Enumeration pointed trees

Mariya I. Soskova
University of Wisconsin–Madison

North American ASL Meeting
Cornell University 2022, April 7-April 10
Work in progress with Jun Le Goh, Josiah Jacobsen-Grocott, and Joseph Miller

Supported by the NSF Grant No. DMS-1762648 and NSF Grant No. DMS-2053848

Outline

- Give a brief review of the enumeration degrees.
- Talk about applications of enumeration reducibility in computable structure theory and introduce enumeration pointed trees.
- Introduce the cototal degrees and their characterization by McCarthy in terms of minimal subshifts.
- Describe my recent work with Goh, Jacobsen-Grocott, and Miller in which we investigate enumeration pointed trees in Baire space and the related class of introenumerable degrees.
- Talk about the proof of one of the separations.

Enumeration reducibility

Friedberg and Rogers introduced enumeration reducibility in 1959.

Informally: $A \subseteq \omega$ is *enumeration reducible* to $B \subseteq \omega$ ($A \leqslant_e B$) if there is a uniform way to enumerate A from an enumeration of B.

Definition. $A \leqslant_e B$ if there is a c.e. set W such that

$$A = \{n \colon (\exists e) \langle n, e \rangle \in W \text{ and } D_e \subseteq B\},$$

where D_e is the eth finite set in a canonical enumeration.

Theorem(Selman 1971). $A \leqslant_e B$ if and only if for every set X if B is c.e. in X then A is c.e. in X.

The degree structure $\mathcal{D}_e$ induced by $\leqslant_e$ is called the *enumeration degrees*. It is an upper semi-lattice with a least element (the degree of all c.e. sets).

The total enumeration degrees

Proposition. $A \leqslant_T B \iff A \oplus \overline{A} \leqslant_e B \oplus \overline{B}$.

This suggests a natural embedding of the Turing degrees into the enumeration degrees.

Proposition. The embedding $\iota\colon \mathcal{D}_T \to \mathcal{D}_e$, defined by

$$\iota(d_T(A)) = d_e(A \oplus \overline{A}),$$

preserves the order and the least upper bound.

Definition. $A \subseteq \omega$ is *total* if $\overline{A} \leqslant_e A$ (equivalently, if $A \equiv_e A \oplus \overline{A}$). An enumeration degree is *total* if it contains a total set.

The image of the Turing degrees under the embedding ι is exactly the set of total enumeration degrees.

It is easy to prove that there are nontotal enumeration degrees. In fact, a sufficiently generic $A \subseteq \omega$ has nontotal degree.

Enumeration reducibility in computable structure theory

Let $\mathcal{A}$ be a countable structure. If $\mathcal{B}$ is isomorphic to $\mathcal{A}$ and has domain ω, we say that $\mathcal{B}$ is a *copy* of $\mathcal{A}$. We identify $\mathcal{B}$ with its atomic diagram.

Definition

We say that $\mathcal{A}$ has *Turing degree* $\deg_T(X)$ if X computes a copy of $\mathcal{A}$ and every copy of $\mathcal{A}$ computes X.

We say that $\mathcal{A}$ has *enumeration degree* $\deg_e(X)$ if every enumeration of X computes a copy of $\mathcal{A}$ and every copy of $\mathcal{A}$ computes an enumeration of X.

Note, if $\mathcal{A}$ has Turing degree $\deg_T(X)$ then its degree spectrum is the Turing cone above $\deg_T(X)$.

If $\mathcal{A}$ has enumeration degree $\deg_e(X)$ then its degree spectrum is the *enumeration cone* above X, i.e. the set of Turing degrees $\mathbf{a}$ such that $\iota(\mathbf{a}) \geqslant \deg_e(X)$.

If $\mathcal{A}$ has Turing degree $\deg_T(X)$ then $\mathcal{A}$ has enumeration degree $\deg_e(X \oplus \overline{X})$.

Examples of structures with enumeration degree but no Turing degree

1. (Calvert, Harizanov, Shlapentokh 07) Torsion-free abelian groups of finite rank always have enumeration degree and every enumeration degree is the degree of some such group.
2. (Frolov, Kalimullin, Miller 09) Fields of finite transcendence degree over $\mathbb{Q}$ always have enumeration degree and every enumeration degree is the degree of some such field.
3. (Steiner 13) Graphs of finite valence with finitely many connected components always have enumeration degree and every enumeration degree is the degree of some such graph.

Note! If the enumeration degree of a structure $\mathcal{A}$ is non-total then $\mathcal{A}$ does not have Turing degree.

Richter proved that if $\mathcal{A}$ has the c.e. embeddability condition then $\mathcal{A}$ does not have enumeration degree unless it is $\mathbf{0}_e$.

Descriptive complexity of degree spectra

Fix a structure $\mathcal{A}$ and consider the set
$D_{2^\omega(\mathcal{A})} = \{X \in 2^\omega | X \text{ computes a copy of } \mathcal{A}\}$.

Question

What is the descriptive complexity of $D_{2^\omega(\mathcal{A})}$?

We can ask the same question about
$D_{\omega^\omega(\mathcal{A})} = \{f \in \omega^\omega | f \text{ computes a copy of } \mathcal{A}\}$.

Theorem (Montalban)

$D_{2^\omega(\mathcal{A})}$ is never the upward closure of an F_σ set in Cantor space unless it is an enumeration cone.
$D_{\omega^\omega(\mathcal{A})}$ is never the upward closure of an F_σ set in Baire space unless it is an enumeration cone.

Definition

A tree $T \subseteq 2^{<\omega}$ (or $\subseteq \omega^{<\omega}$) with no dead ends is *e-pointed* if every branch in T enumerates T.

The cototal enumeration degrees

Question (Jeandel, email from Summer 2015)

If $A \leqslant_e \overline{A}$, what can be said about the enumeration degree of A?

This email inspired a paper on such enumeration degrees (Andrews, Ganchev, Kuyper, Lempp, Miller, A. Soskova, and M. Soskova 2019).

Definition

A set $A \subseteq \omega$ is *cototal* if $A \leqslant_e \overline{A}$. An enumeration degree is *cototal* if it contains a cototal set.

Theorem (Miller, Soskova 2018). The cototal enumeration degrees are a dense substructure of the enumeration degrees.

> Jeandel's interest in these enumeration degrees comes out of *symbolic dynamics*.

Minimal subshifts

Definition

- The *shift operator* is the map $\sigma\colon 2^\omega \to 2^\omega$ that erases the first bit of a given sequence.
- $\mathcal{C} \subseteq 2^\omega$ is a *subshift* if it is closed and shift-invariant.
- $\mathcal{C}$ is *minimal* if there is no nonempty, proper sub-subshift $\mathcal{D} \subset \mathcal{C}$.
- The *language* of subshift $\mathcal{C}$ is the set

$$L_{\mathcal{C}} = \{\sigma \in 2^{<\omega} \colon (\exists X \in \mathcal{C})\ \sigma \text{ is a subword of } X\}.$$

Proposition (Jeandel 2016). Let $\mathcal{C} \subseteq 2^\omega$ be a minimal subshift.

1. Every $\sigma \in L_{\mathcal{C}}$ appears along every member of $\mathcal{C}$.
2. A Turing degree computes a member of $\mathcal{C}$ if and only if it enumerates $L_{\mathcal{C}}$. Therefore, the degrees of members of $\mathcal{C}$ are exactly the total degrees above $\deg_e(L_{\mathcal{C}})$.
3. Furthermore, $L_{\mathcal{C}} \leqslant_e \overline{L_{\mathcal{C}}}$.

E-pointed trees in Cantor space

Theorem (McCarthy 2018)

Every cototal enumeration degree is the degree of the language of a minimal subshift.

McCarthy's proof passes through the notion of *e-pointed trees* in Cantor space.

Theorem (McCarthy 2018)

The following are equivalent for an enumeration degree $\mathbf{a}$:

1. $\mathbf{a}$ is cototal.
2. $\mathbf{a}$ contains a *uniformly* e-pointed tree $T \subseteq 2^{<\omega}$, i.e. a tree T such that for some enumeration operator Γ we have that $T = \Gamma(f)$ for every branch f in $T \subseteq 2^{<\omega}$.
3. $\mathbf{a}$ contains a *uniformly* e-pointed tree $T \subseteq 2^{<\omega}$ *with dead ends*.
4. $\mathbf{a}$ contains an e-pointed tree $T \subseteq 2^{<\omega}$.
5. $\mathbf{a}$ contains an e-pointed tree $T \subseteq 2^{<\omega}$ with *dead ends*.

E-pointed trees in Baire space

We are left with the following question:

Question

What are the degrees of trees $T \subseteq \omega^{<\omega}$ that are:

1. uniformly e-pointed;
2. e-pointed;
3. uniformly e-pointed with dead ends;
4. e-pointed with dead ends?

We know that they contain the cototal degrees, but are they still the same class?

Hyper-enumeration reducibility

Let A and B be sets of natural numbers.

Definition (Sanchis 1978)

$A \leqslant_{he} B$ (A is *hyperenumeration reducible* to B) if and only if there is a c.e. set W such that

$$A = \{x \mid \forall f \in \omega^\omega \exists n \exists v [\langle f \upharpoonright n, x, v \rangle \in W \ \& \ D_v \subseteq B]\}.$$

- We can think of W as defining a labeling l_x of $\omega^{<\omega}$ for every x: $D \in l_x(\sigma)$ if $\langle \sigma, x, D \rangle \in W$.
- B then produces a subtree $T_x \subseteq \omega^{<\omega}$ cutting branches whenever it sees a label D such that $D \subseteq B$.
- We have that $x \in A$ if and only if T_x is well-founded.

Known properties of hyper-enumeration reducibility

Sanchis proved that hyperenumeration reducibility has some natural properties:

1. We can replace the c.e. set W by a Π^1_1 set and we will have the same notion.
2. Hyperenumeration reducibility is a pre-order on $\mathcal{P}(\omega)$, and so it induces $\mathcal{D}_{he}$ the hyper enumeration degrees.
3. It extends enumeration reducibility: furthermore, if $A \leqslant_e B$ then $A \leqslant_{he} B$ and $\overline{A} \leqslant_{he} \overline{B}$.
4. A is Π^1_1 in B if and only if $A \leqslant_{he} B \oplus \overline{B}$.
5. $A \leqslant_h B$ if and only if $A \oplus \overline{A} \leqslant_{he} B \oplus \overline{B}$.

Let's call the hyperenumeration degrees of sets A such that $\overline{A} \leqslant_{he} A$ *hypertotal*.

Theorem (Sanchis 1978)

There are non-hypertotal degrees.

Selman's theorem fails in the hyper enumeration degrees

Theorem (Jacobsen-Groccot (tba))

There are sets A and B such that $A \nleqslant_{he} B$ but for every X if B is $\Pi^1_1(X)$ then A is $\Pi^1_1(X)$.

Proof idea:

Josiah builds a uniformly e-pointed tree $T \subseteq \omega^{<\omega}$ such that $\overline{T} \nleqslant_{he} T$.

If T is $\Pi^1_1(X)$ then there is a branch f in T such that $f \leqslant_h X$.
But then $T \leqslant_e f$ and so

$$\overline{T} \leqslant_{he} \overline{f} \leqslant_{he} f \leqslant_{he} X \oplus \overline{X}.$$

In other words, $\overline{T}$ is $\Pi^1_1(X)$.

Hyper-cotatal sets

Definition

A set A is *hypercototal* if and only if $A \leqslant_{he} \overline{A}$.

- Note, that this is an enumeration degree notion: if $A \equiv_e B$ then $\overline{A} \equiv_e \overline{B}$ so if $A \leqslant_{he} \overline{A}$ then $B \leqslant_{he} \overline{B}$.
- Every Π^1_1 set is hypercototal.
- $\overline{O}$ is not hypercototal because $\overline{O}$ is not Π^1_1.

Theorem (GJMS)

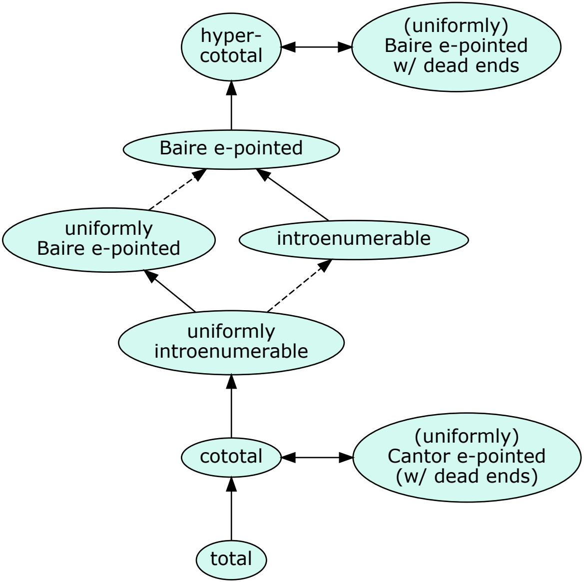
The following are equivalent for an enumeration degree $\mathbf{a}$

1. $\mathbf{a}$ contains a hypercototal set.
2. $\mathbf{a}$ contains a uniformly e-pointed tree $T \subseteq \omega^{<\omega}$ with dead ends.
3. $\mathbf{a}$ contains an e-pointed tree $T \subseteq \omega^{<\omega}$ with dead ends.

Theorem (GJMS)

If an enumeration degree $\mathbf{a}$ contains a 3-generic then $\mathbf{a}$ does not contain an e-pointed tree in Baire space.

The complete picture



Introenumerable enumeration degrees

Definition

A set A is *introenumerable* if $A \leqslant_e S$ for every infinite set $S \subseteq A$.

A is *uniformly introenumerable* if there is an enumeration operator Γ such that $A = \Gamma(S)$ for every infinite set $S \subseteq A$.

- Note, Jockusch (1968) defined (uniform) intreonumerability differently: he asked that A is c.e. in each of its infinite subsets. The uniform versions coincide.
- More recently, Greenberg, Harisson-Trainor, Patey, and Turetsky study introenumerable sets. They show that every uniformly introenumerable set has a uniformly introreducible subset.
- If A is introenumerable then A is enumeration equivalent to the tree if all injective enumerations of infinite subsets of A. This is an e-pointed tree in Baire space with no dead ends.
- Every cototal degree contains a uniformly introenumerable set: for example any uniformly e-pointed tree in Cantor space is uniformly introenumerable.

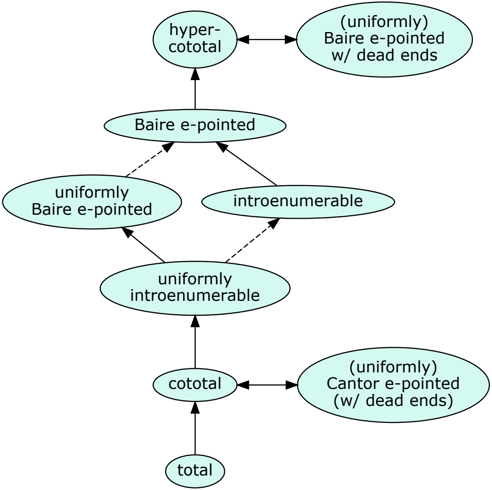
The complete picture

Our main results are:

Thm(GJMS). There is a uniformly introenumerable set that does not have cototal degree.

Thm(GJMS). There is a uniformly Baire e-pointed tree that does not have introenumerable degree.

We are working on the remaining open implications.



A uniformly Baire e-pointed degree that is not cototal

Theorem (GJMS)

There is a uniformly e-pointed tree in Baire space that does not have cototal degree.

Proof sketch:

Let $\{T_\sigma : \sigma \in \omega^{<\omega}\}$ be an effective listing of all finite trees in $\omega^{<\omega}$ where for each $\sigma \in \omega^{<\omega}$ the sequence $T_{\sigma 0}, T_{\sigma 1}, \ldots$ lists each finite tree that contains T_σ infinitely often. Our intention is to build a tree $T \subseteq \omega^{<\omega}$ using forcing, so that for every path $f \in T$ we have that $T = \bigcup_{\sigma \preceq f} T_\sigma$, thus T is uniformly e-pointed.

A forcing condition is a triple $p = (T^p, F^p, L^p)$, where

- T_p is a finite approximation to T;
- F^p consists of nodes we commit to leave out of T;
- $L^p : T_p \times T_p \to \omega \times 2$ is a labelling function that ensures every node on T is eventually enumerated along every path in T.

A uniformly Baire e-pointed degree that is not cototal

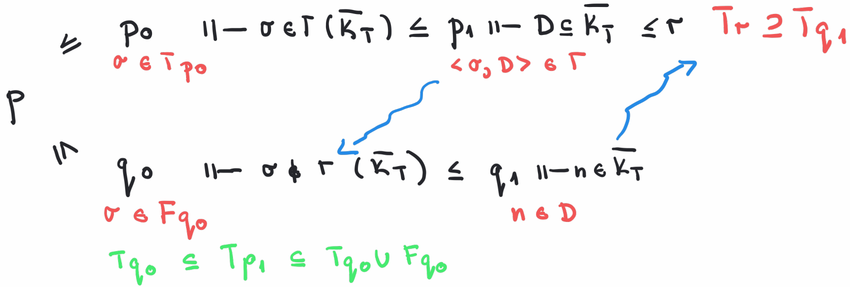
Theorem (GJMS)

There is a uniformly e-pointed tree in Baire space that does not have cototal degree.

Proof sketch:

We say that $q \leqslant p$ if $T^p \subseteq T^q$, $F^p \subseteq F^q$ and $L^p \subseteq L^q$.

To ensure that T does not have cototal degree it is enough to show that for every enumeration operator Γ we have $\Gamma(\overline{K_T}) \neq T$, where $K_T = \bigoplus_e \Gamma_e(T)$.



Thank you!