The definability of the total enumeration degrees and its consequences

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The total enumeration degrees

The structure of the enumeration degrees is an upper semi lattice with jump operation which extends the structure of the Turing degrees. It arises naturally from enumeration reducibility, a notion introduced by Friedberg and Rogers in 1959.

The total enumeration degrees are the image of the Turing degrees under their natural, structure preserving embedding into the enumeration degrees.

Question (Rogers)

Is the set of total enumeration degrees first order definable in the structure of the enumeration degrees?

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Definition

$A \leq_e B$ if there is a c.e. set W, such that

$$A = W(B) = \{x \mid \exists D(\langle x, D\rangle \in W \;\&\; D \subseteq B)\}\,.$$

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- The least upper bound: $d_e(A) \vee d_e(B) = d_e(A \oplus B)$.
- The enumeration jump: $d_e(A)' = d_e(K_A \oplus \overline{K_A})$, where $K_A = \{\langle e, x\rangle \mid x \in W_e(A)\}$.

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A set A is *total* if $A \equiv_e A \oplus \overline{A}$. An enumeration degree is *total* if it contains a total set. The set of total degrees is denoted by $\mathcal{TOT}$.

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$$(\mathcal{D}_T, \leq_T, \vee, ', \mathbf{0}_T) \cong (\mathcal{TOT}, \leq_e, \vee, ', \mathbf{0}_e) \subseteq (\mathcal{D}_e, \leq_e, \vee, ', \mathbf{0}_e)$$

Semi-computable sets

Definition (Jockusch)

A is semi-computable if there is a total computable function s_A, such that $s_A(x, y) \in \{x, y\}$ and if $\{x, y\} \cap A \neq \emptyset$ then $s_A(x, y) \in A$.

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Theorem (Arslanov, Cooper, Kalimullin)

If A is a semi-computable set then for every X:

$$(d_e(X) \vee d_e(A)) \wedge (d_e(X) \vee d_e(\overline{A})) = d_e(X).$$

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- If X is not computable then there is a semi-computable set A with $d_e(X \oplus \overline{X}) = d_e(A) \vee d_e(\overline{A})$.

Kalimullin pairs

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Theorem (Kalimullin)

A pair of sets A, B is a $\mathcal{K}$-pair if and only if their enumeration degrees **a** *and* **b** *satisfy:*

$$\mathcal{K}(\mathbf{a}, \mathbf{b}) \leftrightharpoons (\forall \mathbf{x} \in \mathcal{D}_e)((\mathbf{a} \vee \mathbf{x}) \wedge (\mathbf{b} \vee \mathbf{x}) = \mathbf{x}).$$

Definability of the enumeration jump

Theorem (Kalimullin)

$\mathbf{0}'_e$ is the largest degree which can be represented as the least upper bound of a triple $\mathbf{a}, \mathbf{b}, \mathbf{c}$, such that $\mathcal{K}(\mathbf{a}, \mathbf{b})$, $\mathcal{K}(\mathbf{b}, \mathbf{c})$ and $\mathcal{K}(\mathbf{c}, \mathbf{a})$.

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Corollary (Kalimullin)

1. *The enumeration jump is first order definable in $\mathcal{D}_e$.*

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1. *The enumeration jump is first order definable in $\mathcal{D}_e$.*
2. *The set of total enumeration degrees above $\mathbf{0}_e'$ is first order definable in $\mathcal{D}_e$.*

Definability in the local structure of the enumeration degrees

Theorem (Ganchev, S)

The class of $\mathcal{K}$-pairs below $\mathbf{0}'_e$ is first order definable in $\mathcal{D}_e(\leq \mathbf{0}'_e)$...

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Theorem (Ganchev, S)

The classes of the:

1. *Downwards properly Σ^0_2 enumeration degrees;*
2. *Upwards properly Σ^0_2 enumeration degrees;*
3. *Low enumeration degrees;*

are first order definable in $\mathcal{D}_e(\leq \mathbf{0}'_e)$.

Maximal $\mathcal{K}$-pairs

Definition

A $\mathcal{K}$-pair $\{\mathbf{a}, \mathbf{b}\}$ is maximal if for every $\mathcal{K}$-pair $\{\mathbf{c}, \mathbf{d}\}$ with $\mathbf{a} \leq \mathbf{c}$ and $\mathbf{b} \leq \mathbf{d}$, we have that $\mathbf{a} = \mathbf{c}$ and $\mathbf{b} = \mathbf{d}$.

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Corollary

In $\mathcal{D}_e(\leq \mathbf{0}'_e)$ a nonzero degree is total if and only if it is the least upper bound of a maximal $\mathcal{K}$-pair.

Defining total enumeration degrees in $\mathcal{D}_e$

Theorem (Cai, Ganchev, Lempp, Miller, S)

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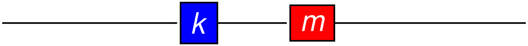
While $(m, k) \notin W$: $\mathbb{Q}$:



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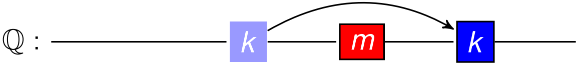

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Corollary (Ganchev, S)

Let $\mathbf{a}$ and $\mathbf{x}$ be Turing degrees such that $\mathbf{a}$ is not c.e. Then $\mathbf{a}$ is c.e. in $\mathbf{x}$ if and only if there is a maximal $\mathcal{K}$-pair $\{\mathbf{c}, \mathbf{d}\}$ such that $\mathbf{c} \leq_e \iota(\mathbf{x})$ and $\iota(\mathbf{a}) = \mathbf{c} \vee \mathbf{d}$.

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If $\mathbf{a} \in \mathcal{D}_T$ is c.e. then for every $\mathbf{b}$ we have that $\mathbf{a} \vee \mathbf{b}$ is c.e. in $\mathbf{b}$.

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Proof: $\mathbf{a}$ is c.e. iff for every $\mathbf{b} \not\leq \mathbf{0}'$ we have that $\mathbf{a} \vee \mathbf{b}$ is c.e. in $\mathbf{b}$.

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Theorem (Cai, Ganchev, Lempp, Miller, S)

The set $\mathcal{CE} = \{\iota(\mathbf{a}) \mid \mathbf{a} \in \mathcal{D}_T \text{ is c.e.}\}$ is first order definable in $\mathcal{D}_e$.

Proof: $\mathbf{a}$ is c.e. iff for every $\mathbf{b} \not\leq \mathbf{0}'$ we have that $\mathbf{a} \vee \mathbf{b}$ is c.e. in $\mathbf{b}$.

Corollary

The image of the relation " c.e. in " in the enumeration degrees is first order definable in $\mathcal{D}_e$.

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Question

Are the continuous degrees definable in $\mathcal{D}_e$*?*

Definable functions in the enumeration degrees

- The only known definable functions in $\mathcal{D}_T$ are on a cone: constant functions and various forms of the jump operator: $f(\mathbf{x}) = \mathbf{x}$; $f(\mathbf{x}) = \mathbf{x}'$; $f(\mathbf{x}) = \mathbf{x}^{(n)}$;$f(\mathbf{x}) = \mathbf{x}^{\omega}$, etc.

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Question

Is there a neat characterization of the definable functions in the enumeration degrees in the spirit of Martin's conjecture?

The automorphism analysis of the Turing degrees

Theorem (Slaman and Woodin (95))

1. *$Aut(\mathcal{D}_T)$ is countable, every member has an arithmetically definable presentation.*
2. *There is an element $\mathbf{g} \leq \mathbf{0}^{(5)}$ such that $\{\mathbf{g}\}$ is an automorphism base for $\mathcal{D}_T$.*
3. *Every relation on $\mathcal{D}_T$ induced by a degree invariant relation definable in Second order arithmetic is definable in $\mathcal{D}_T$ from parameters.*
4. *Every relation on $\mathcal{D}_T$ induced by a degree invariant relation definable in Second order arithmetic and invariant under automorphisms is definable in $\mathcal{D}_T$.*
5. *Every member of $Aut(\mathcal{D}_T)$ is the identity on the cone above $\mathbf{0}''$.*

The total degrees as an automorphism base

Theorem (Selman)

A is enumeration reducible to B if and only if
$\{\mathbf{x} \in \mathcal{TOT} \mid d_e(A) \leq \mathbf{x}\} \supseteq \{\mathbf{x} \in \mathcal{TOT} \mid d_e(B) \leq \mathbf{x}\}$.

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Question

Can every automorphism of $\mathcal{D}_T$ be extended to an automorphism of $\mathcal{D}_e$?

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6. We iterate until we reach the automorphism base below $\mathbf{0}_e^{(5)}$.

The end

Thank you!