Randomness relative to an enumeration oracle

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Algorithmic randomness relative to an oracle

Definition

A $\Sigma^0_1\langle A\rangle$ class U is a subset of 2^ω given by $U = [W]^\prec$ where $W \leq_e A$.

Definition

1. An *$\langle A\rangle$-test* is a uniform sequence of $\Sigma^0_1\langle A\rangle$ classes $\{V_n\}_{n<\omega}$ such that $\mu V_n \leq 2^{-n}$ for every n.
2. A sequence $Z \in 2^\omega$ *passes* the test V if $Z \notin \bigcap_{n<\omega} V_n$.
3. The sequence Z is *$\langle A\rangle$-random* if it passes all $\langle A\rangle$-tests.

Z is $\langle A \oplus \overline{A}\rangle$-random if and only if Z is A-random.

Equivalent forms

Definition

A *Solovay $\langle A\rangle$-test* is a uniform sequence of $\Sigma^0_1\langle A\rangle$ classes $\{V_n\}_{n<\omega}$ such that $\sum_n \mu V_n < \infty$.
Z *passes* a Solovay $\langle A\rangle$-test $\{V_n\}_{n<\omega}$ if Z is in only finitely many members of the test.

Definition

A *Kučera $\langle A\rangle$-test* is a $\Sigma^0_1\langle A\rangle$ class V with $\mu V < 1$.
Z *passes* a Kučera $\langle A\rangle$-test V if not every tail of Z is in V.

Theorem

Z is $\langle A\rangle$-random if and only

- Z passes every Solovay $\langle A\rangle$-test.
- Z passes every Kučera $\langle A\rangle$-test.

Characterizing $\langle A \rangle$-randomness via (super)martingales

Recall that a function $d : 2^{<\omega} \to \mathbb{R}^{\geq 0}$ is a

1. *martingale* if for every $\sigma \in 2^{<\omega}$ we have that $d(\sigma) = \frac{d(\sigma 0)+d(\sigma 1)}{2}$;
2. *super martingale* if for every $\sigma \in 2^{<\omega}$ we have that $d(\sigma) \geq \frac{d(\sigma 0)+d(\sigma 1)}{2}$.

Definition

A (super)martingale d is *$\langle A \rangle$-enumerable* if

$$U_d = \{(\sigma, q) \mid q \in \mathbb{Q} \ \& \ d(\sigma) > q\} \leq_e A.$$

Theorem

Z is $\langle A \rangle$-random if and only no $\langle A \rangle$-enumerable (super)martingale *succeeds* on Z.

We can define d succeeds on Z as:

- $\limsup_n d(Z \upharpoonright n) = \infty$ or as
- $\lim_n d(Z \upharpoonright n) = \infty$.

Comparing $\langle A \rangle$-randomness to randomness relative to total oracles

Theorem

For any set A and sequence Z, consider the following "relative randomness" notions:

1. Z is X-random for some X such that $A \leq_e X \oplus \overline{X}$ (*upwards $\langle A \rangle$-random*),
2. Z is $\langle A \rangle$-random,
3. Z is X-random for every X such that $X \oplus \overline{X} \leq_e A$ (*downwards $\langle A \rangle$-random*).

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3).
Furthermore, each of these implications can be strict.

Lowness for randomness

Definition

A set A is *low for randomness* if every 1-random is $\langle A \rangle$-random;

Proposition

The following two conditions are equivalent:

1. A is low for randomness.
2. Every $\Sigma^0_1\langle A \rangle$ class U with $\mu U < 1$ is covered by a Σ^0_1 class V with $\mu V < 1$.

Proposition

Every 1-generic set is low for randomness.

Every semi-recursive set is low for randomness.

$\langle A \rangle$-randomness cannot be expressed through total oracles

Proposition

If A is weakly 1-generic relative to Z and $A \leq_e X \oplus \overline{X}$ then Z is not X-random.

Fix a weakly 2-generic set A.

- A is weakly 1-generic relative to Chaitin's Ω, so Ω is not random relative to any total oracle above A.
- A is 1-generic and hence low for randomness, so Ω is $\langle A \rangle$-random.

Proposition

If $X \leq_e A$ then X is not $\langle A \rangle$-random.

Fix A such that A is 1-random and of A is of quasiminimal degree.

- A is random with respect to every total oracle below A.
- A is not $\langle A \rangle$-random.

$\langle$self$\rangle$-PA sets

Recall, that a $\Pi^0_1\langle A\rangle$ class is the complement of some $\Sigma^0_1\langle A\rangle$ class.

Definition

An enumeration oracle *$\langle A\rangle$ is PA above $\langle B\rangle$* if every nonempty $\Pi^0_1\langle B\rangle$ class contains an element Z such that $Z \oplus \overline{Z} \leq_e A$.

A is *$\langle$self$\rangle$-PA* if $\langle A\rangle$ is PA above $\langle A\rangle$.

Theorem

1. There is a $\langle$self$\rangle$-PA set A.
2. If A is $\langle$self$\rangle$-PA then the set of total degrees below A is a Scott set.
3. Every countable Scott set can be realized as the set of total degrees below a $\langle$self$\rangle$-PA set.
4. If X is PA above Y if and only if then there is a $\langle$self$\rangle$-PA A such that $Y \oplus \overline{Y} <_e A <_e X \oplus \overline{X}$.

Randomness properties of $\langle\text{self}\rangle$-PA sets

Proposition

If A is $\langle\text{self}\rangle$-PA then there is no universal $\langle A\rangle$-test.

Proposition

If A is $\langle\text{self}\rangle$-PA then every $\Sigma^0_1\langle A\rangle$ class of measure < 1 is covered by a $\Sigma^0_1(Y)$ class of measure < 1, for some Y such that $Y \oplus \overline{Y} \leq_e A$.

If A is $\langle\text{self}\rangle$-PA then Z is $\langle A\rangle$-random if an only if Z is downwards $\langle A\rangle$-random.

Proposition

If A has continuous degree, then there is a universal $\langle A\rangle$-test.

If A has continuous degree, then Z is $\langle A\rangle$-random iff Z is upwards $\langle A\rangle$-random.

Set randomness notions

Definition

1. A *Martin-Löf set $\langle A\rangle$-test* is a sequence $\{W_n\}_{n<\omega}$ that is uniformly enumeration reducible to A, such that for every n $\mathrm{wt}(W_n) \leq 2^{-n}$.
2. A *Solovay set $\langle A\rangle$-test* is a set $W \leq_e A$ with finite weight.

Here $\mathrm{wt}(W) = \sum_{\sigma\in W} 2^{-|\sigma|}$.

Theorem

Upwards $\langle A\rangle$-randomness $\Rightarrow$ $\langle A\rangle$-randomness $\Rightarrow$ Solovay set $\langle A\rangle$-randomness $\Rightarrow$ ML set $\langle A\rangle$-randomness $\Rightarrow$ Downwards $\langle A\rangle$-randomness.

Theorem

There are sets Z and A such that Z is not $\langle A\rangle$-random, but Z is Solovay set $\langle A\rangle$-random.

An incompressibility approach to randomness

A *discrete measure* is a function $m\colon 2^{<\omega} \to \mathbb{Q}$ such that $\sum_{\sigma \in 2^{<\omega}} m(\sigma) \leq 1$.

Definition

A discrete measure m is *$\langle A \rangle$-enumerable* if $U_m = \{(\sigma, q) \mid m(\sigma) > q\} \leq_e A$.

$$K_m(\sigma) = -\log(m(\sigma)).$$

Theorem

Z is Solovay set $\langle A \rangle$-random if and only if $K_m(Z \upharpoonright n) - n \to \infty$ for every $\langle A \rangle$-enumerable discrete measure m.

Z is ML set $\langle A \rangle$-random if and only if $K_m(Z \upharpoonright n) \geq^+ n$ for every $\langle A \rangle$-enumerable discrete measure m.

The end

Thank you!