Extensions of the Turing model for relative definability

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Preliminaries: Enumeration reducibility

Definition

$A \leq_e B$ if and only if there is a c.e. set W, such that $x \in A$ if and only if $\exists u(\langle x, u\rangle \in W \wedge D_u \subseteq B)$.

$\mathcal{D}_e = \langle D_e, \leq, \vee, ', \mathbf{0}_e\rangle$ is an upper semi-lattice with jump operation and least element.

Note that $A \leq_T B$ if and only if $A \oplus \overline{A} \leq_e B \oplus \overline{B}$.

Proposition

The embedding $\iota : \mathcal{D}_T \to \mathcal{D}_e$, defined by $\iota(d_T(A)) = d_e(A \oplus \overline{A})$, preserves the order, the least upper bound and the jump operation.

The substructure of the total e-degrees is defined as $\mathcal{TOT} = \iota(D_T)$.

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Three examples

1. Computable model theory: degree spectra of torsion free abelian groups of rank 1.
2. Definability of the jump operation.
   - Slaman and Shore: definability in $\mathcal{D}_T$.
   - Kalimullin: definability in $\mathcal{D}_e$.
3. Local definability.
   - Shore: Biinterpretability up to double jump in the Δ^0_2 Turing degree degrees. The jump classes H_n and L_{n+1} are definable.
   - Ganchev and S: The *total* degrees and L_1 are definable in the local structure of the Σ^0_2 enumeration degrees.

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$\mathcal{D}_T$: Coles, Downey and Slaman: "Every set has a least jump enumeration"

Given a set A let $\mathcal{C}(A) = \{X \mid A \text{ is c.e. in } X\}$.

Theorem (Richter)

There is a set A, such that $\mathcal{C}(A)$ does not have a member of least degree.

Theorem (Coles, Downey, Slaman)

For every sets A the set: $\mathcal{C}(A)' = \{X' \mid A \text{ is c.e. in } X\}$ has a member of least degree: $c'_\mu(A)$.

A set of the degree $c'_\mu(A)$ is obtained using forcing with finite conditions.

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Motivation: torsion-free abelian groups of rank 1

A torsion free abelian group of rank 1 G is (isomorphic to) a subgroup of $(\mathbb{Q}, +, =)$.

Definition

Let p be a prime number and $a \in G$.

$$h_p(a) = \begin{cases} \text{the largest } k, & \text{such that } p^k | a \text{ in } G; \\ \infty, & \text{if } \forall k (p^k | a \text{ in } G) \ . \end{cases}$$

Here $p^k | a$ in G if there exists $b \in G$ such that $p^k . b = a$.

If $a, b \neq 0$ then for all but finitely many p, $h_p(a) = h_p(b)$.

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The type of G

Definition

The characteristic of an element $a \in G$ is the sequence:

$$\chi(a) = (h_{p_0}(a), h_{p_1}(a), \dots h_{p_n}(a), \dots)$$

So if $a, b \neq 0$ then $\chi(a) =^* \chi(b)$.
The type of G, denoted $\chi(G)$ is the equivalence class of $\chi(a)$ for any $a \neq 0$ in G.

Baer noticed that there is a TFA1 group of every possible type.

Theorem (Baer)

Two torsion-free abelian groups of rank 1 are isomorphic if and only if they have the same type.

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Let $S(G) = \{\langle i, j\rangle \mid j \leq$ the i-th element of $\chi(G)\}$.

Note that every S can be coded as (is m-equivalent to) $\{\langle i, j\rangle \mid j = 0 \vee i \in S \;\&\; j = 1\}$.

Theorem (Downey, Jockusch)

The degree spectrum of G: $\{d_T(H) \mid H \cong G\}$ *is precisely* $\{deg_T(Y) \mid S(G)$ is c.e. in $Y\}$.

$c'_\mu(A)$ *exists for all A if and only if for every torsion-free abelian group G the jump spectrum of G, the set* $\{d_T(H)' \mid H \cong G\}$ *has a least element.*

Corollary (Coles, Downey, Slaman)

Every every torsion-free abelian group of rank 1 G has a jump degree.

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More connections between $\mathcal{D}_T$ and $\mathcal{D}_e$

- B is c.e. in A if and only if $B \leq_e A \oplus \overline{A}$.
- Selman's Theorem: $A \leq_e B$ if and only if

$$\{X \mid B \text{ is c.e. in } X\} \subseteq \{X \mid A \text{ is c.e. in } X\}.$$

$$\left\{d_e(X \oplus \overline{X}) \mid B \leq_e X \oplus \overline{X}\right\} \subseteq \left\{d_e(X \oplus \overline{X}) \mid A \leq_e X \oplus \overline{X}\right\}$$

Recall that $d_e(A)' = d_e(K_A \oplus \overline{K_A})$ is by definition total.

Theorem (Soskov's JIT)

For every degree $\mathbf{x}$ exists a total degree $\mathbf{a} \geq \mathbf{x}$ and $\mathbf{a}' = \mathbf{x}'$.

Theorem (McEvoy's JIT)

For every total $\mathbf{q} \geq \mathbf{0}_e'$, there is a non-total degree $\mathbf{a}$ such that $\mathbf{a}' = \mathbf{q}$.

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$\mathcal{D}_e$: Soskov "Degree spectra and co-spectra of structures"

Consider a structure $\mathcal{A} = (\mathbb{N}, R_1 \dots R_k)$.

Definition

An enumeration is any bijective mapping $f : \mathbb{N} \to \mathbb{N}$.

If R is an n-ary relation then
$f^{-1}(R) = \{\langle x_1, \dots, x_n \rangle \mid (f(x_1), \dots f(x_n)) \in R\}$.

The pullback of $\mathcal{A}$ is the set $f^{-1}(\mathcal{A}) = f^{-1}(R_1) \oplus f^{-1}(R_2) \dots \oplus f^{-1}(R_k)$.

- Consider the structure $\mathcal{A}^+ = (\mathbb{N}, R_1, \overline{R_1} \dots R_k, \overline{R_k})$.
- Then $f^{-1}(\mathcal{A}^+)$ is total set and corresponds to an isomorphic presentation of $\mathcal{A}$ with isomorphism induced by f and atomic diagram e-equivalent $f^{-1}(\mathcal{A}^+)$.

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$\mathcal{D}_e$: Soskov "Degree spectra and co-spectra of structures"

Consider a structure $\mathcal{A} = (\mathbb{N}, R_1 \dots R_k)$.

Definition

An enumeration is any bijective mapping $f : \mathbb{N} \to \mathbb{N}$.

If R is an n-ary relation then
$f^{-1}(R) = \{\langle x_1, \dots, x_n \rangle \mid (f(x_1), \dots f(x_n)) \in R\}$.

The pullback of $\mathcal{A}$ is the set $f^{-1}(\mathcal{A}) = f^{-1}(R_1) \oplus f^{-1}(R_2) \cdots \oplus f^{-1}(R_k)$.

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Enumeration degree spectrum

Fix $\mathcal{A} = (\mathbb{N}, R_1 \dots R_k)$.

Definition

The e-degree spectrum of $\mathcal{A}$ is
$DS_e(\mathcal{A}) = \left\{ d_e(f^{-1}(\mathcal{A})) \mid f \text{ in an enumeration} \right\}$.

If $DS_e(\mathcal{A})$ has a least member, it is the (enumeration) degree of $\mathcal{A}$.

- In fact $DS_e(\mathcal{A}^+) = \{\iota(\mathbf{a}) \mid \mathbf{a} \in DS_T(\mathcal{A})\}$.
- $\mathcal{A}$ has T-degree $\mathbf{a}$ if and only if $\mathcal{A}^+$ has e-degree $\iota(\mathbf{a})$.

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- If $\mathcal{A}$ has degree $\mathbf{a}$ then it has co-degree $\mathbf{a}$.
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Every countable ideal of enumeration degrees can be represented as the co-spectrum of a structure.

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The case of principal ideals

Let $(G, +, =)$ be a torsion-free abelian group of rank 1.

Note that for every enumeration f, $f^{-1}(G) \equiv_e f^{-1}(G^+)$.

Recall that the degree spectrum of G: $\{d_T(H) \mid H \cong G\}$ is precisely $\{deg_T(Y) \mid S(G)$ is c.e. in $Y\}$.

So the enumeration degree spectrum of G is $DS_e(G) = \{\mathbf{a} \mid \mathbf{a} \in \mathcal{TOT} \ \& \ d_e(S(G)) \leq_e \mathbf{a}\}$.

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$$DS_e(G) = \{\mathbf{a} \mid \mathbf{a} \in \mathcal{TOT} \;\&\; \mathbf{s}_G \leq \mathbf{a}\}$$

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The end

Thank you!