Definability and automorphisms in the enumeration degrees

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The structure of the enumeration degrees

Definition

$A \leq_e B$ if there is a c.e. set W, such that

$$A = W(B) = \{x \mid \exists D(\langle x, D\rangle \in W \;\&\; D \subseteq B)\}.$$

- The least upper bound: $d_e(A) \vee d_e(B) = d_e(A \oplus B)$.
- The enumeration jump: $d_e(A)' = d_e(K_A \oplus \overline{K_A})$, where $K_A = \{\langle e, x\rangle \mid x \in W_e(A)\}$.

The embedding $\iota : \mathcal{D}_T \to \mathcal{D}_e$, defined by $\iota(d_T(A)) = d_e(A \oplus \overline{A})$, preserves the order, the least upper bound and the jump operation.

$$(\mathcal{D}_T, \leq_T, \vee, ', \mathbf{0}_T) \cong (\mathcal{TOT}, \leq_e, \vee, ', \mathbf{0}_e) \subseteq (\mathcal{D}_e, \leq_e, \vee, ', \mathbf{0}_e)$$

The importance of definability

Let $\mathcal{D} \in \{\mathcal{D}_T, \mathcal{D}_e\}$.

Theorem (Slaman, Woodin)

The following are equivalent:

1. $\mathcal{D}$ is rigid, i.e. has no nontrivial automorphisms.
2. The definable relations in $\mathcal{D}$ are the ones induced by degree-invariant definable relations on sets in second order arithmetic.
3. $\mathcal{D}$ is biinterpretable with second order arithmetic.

Theorem (Slaman, Woodin)

There is an arithmetical parameter $\mathbf{g}$, such that:

1. The only automorphism of $\mathcal{D}$ that fixes $\mathbf{g}$ is the identity.
2. Every relation on $\mathcal{D}$, induced by a degree invariant relation on sets in second order arithmetic is definable with parameter $\mathbf{g}$.

Definability in the Turing degrees

Theorem (Jockusch and Shore)

A Turing degree is arithmetical if and only if it is bounded by a degree that is not a minimal cover relative to any other Turing degree.

Theorem (Shore and Slaman)

The Turing jump is first order definable.

Theorem (Ambos-Spies, Jockusch, Shore and Soare)

The promptly simple c.e. degrees are exactly the non-cappable c.e. degrees.

Theorem (Downey, Greenberg and Weber)

The totally ω-c.e. degrees are the ones that do not bound a critical triple.

Theorem (Nies, Shore and Slaman; Shore)

All jump classes apart from low_1 are first order definable in $\mathcal{R}$ and in $\mathcal{D}_T(\leq \mathbf{0}')$.

$\mathcal{K}$-pairs

Definition

A pair of non-c.e. sets A and B are a $\mathcal{K}$-pair if there is a c.e. set W, such that $A \times B \subseteq W$ and $\overline{A} \times \overline{B} \subseteq \overline{W}$.

For example if $L_A = \{\sigma \in 2^{<\omega} \mid \sigma \leq_{lex} A\}$ and $R_A = \overline{L_A}$ then L_A and R_A are a $\mathcal{K}$-pair via $W = \{\langle \sigma, \tau \rangle \mid \sigma \leq_{lex} \tau\}$.

Theorem (Kalimullin)

A and B are a $\mathcal{K}$-pair if and only if $d_e(A) = \mathbf{a}$ and $d_e(B) = \mathbf{b}$ satisfy:

$$\forall \mathbf{x}(\mathbf{x} = (\mathbf{a} \vee \mathbf{x}) \wedge (\mathbf{b} \vee \mathbf{x})).$$

Definability of the enumeration jump

Theorem (Kalimullin)

$\mathbf{0}'_e$ is the largest enumeration degree that can be represented as $\mathbf{a} \vee \mathbf{b} \vee \mathbf{c}$, where $\{\mathbf{a}, \mathbf{b}\}$, $\{\mathbf{b}, \mathbf{c}\}$ and $\{\mathbf{a}, \mathbf{c}\}$ are $\mathcal{K}$-pairs.

Relativizing the notion of a $\mathcal{K}$-pair Kallimulin showed that the enumeration jump is first order definable.

Defining totality

Definition

A $\mathcal{K}$-pair $\{\mathbf{a}, \mathbf{b}\}$ is maximal if and only if no degree above $\mathbf{a}$ forms a $\mathcal{K}$-pair with $\mathbf{b}$ and no degree above $\mathbf{b}$ forma a $\mathcal{K}$-pair with $\mathbf{a}$.

Theorem (Cai, Ganchev, Lempp, Miller, S)

A nonzero enumeration degree is total if and only if it is the join of a maximal $\mathcal{K}$-pair.

Theorem (Cai, Ganchev, Lempp, Miller, S)

The image of the relation ‘c.e. in’ on Turing degrees is first order definable in $\mathcal{D}_e$.

Corollary (Cai)

The image of array noncomputable Turing degrees is first order definable in $\mathcal{D}_e$.

Local and global structural interaction

Theorem (Slaman, S)

$\mathcal{D}_e$ is rigid if any of the following structures are:

1. $\mathcal{R}$, the c.e. Turing degrees.
2. $\mathcal{D}_T(\leq \mathbf{0}')$, the Δ^0_2 Turing degrees.
3. $\mathcal{D}_e(\leq \mathbf{0}'_e)$, the Σ^0_2 enumeration degrees.

Start with a coded copy of the standard model of arithmetic $\mathcal{M}$ and a function $\psi : \mathbb{N}^{\mathcal{M}} \to \mathcal{D}_e(\leq \mathbf{0}'_e)$, such that $\psi(i^{\mathcal{M}}) = d_e(\overline{W}_i)$.

- Extend to index all total Δ^0_2 enumeration degrees.
- Extend to index all total degrees in $[\mathbf{x}, \mathbf{x}']$ where $\mathbf{x} \leq \mathbf{0}'_e$ is total.
- Extend to index all total enumeration degrees below $\mathbf{0}''_e$.

Definability in $\mathcal{D}_e(\leq \mathbf{0}'_e)$

Theorem (Ganchev, S)

$\mathcal{K}$-pairs are first order definable in $\mathcal{D}_e(\leq \mathbf{0}'_e)$ …

Theorem (Cai, Lempp, Miller, S)

… by the same first order formula as in $\mathcal{D}_e$.

Theorem (Ganchev, S)

The following are first order definable in $\mathcal{D}_e(\leq \mathbf{0}'_e)$:

1. The upwards properly Σ^0_2 e-degrees: not bounded by the join of $\mathcal{K}$-pair.
2. The downwards properly Σ^0_2 e-degrees: do not bound a $\mathcal{K}$-pair.
3. The total e-degrees: joins of maximal $\mathcal{K}$-pairs.
4. The Low_1 e-degrees: every nonzero degree bounded by them bounds a $\mathcal{K}$-pair.
5. The Low_{n+1} and High_n e-degrees.

Continuous degrees

Definition (Miller)

Let $\{\alpha_i\}_i$ be a sequence of real numbers. The enumeration degree of the set:
$\bigoplus_i(\{q \in \mathbb{Q} \mid q < \alpha_i\} \oplus \{q \in \mathbb{Q} \mid q > \alpha_i\})$
is called a continuous degree.

Theorem (Miller)

1. Every total enumeration degree is continuous.
2. There are non-total continuous enumeration degrees.
3. For $\mathbf{a}$, $\mathbf{b}$-total, "$\mathbf{b}$ is PA above $\mathbf{a}$" if and only if there is a non total continuous degree $\mathbf{x}$ such that $\mathbf{a} \leq \mathbf{x} \leq \mathbf{b}$.

Definition (Cai, Lempp, Miller, S)

A degree $\mathbf{x}$ is *almost total* if for every total enumeration degree $\mathbf{a} \not\leq \mathbf{x}$, we have that $\mathbf{x} \vee \mathbf{a}$ is total.

Cototal enumeration degrees

Definition

A set A is *cototal* if $A \leq_e \overline{A}$. A degree is cototal if it contains a cototal set.

The following degrees are co-total:

1. Total degrees.
2. Continuous degrees.
3. The n-c.e.a degrees.
4. Joins of $\mathcal{K}$-pairs.
5. The degree of the language of a minimal subshift.
6. The degree of the nonzero words in a finitely generated simple group.
7. The complements of maximal independent subsets for graphs on ω.

The skip operator

The enumeration jump of a set A is $K_A \oplus \overline{K_A}$, where $K_A = \bigoplus_{e<\omega} W_e(A)$.

Definition

The *skip* of a set A is $A^\diamond = \overline{K_A}$.

Theorem (Andrews, Ganchev, Kuyper, Lempp, Miller, Soskova, S)

- $A \leq_e B$ if and only if $A^\diamond \leq_1 B^\diamond$.
- Every degree above $\mathbf{0}'_e$ is the skip of some enumeration degree.
- $\mathbf{a}$ is cototal if and only if $\mathbf{a} \leq \mathbf{a}^\diamond$ if and only if $\mathbf{a}^\diamond = \mathbf{a}'$.
- The double skip has a fixed point: there are degrees $\{\mathbf{a}, \mathbf{b}\}$, such that $\mathbf{a} = \mathbf{b}^\diamond$ and $\mathbf{b} = \mathbf{a}^\diamond$.

The skip operator restricted to $\mathcal{K}$-pairs is first order definable: if $\{\mathbf{a}, \mathbf{b}\}$ is a $\mathcal{K}$-pair then $\mathbf{a}^\diamond = \mathbf{b} \vee \mathbf{0}'_e$.

The end

Thank you!