal

Using parameters $\vec{p}$ that index the image of the c.e. degrees define an indexing of the images of all Turing degrees that are c.e. in and above some Δ^0_2 Turing degree.

$\psi(e_0^{\mathcal{M}}, e_1^{\mathcal{M}}) = \iota(d_T(Y))$, where $Y = W_{e_0}^X$ and $X = \{e_1\}^{\emptyset'}$.

- If we succeed then relativizing the previous step to any total Δ^0_2 enumeration degree we can extend this to an indexing of the image of $\bigcup_{\mathbf{x} \leq_T \mathbf{0}'} [\mathbf{x}, \mathbf{x}']$.
- We will use that the image of the relation 'c.e. in' and the enumeration jump are definable.

C.e. in and above a Δ^0_2 degree

Suppose that $\mathbf{x}$ is Δ^0_2 and $\mathbf{y}$ is c.e. in and above $\mathbf{x}$.

1. If $\mathbf{y} \geq \mathbf{0}'$ then we use Shoenfield's jump inversion theorem to find a Δ^0_2 degree $\mathbf{z}$ such that $\mathbf{z}' = \mathbf{y}$.
2. Otherwise using Sacks' splitting theorem we can represent $\mathbf{y}$ as $\mathbf{a}_1 \vee \mathbf{a}_2$, where $\mathbf{a}_1$ and $\mathbf{a}_2$ are low and c.e.a. relative to $\mathbf{x}$ which avoid the cone above $\mathbf{0}'$.
3. Define an indexing of all low and c.e.a. relative to $\mathbf{x}$ such avoid the cone above $\mathbf{0}'$.

- We can define the set of images of low relative to $\mathbf{x}$ degrees that are c.e. in and above $\mathbf{x}$ and avoid the cone above $\mathbf{0}'$.

C.e. in and above a Δ^0_2 degree: complicated case

Theorem

If Y and W are c.e. sets and A is a low c.e. set such that $W \not\leq_T A$ and $Y \not\leq_T A$ then there are sets U and V computable from W such that:

1. $V \leq_T Y \oplus U$
2. $V \not\leq_T A \oplus U$

Relative to X and with $W = \emptyset'$ we get:

Within the class of low and c.e.a degrees relative to $\mathbf{x}$ which do not compute $\emptyset'$, $\mathbf{y}$ is uniquely positioned with respect to the Δ^0_2 Turing degrees.

The rest of the total enumeration degrees

Theorem

Let $\vec{p}$ are parameters that index the image of the c.e. Turing degrees then $\vec{p}$ index $\bigcup_{\mathbf{x} \leq_T \mathbf{0}'} [\mathbf{x}, \mathbf{x}']$.

Next Goal

Extend to an indexing of the image of all Δ^0_3 Turing degrees.

Theorem

There are high Δ^0_2 degrees $\mathbf{h}_1$ and $\mathbf{h}_2$ such that every 2-generic Δ^0_3 Turing degree $\mathbf{g}$ satisfies $(\mathbf{h}_1 \vee \mathbf{g}) \wedge (\mathbf{h}_2 \vee \mathbf{g}) = \mathbf{g}$.

Note that $\mathbf{h}_i \vee \mathbf{g} \in [\mathbf{h}_i, \mathbf{h}_i']$ thus we have a way to identify this degree and hence we have a way to identify $\mathbf{g}$.

And now we iterate!

Theorem

Let n be a natural number and $\vec{p}$ be parameters that index the image of the c.e. Turing degrees. There is a definable from $\vec{p}$ indexing of the total Δ^0_{n+1} sets.

Consequences

1. There is a finite automorphism base for the enumeration degrees consisting of total Δ^0_2 enumeration degrees:
2. The image of the c.e. Turing degrees is an automorphism base for $\mathcal{D}_e$.
3. If the structure of the c.e. Turing degrees is rigid then so is the structure of the enumeration degrees.

Question

1. *Can every automorphism of the Turing degrees be extended to an automorphism of the enumeration degrees?*
2. *Can we extend automorphisms of the c.e. degrees to automorphisms of $\mathcal{D}_T$ or of $\mathcal{D}_e$?*