Generic Muchnik reducibility and expansions of the reals

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Capturing the effective content of uncountable structures

If $\mathcal{A}$ and $\mathcal{B}$ are countable structures, then $\mathcal{A}$ is Muchnik reducible to $\mathcal{B}$ (written $\mathcal{A} \leq_w \mathcal{B}$) if every ω-copy of $\mathcal{B}$ computes an ω-copy of $\mathcal{A}$.

Definition (Schweber)

If $\mathcal{A}$ and $\mathcal{B}$ are (possibly uncountable) structures, then $\mathcal{A}$ is generically Muchnik reducible to $\mathcal{B}$ (written $\mathcal{A} \leq_w^* \mathcal{B}$) if $\mathcal{A} \leq_w \mathcal{B}$ in some forcing extension of the universe in which $\mathcal{A}$ and $\mathcal{B}$ are countable.

It follows from Shoenfield absoluteness that generic Muchnik reducibility is robust: If $\mathcal{A} \leq_w^* \mathcal{B}$, then $\mathcal{A} \leq_w \mathcal{B}$ in *every* forcing extension that makes $\mathcal{A}$ and $\mathcal{B}$ countable.

In particular, for countable structures, $\mathcal{A} \leq_w^* \mathcal{B} \iff \mathcal{A} \leq_w \mathcal{B}$.

Main examples

Definition

1. Cantor space is the structure $\mathcal{C}$ with domain 2^ω and predicates $P_n(X)$ that hold if and only if $X(n) = 1$.
2. The ordered field of the reals is the structure $\mathbb{R} = (\mathbb{R}; 0, 1, +, *, <)$.
3. Baire space is the structure $\mathcal{B}$ with domain ω^ω and predicates $P_{n,m}(X)$ that hold if and only if $X(n) = m$.

Knight, Montalbán and Schweber proved that all these examples have higher complexity than any countable structure.

Further, $\mathcal{C} \leq_w^* \mathbb{R}$. Let $\mathcal{I}$ be the countable Turing ideal of sets from the ground model. Any copy of $\mathbb{R}$ computes a listing of $\mathcal{I}$. Any listing of $\mathcal{I}$ computes a copy of $\mathcal{C}$.

$$\mathcal{C} \leq_w^* \mathbb{R}$$

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Igusa, Knight and Downey, Greenberg, J. Miller independently showed that $C <_w^* \mathbb{R}$.

Downey, Greenberg, and J. Miller showed that $\mathbb{R} \equiv_w^* \mathcal{B}$. Computing a copy of $\mathcal{B}$ is equivalent to computing a listing of the functions in $\mathcal{I}$.

$\mathcal{I}$ is a countable Scott ideal.There is a listing of the sets in $\mathcal{I}$ that does not compute a listing of the functions in $\mathcal{I}$.

$$\mathcal{C} <_w^* \mathcal{B} \equiv_w^* \mathbb{R}$$

Expansions of $\mathbb{R}$

Definition

Let f be any function on the reals. $\mathbb{R}_f$ is the ordered field of the reals $\mathbb{R}$ augmented by the function f.

Igusa, Knight, and Schweber investigated the Muchnik degree of expansions of the reals. The proved that $\mathbb{R}_{e^x} \equiv_w^* \mathbb{R}$. The main tool that they use is o-minimality. Building on that, they show that for every analytical f we have that $\mathbb{R}_f \equiv_w^* \mathbb{R}$.

Is there a continuous expansion of $\mathbb{R}$ that has strictly higher generic Muchnick degree than $\mathbb{R}$?

Theorem (AKKMS)

Any expansion of $\mathbb{R}$ by countably many continuous functions f is generically Muchnik equivalent to $\mathbb{R}$.

The running jump

Definition

Let $\{X_n\}_{n\in\omega}$ be a sequence of sets. The corresponding running jump is the sequence $\left\{\left(\bigoplus_{i\leq n} X_i\right)'\right\}_{n\in\omega}$.

Note that computing the running jump is equivalent to uniformly being able to compute the jump of any join of members of the list.

Suppose we can compute a listing of $\mathcal{I}$(the ground model elements of 2^ω) along with running jump.

For $X \in 2^\omega$, let $0.X$ denote the real number in $[0,1]$ with binary expansion X. For $z \in \mathbb{Z}$, let $z.X$ denote $z + 0.X$. Using $(X_0 \oplus X_1)'$, we can check if $z_0.X_0 = z_1.X_1$.

A continuous function f (from the ground model) can be coded by a parameter $P \in \mathcal{I}$. Using $(P \oplus X_0 \oplus X_1)'$, we can check if $f(z_0.X_0) = z_1.X_1$.

We can compute a copy of $\mathbb{R}_f$

Listing of $\mathcal{I}$ with the running jump

Lemma (AKKMS)

Let $\mathcal{I}$ be a countable jump ideal. Every listing of the functions in $\mathcal{I}$ computes a listing of the sets in $\mathcal{I}$ along with the running jump.

To compute the next set in the running jump, we guess a function in $\mathcal{I}$ that majorizes the corresponding settling-time function. If we are wrong, there is an injury (and a new guess).

When an injury occurs, we use the low basis theorem to "patch up" the listing consistently and keep control of the jumps.
Note:

- We need to start with a listing of the functions in $\mathcal{I}$ so that we can search for settling-time functions.
- We can only hope to compute the running jump for a listing of the sets in $\mathcal{I}$. (We can't use the low basis theorem in Baire space.)

Continuous expansions of $\mathcal{C}$

Theorem (AKKMS)

Any expansion of $\mathcal{C}$ by countably many continuous functions is generically Muchnik reducible to $\mathcal{B}$.

But even in some simple cases, they are equivalent.

Let $\sigma\colon \omega^\omega \to \omega^\omega$ denote the shift: i.e., $\sigma(n_0 n_1 n_2 n_3 \cdots) = n_1 n_2 n_3 \cdots$.
Let $\oplus\colon \omega^\omega \times \omega^\omega \to \omega^\omega$ denote the join. Both are continuous and both restrict to functions on 2^ω.

Proposition (AKKMS)

$(\mathcal{C}, \sigma) \equiv_w^* (\mathcal{C}, \oplus) \equiv_w^* \mathcal{B}$.

Theorem (Andrews, J. Miller, Schweber, S)

Every continuous expansion of $\mathcal{C}$ is either generically Muchnick equivalent to $\mathcal{C}$ or to $\mathcal{B}$.

Continuous expansions of $\mathcal{B}$

$Z = \{(f \oplus g) \oplus h \colon h$ is the settling-time function witnessing that $g = f'\}$ is a closed subset of ω^ω.

Let F be a continuous function on ω^ω such that $Z = F^{-1}(0^\omega)$.

Theorem (AKKMS)

$(\mathcal{B}, \oplus, F) \equiv^*_w (\mathcal{B}, \oplus, ') \equiv^*_w (\mathcal{C}, \oplus, ')$.

Let $\mathcal{I}$ be the countable jump ideal of ground model sets after collapse. The following are equivalent for a degree $\mathbf{d}$:

1. $\mathbf{d}$ computes a copy of $(\mathcal{C}, \oplus, ')$,
2. $\mathbf{d}$ computes a listing of the sets in $\mathcal{I}$ along with join and jump as functions on indices,
3. $\mathbf{d}$ computes a listing of the functions in $\mathcal{I}$ along with the running jump.

Hyper-Scott ideals

Definition

An ideal $\mathcal{I}$ is a hyper-Scott ideal if whenever a tree $T \subseteq \omega^{<\omega}$ in $\mathcal{I}$ has an infinite path, it has an infinite path in $\mathcal{I}$.

If $\mathcal{I}$ is the Turing ideal of sets from the ground model, then it is a hyper-Scott ideal.

Theorem (AKKMS)

If $\mathcal{I}$ is a countable hyper-Scott ideal then there is a listing of the functions in $\mathcal{I}$ that does not compute a listing of the functions in $\mathcal{I}$ along with join and jump as functions on indices.

$$\mathcal{C} \quad <^*_w \quad \mathbb{R} \equiv^*_w \mathbb{R}_f \equiv^*_w \mathcal{B} \quad <^*_w \quad (\mathcal{C}, \oplus, ').$$

The Borel degree

Definition

Let $(D, E, f_1, f_2, \dots)$ be a structure, such that $D \subseteq \omega^\omega$ is Borel, E is a Borel equivalence relation on D, and $f_1, f_2, \dots$ are Borel functions on D that are compatible with E. The induced structure with domain D/E is called a Borel structure.

Examples: the Turing degrees with join and jump, the automorphism group of any countable structure.

Theorem (AKKMS)

Every Borel structure is $\leq_w^* (\mathcal{C}, \oplus, ')$. We call the degree of $(\mathcal{C}, \oplus, ')$ the Borel degree.

Theorem (Andrews, J. Miller, Schweber, S)

Every continuous expansion of $\mathcal{B}$ is either generically Muchnick equivalent to $\mathcal{B}$ or its degree is the Borel degree.

The end

Thank you!