NONDENSITY OF DOUBLE BUBBLES IN THE D.C.E. DEGREES

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Abstract. In this paper, we show that the so-called "double bubbles" are not downward dense in the d.c.e. degrees. Here, a pair of d.c.e. degrees $\mathbf{d}_1 > \mathbf{d}_2 > \mathbf{0}$ forms a *double bubble* if all d.c.e. degrees below $\mathbf{d}_1$ are comparable with $\mathbf{d}_2$.

1. Introduction

In this paper, we study a fundamental structural property of the d.c.e. degrees. The d.c.e., and more generally the n-c.e., sets and degrees were introduced by Putnam [13] and Gold [8] as a generalization of the c.e. sets and degrees. A set A is n-c.e. if it has an approximation that can change the value of $A(x)$ at most n times for every natural number x, starting with $x \notin A$. When $n = 1$, we obtain the c.e. sets, and when $n = 2$, we obtain the difference of c.e. sets—the d.c.e. sets. Later on, this hierarchy was extended by Ershov [5, 6, 7] to arbitrary computable ordinals. The difference hierarchy gives rise to a corresponding nested hierarchy of degree structures, all contained in the Δ^0_2-Turing degrees. Naturally, one wonders if these structures are different. Lachlan showed that every nonzero n-c.e. degree bounds a nonzero c.e. degree, thus the structures of the n-c.e. degrees are different from that of the Δ^0_2-degrees, which contains minimal elements. Lachlan's proof relies on a particular set that is fairly easy to define: If A is d.c.e. and $\{A_s\}_{s<\omega}$ is a d.c.e. approximation to A, then the *Lachlan set* $L(A)$ is the c.e. set of stages s at which some element x enters A which then later leaves A.

Next, Arslanov [1] found an elementary difference between the structures of the c.e. degrees and the d.c.e degrees: Cooper and Yates (see [3], [12]) had constructed a noncuppable nonzero c.e. degree, whereas Arslanov [1] showed that every nonzero d.c.e. degree is cuppable. Downey [4] found a further difference between the two structures: He showed that the diamond can be embedded into the d.c.e. degrees, in contrast to the Lachlan Non-Diamond Theorem for the c.e. degrees [10]. Downey's

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work lead him to conjecture that for every $n, m > 1$, the structures of the n-c.e. degrees and the m-c.e. degrees are elementarily equivalent.

Arslanov, Kalimullin and Lempp [2] disproved this conjecture. They showed that the structure of d.c.e. degrees contains special pairs of degrees which they informally called a *double bubble*. They used this notion and a generalization of this notion to the 3-c.e. degrees to refute Downey's Conjecture by showing that the partial orders of the d.c.e. and the 3-c.e. Turing degrees are not elementarily equivalent. A pair of d.c.e. degrees $\mathbf{d}_1 > \mathbf{d}_2 > \mathbf{0}$ forms a *double bubble* if all d.c.e. degrees below $\mathbf{d}_1$ are comparable with $\mathbf{d}_2$. We call $\mathbf{d}_2$ the *middle* of the bubble and $\mathbf{d}_1$ the *top* of the bubble. Double bubbles play an important role in the study of the properly d.c.e. degrees. They have many nontrivial properties and have sparked a lot of interest. It is easy to see, by relativizing the Sacks Splitting Theorem [14], that the top of a bubble must always be properly d.c.e. On the other hand, it was shown in [2] that the middle is always a c.e. degree. A more elaborate property is related to the notion of an *exact d.c.e. degree*. Exact degrees were introduced and first studied by Ishmukhametov [9]; a d.c.e. degree $\mathbf{d}$ is called *exact* if all Lachlan sets of d.c.e. members of $\mathbf{d}$ have the same degree. It follows from [2] that the top of every bubble is an exact degree and the middle of the bubble is the degree of the Lachlan set of any member of the top of the bubble (a proof can be found, e.g., in [16]).

Liu, Wu and Yamaleev [11] investigated the possibility of combining the construction of a double bubble in the d.c.e. degrees with other properties, such as upward and downward density in the c.e. degrees. They noted that a positive answer to the full density question would allow us to define the c.e. degrees within the d.c.e. degrees.[1] If $\mathbf{d}$ is a properly d.c.e. degree and $\mathbf{d}_1, \mathbf{d}_2$ form a nontrivial splitting of $\mathbf{d}$ in the d.c.e. degrees, then at least one of the intervals $(\mathbf{d}_1, \mathbf{d})$ or $(\mathbf{d}_2, \mathbf{d})$ must be free of c.e. degrees and hence bubbles, otherwise $\mathbf{d}$ would be c.e. On the other hand, if every nonempty interval of c.e. degrees contained a bubble, then, by the Sacks Splitting Theorem, every nonzero c.e. degree $\mathbf{c}$ would have a nontrivial splitting $\mathbf{c}_1, \mathbf{c}_2$, such that both intervals $(\mathbf{c}_1, \mathbf{c})$ and $(\mathbf{c}_2, \mathbf{c})$ contain a bubble. Liu, Wu and Yamaleev [11] showed that exact degrees are downward dense in the c.e. degrees and left the downward density of double bubbles as an open question. In this paper, we show that double bubbles are not downward dense in the d.c.e. degrees. Of course, it suffices to show that double bubbles do not necessarily exist below any nonzero c.e. degree:

Theorem 1.1. *There exists a c.e. degree* $\mathbf{a}$ *such that there are no d.c.e. degrees* $\mathbf{d_2} < \mathbf{d}_1 \leq \mathbf{a}$ *which form a double bubble in the d.c.e. degrees.*

The rest of this paper is devoted to the proof of this theorem. Our notation and terminology is standard and generally follows Soare [15]. We also use standard notation and terminology for priority constructions.

2. Strategies

2.1. **Requirements.** Recall that the top of a double bubble is always an exact degree. We will give a more formal definition of what this means. Fix a d.c.e. set D and a d.c.e. enumeration $\{D_s\}_{s\in\omega}$ of D. For technical reasons, we will assume from now on that at any stage, any set D changes at at most one number, thus

[1] In fact, this idea goes back to Arslanov, who noted it in private communication with Shore. Later, he publicized this idea in conference talks.

$|D_s \triangle D_{s-1}| \leq 1$ for any $s \in \omega$. We define the partial computable function $s^D(x)$ as the stage of entry of x into D, i.e., $s^D(x) \downarrow = s$ is defined if x is enumerated into D at stage s. If x is never enumerated into D then $s^D(x) \uparrow$. The *Lachlan set of* D *with respect to the enumeration* $\{D_s\}_{s<\omega}$ is defined as

$$L(D) = \{s \mid (\exists x)(s^D(x) \downarrow = s \ \& \ x \notin D)\}.$$

Although it may not be immediately obvious from the definition, every Lachlan set is c.e. This follows from the fact that it is defined with respect to a d.c.e. approximation and so '$x \notin D$' can be substituted by '$(\exists t > s)(x \notin D_t)$'. Furthermore, it is not difficult to see that the degree of $L(D)$ does not depend on the particular choice of a d.c.e. approximation for D.

If $\mathbf{d} = \deg(D)$ is a d.c.e. degree, then the *set of Lachlan degrees of* $\mathbf{d}$ is the set

$$L[\mathbf{d}] = \{\deg(L(B)) \mid \ B \in \mathbf{d} \text{ and } B \text{ is d.c.e.}\}.$$

A d.c.e. degree $\mathbf{d}$ is *exact* if $|L[\mathbf{d}]| = 1$.

Fix a double bubble $\mathbf{d}_1 > \mathbf{d}_2 > \mathbf{0}$. If $D \in \mathbf{d}_1$ is d.c.e., then $L(D) \in \mathbf{d}_2$ (by Ishmukhametov [9, Proposition 1.2] and Arslanov/Kalimullin/Lempp [2, Theorem 5]). So in order to prove the theorem, we must construct a noncomputable c.e. set A such that for any noncomputable d.c.e. set $D \leq_T A$, if $\mathbf{0} < \deg(L(D)) < \deg(D)$, then there is a d.c.e. set $E \leq_T D$ that is Turing incomparable with $L(D)$. Fix a computable listing of all tuples $\langle \Phi, \Psi, \Theta, \Omega, D\rangle$ of partial computable functionals $\Phi, \Psi, \Theta, \Omega$ and d.c.e. sets D. It suffices to build a c.e. set A satisfying the following list of requirements:

$$\mathcal{P}_\Theta \colon A \neq \Theta;$$

$$\mathcal{R}_{\Phi,D} \colon D = \Phi^A \Rightarrow$$

$$\exists E\, \exists \Lambda_{\Phi,D}\, (E = \Lambda^D_{\Phi,D} \wedge E \mid_T L(D)) \vee D \leq_T L(D) \vee L(D) \leq_T \emptyset,$$

where each $\mathcal{R}$-requirement has its own infinite list of subrequirements:

$$\mathcal{T}_\Psi \colon E = \Psi^{L(D)} \Rightarrow \exists \Gamma_\Psi\, (D = \Gamma_\Psi^{L(D)});$$

$$\mathcal{S}_\Omega \colon L(D) = \Omega^E \Rightarrow \exists \Delta_\Omega\, (L(D) = \Delta_\Omega) \vee \exists \Gamma_\Omega\, (D = \Gamma_\Omega^{L(D)}).$$

(We will usually suppress the subscripts on the functionals above when they are clear from the context.) We will construct A using a tree of strategies and the gap/co-gap method. The proof will be a $\mathbf{0}'''$-priority argument. We will first describe the intuition behind the construction, starting with each strategy in isolation.

2.2. **Strategies in isolation.** Recall our convention that at every stage, any of the given sets can change at at most one element.

The basic $\mathcal{P}$*-strategy.* The basic $\mathcal{P}$-strategy is a variant of the standard Friedberg–Muchnik strategy. We choose a fresh witness a, wait for a stage s such that $\Theta(a)[s] \downarrow = 0$, and enumerate a into A.

The basic $\mathcal{R}$*-strategy.* An $\mathcal{R}$-strategy ρ serves as the mother strategy for all of its substrategies. It monitors the length of agreement between D and Φ^A. At non-expansionary stages, it takes the finitary outcome *fin*. At expansionary stages, it makes progress towards building the functional Λ so that $\Lambda^D = E$ and takes its infinite outcome ∞, allowing its $\mathcal{S}$- and $\mathcal{T}$-substrategies to act.

The basic $\mathcal{T}$-strategy. A $\mathcal{T}$-strategy τ is a child strategy of some $\mathcal{R}$-strategy. In isolation, it checks the length of agreement between E and $\Psi^{L(D)}$. At expansionary stages, τ builds Γ_τ so that $\Gamma_\tau^{L(D)} = D$. The strategy has two possible outcomes, Γ and *fin*.

The basic $\mathcal{S}$-strategy. An $\mathcal{S}$-strategy σ, say, is a child strategy of some $\mathcal{R}$-strategy. In isolation, it checks the length of agreement between $L(D)$ and Ω^E, and if the stage is expansionary, then σ first tries to build Δ_σ so that $\Delta_\sigma = L(D)$. This strategy exhibits an interesting behavior in response to other strategies. We will describe this in the next subsection and see how this response may cause σ to build a backup functional Γ_σ so that $\Gamma_\sigma^{L(D)} = D$ at expansionary stages. The strategy has three possible outcomes, Γ, Δ, and *fin*.

2.3. **Interactions between strategies.** In this section, we consider nontrivial interactions between strategies and describe how to overcome the corresponding problems. Since all problems begin when a $\mathcal{P}$-strategy enumerates an element into A, we will always assume that there is a $\mathcal{P}$-strategy below the other strategies we consider.

A $\mathcal{T}$-strategy τ below its mother $\mathcal{R}$-strategy ρ. The nontrivial case is when a $\mathcal{P}$-strategy π below the Γ-outcome of τ acts. Let us consider τ in more detail. For every x, we need to correctly define $\Gamma^{L(D)}(x) = D(x)$. We pick a big $y = y_x$ first and wait until the length of agreement between $\Psi^{L(D)}$ and E is larger than y. At the first expansionary stage s at which this happens, we define $\Gamma^{L(D)}(x)[s] = D(x)[s]$ with use $\gamma(x)[s] = s > \psi(y)[s]$. From now on (assuming τ is along the true path), the equality between $\Gamma^{L(D)}(x)$ and $D(x)[s]$ can be broken only if a witness a of the $\mathcal{P}$-strategy $\pi \supseteq \tau\widehat{\ }\Gamma$ is enumerated into A. It is worth noting that a must have been chosen before stage s, and so this can happen at most finitely many times (since all new witnesses of $\mathcal{P}$-strategies after initialization will be chosen big enough and there are only finitely many old witnesses).

The change in A allows a change in D on any x with Φ-use $\varphi(x)[s] \geq a$. We have the following possible cases:

(1) x enters D but there is no change in $L(D) \upharpoonright (\gamma(x)+1)$: Then we enumerate $y = y_x$ into E and we initialize all strategies below τ. So we have $1 = E(y) \neq \Psi^{L(D)}(y) = \Psi^{L(D)}(y)[s] = 0$, and τ wins. Initialized strategies must pick fresh witnesses, so from this moment on only strategies of higher priority than τ can enumerate numbers into A that allow changes of $\Psi^{L(D)}(y)[s]$. Indeed, if $\Psi^{L(D)}(y)[s]$ changes at a stage $s_1 > s$, then a number x_1 is extracted from D where $s^D(x_1) \leq \psi(y) < s$. It follows that some $a_1 \leq \varphi(x_1)[s] < s$ entered A after stage s, so a_1 must have been chosen before stage s.

(2) x enters or leaves D and there is a change in $L(D) \upharpoonright (\gamma(x)+1)$: In this case, we can update $D(x) = \Gamma^{L(D)}(x)$ with new big use $\gamma(x)$. Note that a new update of $\Gamma^{L(D)}(x)$ can only be caused by a number $a_1 < a$ entering A. This is because when a is enumerated into A by a $\mathcal{P}$-strategy, we initialize all lower-priority strategies, and hence all strategies with witnesses greater than a. New witnesses will be greater than the current use $\varphi(x)$ and will not be able to change computations related to x. So an increase in $\varphi(x)$ can

only be caused by the enumeration of some $a_1 < a$, and as we noted above, this can happen at most finitely often.

Note that if x leaves D, then there must be a change in $L(D) \upharpoonright (\gamma(x)+1)$. This is because we defined $\Gamma^{L(D)}(x)$ correctly at stage s, when we have that x is already in D, and so $s^D(x) < s = \gamma(x)$. It follows that the two cases above exhaust all possibilities.

In what follows, when we build a functional Γ, we can think of it as *opening a gap* and allowing for some number a to enter A. Hence, either a gap will be *closed successfully*, namely, at some point we have case (1) and a diagonalization at τ, or all gaps will be *closed unsuccessfully*, namely, we always have case (2), in which case we will correctly reduce D to $L(D)$. In the construction, we will create a link from τ to ρ. This link allows us to jump directly from ρ to τ and decide whether we want to enumerate y into E while keeping $E = \Lambda^D_\rho$ correct. So we will enumerate a number into or extract a number from E only when we come to a substrategy of ρ using a link (if there is no link at ρ then we change E at ρ); otherwise, we will not need to change E at ρ, since at ρ we will not be in a position in which we must change E back due to D returning to an old initial segment (except for the situation when some $\mathcal{P}$-strategy between ρ and τ enumerates a small number into A, which allows a D-change which can force us to change E back at ρ but also causes τ to be initialized).

A $\mathcal{T}$-strategy τ below an $\mathcal{S}$-strategy σ below their mother $\mathcal{R}$-strategy ρ. The real conflict, which also causes this priority argument to be a $\mathbf{0}'''$-argument rather than just an infinite-injury argument, first arises in the following scenario: Suppose we have an $\mathcal{R}$-strategy ρ with an $\mathcal{S}$-substrategy σ and a $\mathcal{T}$-substrategy τ below such that τ is below the finite outcome of σ. Furthermore, assume we have three $\mathcal{P}$-strategies π_2, π_1 and π_0 below the Γ-outcome of τ, the Δ-outcome of σ and the Γ-outcome of σ, respectively. Suppose now the following sequence of events:

First, the $\mathcal{P}$-strategy π_2 enumerates a witness a_2 into A, allowing a number x to enter D and causing τ to enumerate a number $y = y_x$ into E in order to diagonalize τ. Next, the $\mathcal{P}$-strategy π_1 enumerates a witness $a_1 < a_2$ into A, allowing x to leave D, which would normally force y to be extracted from E in order to keep Λ correct. However, for the stage $s^D(x)$ at which x entered D, $s^D(x)$ will enter $L(D)$ when x leaves D, while σ has possibly already defined $\Delta(s^D(x)) = 0$, which cannot be corrected. We resolve this conflict by threatening to let σ build a Turing functional $\Gamma^{L(D)} = D$ to permanently satisfy ρ.

However, letting σ build Γ (and taking an infinite Γ-outcome to the left of the infinite Δ-outcome) creates a new problem: Suppose our $\mathcal{P}$-strategy π_0 below the Γ-outcome of σ next enumerates a number a_0 into A, allowing D to change at a number on which $\Gamma^{L(D)}$ is already defined and now possibly wrong. The strategy for σ can use the following procedure to force an $L(D)$-change and correct $\Gamma^{L(D)}$: Before letting π_0 choose its witness a_0, we have a number x from some $\mathcal{P}$-strategy π_1 ready that just left D and caused the function Δ of σ to be incorrect. We will have a link from ρ to σ so that we can visit σ directly before ρ has a chance to extract y from E, allowing Λ^D to be temporarily incorrect. If the functional Δ is now wrong on $s^D(x)$, then we create a second link from ρ to σ and move to outcome Γ, only then allowing a_0 to be enumerated in A. Suppose that this causes a change in $D(x')$.

(1) If x' enters D, then there need not be any $L(D)$-change and thus $\Gamma^{L(D)}(x')$ may now be incorrect. If $\Gamma^{L(D)}(x')$ is defined, then this means that x' is small enough to allow us to preserve y in E while still keeping Λ^D correct. This causes a permanent disagreement between Ω^E and $L(D)$ at $s^D(x)$, since the old definition of $\Omega^E(s^D(x)) = 0$ is still valid while $s^D(x) \in L(D)$; this disagreement can only be undone by an action of a strategy of higher priority than σ, since σ can now switch to a permanent finitary diagonalization outcome unless initialized later.

(2) If x' leaves D (and had previously entered D at a stage $s^D(x')$), then it will follow from the way we construct Γ that $\gamma(x') \geq s^D(x')$. So x' leaving D will cause $s^D(x')$ to enter $L(D)$ and allow $\Gamma^{L(D)}(x')$ to be corrected.

Similarly to the previous case, we *open a second gap* when we allow the number a_0 to enter A. Either one of these gaps will be *closed successfully* (i.e., at some point, we have case (1)) and we have a permanent win at σ, or all gaps will be *closed unsuccessfully* (i.e., we always have case (2)), then we correctly reduce D to $L(D)$. Again, in the construction, we will create a link from σ to ρ since we jump from ρ to σ when we need to decide whether to enumerate y into E or not, and the link allows us to keep $E = \Lambda^D_\rho$ correct.

2.4. **Several $\mathcal{R}$-strategies.** Now we consider several $\mathcal{R}$-strategies with their substrategies. In our intuitive analysis, we restrict ourselves to two $\mathcal{R}$-strategies ρ_0 and ρ_1. Assume that we have $\rho_0 \subset \rho_1$, and that they have substrategies σ_0 and σ_1, respectively (also assume that σ_0 and σ_1 have Γ-outcome). The conceivable relative priorities for these strategies are as follows:

(1) $\rho_0 \subset \sigma_0 \subset \rho_1 \subset \sigma_1$,
(2) $\rho_0 \subset \rho_1 \subset \sigma_1 \subset \sigma_0$, and
(3) $\rho_0 \subset \rho_1 \subset \sigma_0 \subset \sigma_1$.

The third case could produce non-nested links; so we disallow it as follows: When σ_0 changes the global outcome of ρ_0 along the true path, we introduce another version of ρ_1, say, an $\mathcal{R}$-strategy ρ_1', first, and only allow substrategies of ρ_1' but not of ρ_1 below ρ_1'. This reduces the third case above to the first, in the usual manner of $\mathbf{0}'''$-arguments.

In the first case, there is no real conflict, since ρ_1 already knows that σ_0 will build its Γ, which permanently satisfies ρ_0. In the second case, there may be links from ρ_0 directly to σ_0, over ρ_1 and σ_1; but if σ_0 truly has Γ-outcome, then we again introduce another version of ρ_1, say, an $\mathcal{R}$-strategy ρ_1', below σ_0 and only allow substrategies of ρ_1' but not of ρ_1 below σ_0.

3. Construction

3.1. **Outcomes and the tree of strategies.** Throughout the construction, we insert comments in brackets which we hope will help the reader connect the formal construction back to the intuition given above.

Let ListFunc $= \{\langle \Phi, \Psi, \Theta, \Omega, D\rangle\}$ be the above-mentioned computable listing of all tuples of p.c. functionals $\Phi, \Psi, \Theta, \Omega$ and d.c.e. sets D.

Let ListReq be a computable listing of all requirements defined as follows: First, we fix the least element $\langle \Phi_0, \Psi_0, \Theta_0, \Omega_0, D_0\rangle$ in ListFunc. Then we set

$$\begin{aligned}\text{ListReq} = \{&\mathcal{P}_\Theta \mid \langle \Phi_0, \Psi_0, \Theta, \Omega_0, D_0\rangle \in \text{ListFunc}\} \cup \\ \{&\mathcal{R}_{\Phi,D} \mid \langle \Phi, \Psi_0, \Theta_0, \Omega_0, D\rangle \in \text{ListFunc}\} \cup \\ \{&\mathcal{T}_{\Phi,D,\Psi} \mid \langle \Phi, \Psi, \Theta_0, \Omega_0, D\rangle \in \text{ListFunc}\} \cup \\ \{&\mathcal{S}_{\Phi,D,\Omega} \mid \langle \Phi, \Psi_0, \Theta_0, \Omega, D\rangle \in \text{ListFunc}\}\end{aligned}$$

For $\mathcal{X} \in \text{ListReq}$, let $\text{ind}(\mathcal{X}) \in \text{ListFunc}$ be the corresponding tuple for $\mathcal{X}$. Now we say that $\mathcal{X} < \mathcal{Y}$ for $\mathcal{X}, \mathcal{Y} \in \text{ListReq}$ if and only if either $\text{ind}(\mathcal{X}) \neq \text{ind}(\mathcal{Y})$ and $\text{ind}(\mathcal{X})$ is listed before $\text{ind}(\mathcal{Y})$ in ListFunc, or if $\text{ind}(\mathcal{X}) = \text{ind}(\mathcal{Y})$ and $\mathcal{X} \neq \mathcal{Y}$, then we use the following ordering of the requirements when we compare $\mathcal{X}$ and $\mathcal{Y}$: $\mathcal{P} < \mathcal{R} < \mathcal{T} < \mathcal{S}$.

The strategies for these requirements have the following outcomes, where $L = \{d, \infty, \Gamma, \Delta, \mathit{fin}\}$ is the set of outcomes (we have added one more outcome d that was not mentioned in the intuition, meant to isolate the situation when a strategy has a permanent win by diagonalization, i.e., by successfully closing a gap):

- A $\mathcal{P}$-strategy has two possible outcomes: $d <_L \mathit{fin}$.
- An $\mathcal{R}$-strategy has two possible outcomes: $\infty <_L \mathit{fin}$.
- A $\mathcal{T}$-strategy has three possible outcomes: $d <_L \Gamma <_L \mathit{fin}$.
- An $\mathcal{S}$-strategy has four possible outcomes: $d <_L \Gamma <_L \Delta <_L \mathit{fin}$.

The *tree of strategies* $T \subset L^{<\omega}$ is defined by induction as follows. When we assign a requirement to some node, then the strategy of this requirement will work at this node. For the empty node λ, we set $\text{ListReq}_\lambda = \text{ListReq}$. Given a node $\xi \in T$, we assign to it the highest-priority (sub)requirement from ListReq_ξ. If it is a subrequirement of an $\mathcal{R}$-requirement, then there will be a longest strategy $\rho \subset \xi$ assigned to the corresponding $\mathcal{R}$-requirement, and we call ξ a *child node of* ρ and ρ the *mother node of* ξ. Depending on the requirement assigned to ξ, we next define the list of requirements yet to be satisfied as follows.

- If it is a $\mathcal{P}_\Theta$-requirement, then define
$$\text{ListReq}_{\xi^\frown d} = \text{ListReq}_{\xi^\frown \mathit{fin}} = \text{ListReq}_\xi - \{\mathcal{P}_\Theta\}.$$
- If it is an $\mathcal{R}_{\Phi,D}$-requirement, then define
$$\begin{aligned}\text{ListReq}_{\xi^\frown \infty} &= \text{ListReq}_\xi - \{\mathcal{R}_{\Phi,D}\}, \text{ and}\\ \text{ListReq}_{\xi^\frown \mathit{fin}} &= \text{ListReq}_\xi - \{\mathcal{R}_{\Phi,D}\}\\ &\qquad - \{\mathcal{T}_{\Phi,D,\Psi} \mid \text{ind}(\mathcal{T}_{\Phi,D,\Psi}) \in \text{ListFunc}\}\\ &\qquad - \{S_{\Phi,D,\Omega} \mid \text{ind}(\mathcal{S}_{\Phi,D,\Omega}) \in \text{ListFunc}\}.\end{aligned}$$
[So, for the infinite outcome, we remove only $\mathcal{R}_{\Phi,D}$, but for the finite outcome, we remove $\mathcal{R}_{\Phi,D}$ and all its subrequirements.]
- If it is a $\mathcal{T}_{\Phi,D,\Psi}$-subrequirement and the mother node of ξ is ρ, then define
$$\begin{aligned}\text{ListReq}_{\xi^\frown \Gamma} &= \text{ListReq}_{\rho^\frown \mathit{fin}}, \text{ and}\\ \text{ListReq}_{\xi^\frown d} &= \text{ListReq}_{\xi^\frown \mathit{fin}} = \text{ListReq}_\xi - \{\mathcal{T}_{\Phi,D,\Psi}\}.\end{aligned}$$
- If it is an $\mathcal{S}_{\Phi,D,\Omega}$-subrequirement and the mother node of ξ is ρ, then define
$$\begin{aligned}\text{ListReq}_{\xi^\frown \Gamma} &= \text{ListReq}_{\xi^\frown \Delta} = \text{ListReq}_{\rho^\frown \mathit{fin}}, \text{ and}\\ \text{ListReq}_{\xi^\frown d} &= \text{ListReq}_{\xi^\frown \mathit{fin}} = \text{ListReq}_\xi - \{\mathcal{S}_{\Phi,D,\Omega}\}.\end{aligned}$$

[Namely, in both these cases, under outcomes d and *fin*, we remove only the subrequirement itself, whereas under outcomes Γ and Δ, we consider the same list of requirements as under the finite outcome of the mother node.]

Now we define the expansionary stages. A stage s is called a *ξ-stage* if the node ξ is visited at this stage. The *length of agreement functions* for an $\mathcal{R}$-strategy ρ and for an $\mathcal{S}$-strategy σ are defined as follows:

$$l(\rho)[s] = \max\big\{t < s \mid \forall x < t\,(D(x)[s] = \Phi_\rho^A(x)\downarrow[s])\big\},$$
$$l(\sigma)[s] = \max\big\{t < s \mid \forall z < t\,(L(D)(z)[s] = \Omega_\sigma^E(z)\downarrow[s])\big\}.$$

A stage s is *ξ-expansionary* (for $\xi \in \{\rho, \sigma\}$) if $l(\xi)[s] > s^-[s]$, where $s^-[s] = s^-_\xi[s]$ is the previous ξ-expansionary stage at stage s, and where stage 0 is always ξ-expansionary. (Note that we will not need the notion of a ξ-expansionary stage for $\mathcal{T}$-strategies ξ, where we will use a version of the Sacks coding strategy instead.) During the construction, *initializing* a node will mean canceling its satisfaction, its witnesses, and its associated functionals.

3.2. **Full Construction.** We build a computable approximation TP_s to the true path TP at each stage. Meanwhile we define approximations of all sets and functionals at these stages (keeping sets and functionals the same unless we redefine them explicitly). The construction proceeds as follows.

Stage $s = 0$. We set $A_0 = \emptyset$ and initialize all strategies.

Stage $s+1$. We work in substages $t \le s+1$, possibly skipping over some substages. Let $TP_{s,0} = \lambda$. Given $TP_{s+1,t}$ at a substage $t+1$, we define $TP_{s+1,t'}$ for some $t' > t$ (usually $t' = t+1$). After we define $TP_{s+1,t+1}$, if $t < s$, then we proceed to substage $t+2$ unless explicitly stated otherwise. If $t = s$, then we define $TP_{s+1} = TP_{s+1,t+1}$, proceed to the next stage, and initialize all nodes $\xi \not\leq TP_{s+1}$. At substage $t+1$, the construction depends on the requirement assigned to $TP_{s+1,t}$:

Case 1: $TP_{s+1,t} = \pi$ is a $\mathcal{P}$-strategy: Go to the first subcase which applies.

- $\pi 1$. If no witness is defined for π, then define $a = a_\pi$ to be a big number and let $TP_{s+1,t+1} = \pi^\frown \mathit{fin}$.
- $\pi 2$. Otherwise, if $\Theta(a)[s]\uparrow$ or $\Theta(a)[s] \neq 0$, then define $TP_{s+1,t+1} = \pi^\frown \mathit{fin}$.
- $\pi 3$. Otherwise, if $\Theta(a)[s] = 0$ and $a \notin A_s$, then enumerate a into A and define $TP_{s+1,t+1} = \pi^\frown d$.
- $\pi 4$. Otherwise, $\Theta(a)[s] = 0$ and $a \in A_s$, so define $TP_{s+1,t+1} = \pi^\frown d$.

Case 2: $TP_{s+1,t} = \rho$ is an $\mathcal{R}$-strategy: Go to the first subcase which applies. [The goal of ρ is to use links and to define the reduction $E \le_T D$.]

- $\rho 1$. If stage s is not ρ-expansionary, then define $TP_{s+1,t+1} = \rho^\frown \mathit{fin}$.
- $\rho 2$. Otherwise, fix the previous ρ-expansionary stage $s^-[s+1] = s$ and consider the following subcases.
- $\rho 2.1$. If there is no link to ρ, then extract the necessary numbers from $E = E_\rho$ in order to keep $E(y)[s+1] = \Lambda_\rho^D(y)[s]$ correct for all $y \in \mathrm{dom}(\Lambda_\rho^D[s])$, define $\Lambda_\rho^D(y_0)[s+1] = E(y_0)[s+1]$ for the least $y_0 \notin \mathrm{dom}(\Lambda_\rho^D[s])$, with use $\lambda_\rho(y_0)[s+1] = y_0$. Let $TP_{s+1,t+1} = \rho^\frown\infty$. [It is easy to see that each $\lambda_\rho(y_0)$ will not increase and is bigger than x_0, the number to which it is potentially related by some $\mathcal{T}$-strategy.]

ρ2.2. Otherwise, travel the link to the child node η, say, which created the link, define $TP_{s+1,|\eta|} = \eta$, and proceed to substage $|\eta| + 1$.

Case 3: $TP_{s+1,t} = \tau$ is a $\mathcal{T}$-strategy: The strategy works in cycles. Let ρ be the mother node of τ. Proceed as in the first subcase which applies. [The goal of τ is to diagonalize against $E = \Psi^{L(D)}$ or to define a reduction $D \leq_T L(D)$.]

τ1. If τ is visited through a link, then the link must be from the mother node ρ. Cancel this link and consider the following subcases. [This means that at the previous τ-stage we had outcome Γ, and now we either diagonalize or extend the Γ-functional.]

τ1.1. If there is x such that $\Gamma^{L(D)}(x)[s]$ is defined and such that $D(x)[s] \neq \Gamma^{L(D)}(x)[s]$, then put y_x into E and define $TP_{s+1,t+1} = \tau^\frown d$. Declare τ *satisfied*. [This is Case (1) in the intuition, which allows us to change $E(y_x)$ since $\Lambda^D(y_x)$ becomes undefined because of x, hence it allows a diagonalization at τ.]

τ1.2. Otherwise, let x be the greatest opened cycle (if there is none, set $x = 0$). Open cycle $x + 1$ and, for all $u \leq x$, define $\Gamma_\tau^{L(D)}(u)[s + 1] = D(u)[s + 1]$ (if $\Gamma_\tau^{L(D)}(u)[s] \uparrow$) with use $\gamma(u) = s + 1$, and define $TP_{s+1,t+1} = \tau^\frown fin$. [This is Case (2) in the intuition.]

τ2. Otherwise, if τ is satisfied, then define $TP_{s+1,t+1} = \tau^\frown d$

τ3. Otherwise, let x be the greatest opened cycle (if there is none, set $x = 0$). Consider the following subcases.

τ3.1. If there is no attacker $y = y_x$, then choose y big, and define $TP_{s+1,t+1} = \tau^\frown fin$.

τ3.2. Otherwise, if $E(y)[s] \neq \Psi_\tau^{L(D)}(y)[s]$, then define $TP_{s+1,t+1} = \tau^\frown fin$.

τ3.3. Otherwise, we have $E(y)[s] = \Psi_\tau^{L(D)}(y)[s] \downarrow$, so we create a link with ρ and define $TP_{s+1,t+1} = \tau^\frown \Gamma$.

Case 4: $TP_{s+1,t} = \sigma$ is an $\mathcal{S}$-strategy: The strategy works in cycles. Let ρ be the mother node of σ. Go to the first subcase which applies. [The goal of σ is to diagonalize against $L(D) = \Omega^E$ or to either define a reduction $L(D) \leq_T \emptyset$ or to define a reduction $D \leq_T L(D)$. Also note that cycles here are analogues of cycles in $\mathcal{T}$-strategies; however, inside these cycles, we use σ-expansionary stages, which could be considered as inner cycles.]

σ1. If σ is visited through a link, then the link must be from the mother node ρ, and before creating the link at the previous σ-expansionary stage $s^- = s_\sigma^-[s]$, the node σ had either outcome Δ or Γ. Cancel this link and consider the following subcases.

σ1.1. If the previous outcome was Δ, then if there is some $z \leq s^-[s^-[s]]$ with $\Delta_\sigma(z)[s] \neq L(D)(z)[s]$, then keep $\Lambda_\rho^D(y)$ undefined (where $y = y_x$ is the number which entered E earlier due to x entering D but now x has left D again, and $z = s^D(x)$), create a link between σ and ρ again, and define $TP_{s+1,t+1} = \sigma^\frown \Gamma$. [The first gap is closed successfully.]

σ1.2. Otherwise, if the previous outcome was Δ, but the functional Δ_σ agrees with $L(D)$ on its domain, then, for all $z \leq s^-[s]$, define $\Delta_\sigma(z)[s + 1] = L(D)(z)[s + 1]$, and define $TP_{s+1,t+1} = \sigma^\frown fin$. [The first gap is closed unsuccessfully.]

σ1.3. Otherwise, the previous outcome was Γ. If there is an opened cycle x such that $\Gamma_\sigma^{L(D)}(x)[s]$ is defined and $D(x)[s] \neq \Gamma_\sigma^{L(D)}(x)[s]$, then declare σ

satisfied, keep E unchanged, and redefine $\Lambda_\rho^D(y) = E(y)$ with old use $\lambda(y)$ for all y (which is possible due to the fresh $x \in D$), and define $TP_{s+1,t+1} = \sigma^\frown d$. [The second gap is closed successfully. This is Case (1) in the intuition which allows to keep $E(y)$ unchanged since $\Lambda^D(y)$ became undefined because of x, hence it allows diagonalization at σ.]

σ1.4. Otherwise, the previous outcome was Γ, and $\Gamma^{L(D)}$ is correct on its domain. Let x be the greatest opened cycle (if there is none, set $x = 0$). Then open cycle $x+1$, and for all $u \leq x$, define $\Gamma_\sigma^{L(D)}(u)[s+1] = D(u)[s+1]$ (if $\Gamma_\sigma^{L(D)}(u)[s]\uparrow$) with use $\gamma(u) = s+1$, cancel Δ, and define $TP_{s+1,t+1} = \sigma^\frown \mathit{fin}$. [The second gap is closed unsuccessfully. This is Case (2) in the intuition.]

σ2. Otherwise, if there is an opened cycle x such that σ is satisfied at cycle x, then define $TP_{s+1,t+1} = \sigma^\frown d$.

σ3. Otherwise, if stage s is not σ-expansionary, or if this is the first visit of σ after initialization, then $TP_{s+1,t+1} = \sigma^\frown \mathit{fin}$.

σ4. Otherwise, s is a σ-expansionary stage, then fix the previous expansionary stage $s_\sigma^-[s+1] = s$, create a link with ρ, and define $TP_{s+1,t+1} = \sigma^\frown \Delta$. [Note that this feature introduces a small delay into the definition of Δ.]

4. Verification

Define the *true path* $TP = \liminf_s TP_s$. We will show that TP exists and that each requirement is satisfied by some node along TP.

Lemma 4.1. *The true path TP exists.*

Proof. This is clear by definition since the tree is finitely branching and $|TP_s| = s$ for all $s \in \omega$. □

Lemma 4.2. *Each node along the true path TP is initialized only finitely often.*

Proof. At stage s, we initialize only the nodes $\xi \not\leq TP_s$. So eventually, every node along TP will not be initialized. □

Lemma 4.3. *There are no nodes along TP which are part of a permanent link.*

Proof. A link connects a mother node and one of its children. So, if a mother node is along TP and is part of a link, then when we visit the mother node, we travel the link to its child node and cancel the link if the child node is a $\mathcal{T}$-substrategy, or cancel the link after traveling it at most twice if the child node is an $\mathcal{S}$-substrategy. □

Lemma 4.4. *For every $\mathcal{R}$-strategy ρ along the true path and with true outcome ∞, there are infinitely many stages at which it does not travel links and $\rho^\frown\infty$ is visited. More generally, each node on TP is visited infinitely often.*

Proof. Suppose ρ travels a link to a child strategy ξ. It follows that ξ has not yet been satisfied, and so all strategies below $\xi^\frown d$ are in initial state. By construction, when we previously visited ξ it took an infinite outcome Γ or Δ, and so all strategies extending $\xi^\frown \mathit{fin}$ are also in initial state. If ξ is a $\mathcal{T}$-strategy τ, the link is canceled and τ has either outcome d for the first time since initialization or outcome *fin*, visiting in each case strategies in their initial state. If ξ is an $\mathcal{S}$-strategy σ, the link is canceled and σ has either outcome *fin*, visiting strategies in their initial state, or creates a second link and has outcome Γ. At the next expansionary ρ-stage, the second link is traveled and canceled, and σ has outcome d or *fin*, visiting in each case strategies in their initial state. No strategy in its initial state can create a link on its first visit. So, when ρ is next visited, it will not travel a link, and at its next expansionary stage, ρ will have outcome ∞.

The second part of this lemma is now an easy induction. Consider a node $\eta \in TP$ and assume that lemma holds for all $\xi \subset \eta$. The case of the empty node η is trivial, so let $\eta = \xi^\frown o$. The case when ξ is an $\mathcal{R}$-strategy and $o = \infty$ was just proved. If $o = fin$ or $o = d$, then η is visited at all but finitely many ξ-stages. If $o = \Gamma$ or $o = \Delta$, then unless ξ has outcome o at infinitely many stages, η is not along the true path, contradicting our choice of η. □

Lemma 4.5. *Each $\mathcal{P}$-requirement is satisfied by a node along the true path TP.*

Proof. Consider a requirement $\mathcal{P}_\Theta$. By the definition of the tree of strategies, we can choose a node $\pi \subset TP$ assigned to $\mathcal{P}_\Theta$ of maximal length. By Lemma 4.4, π is visited at infinitely many stages. By Lemma 4.2, we fix a stage s_0 such that π is not initialized after stage s_0 (even though links starting at $\rho \subset \pi$ to some child node $\tau \supset \pi$ of ρ may be traveled after stage s_0, this would not initialize π). It follows that π has a final witness a. Now, clearly, the requirement will be satisfied: Either $\Theta(a) \neq 0$ and $a \notin A$; or at some stage $s_1 > s_0$, $\Theta(a)[s_1] = 0$ and when we next visit π, we enumerate a into A. □

Lemma 4.6. *If a $\mathcal{T}_{\Phi,D,\Psi}$-strategy of maximal length along TP has outcome d or fin, then its requirement is satisfied.*

Proof. By the definition of the tree of strategies, we can choose a node $\tau \subset TP$ assigned to $\mathcal{T}_{\Phi,D,\Psi}$ of maximal length. By Lemma 4.2, we fix a stage s_0 such that τ is not initialized after stage s_0. Now consider the following two cases:

$\tau^\frown fin \subset TP$: Then fix a stage $s_1 > s_0$ such that τ does not take an outcome to the left of *fin* after stage s_1. Now let x be the greatest opened cycle, so for all τ-stages $t > s_1$ we have $0 = E(y_x) = E(y_x)[t] \neq \Psi^{L(D)}(y_x)[t]$ (otherwise, we would have outcome Γ at least once). Hence, $\mathcal{T}_{\Phi,D,\Psi}$ is satisfied.

$\tau^\frown d \subset TP$: Then fix a stage $s_1 > s_0$ such that τ takes outcome d at stage s_1 and diagonalizes via cycle x. Then we have that $1 = E(y_x) = E(y_x)[s_1+1] \neq \Psi^{L(D)}(y_x)[s_1] = 0$. Furthermore, for all τ-stages $t > s_1$, the only way this can change is if $L(D) \upharpoonright (\psi(y_x)[s_1]+1)$ changes. However, this can happen only if a number x_0 leaves D after stage s_1 and $s^D(x_0) \leq \psi(y_x)[s_1] < s_1$, but this means that $A \upharpoonright (\varphi(x_0)[s_1]+1)$ has changed after stage s_1, and this can happen only due to a node $\xi < \tau$ (and so τ would be initialized in that case). Indeed, after stage s_1, τ will always take outcome d, below which every $\mathcal{P}$-strategy will choose a fresh witness greater than $s_1 > \varphi(x_0)[s_1]$. (Moreover, since we visited τ at stage s_1, there was no link which crossed over $\tau^\frown fin$ at stage s_1. Also, if new links cross over $\tau^\frown d$ later, they don't affect our restraint on A.) So $\mathcal{T}_{\Phi,D,\Psi}$ is satisfied. □

Lemma 4.7. *If an $\mathcal{S}_{\Phi,D,\Omega}$-strategy of maximal length along TP has outcome d or fin, then its requirement is satisfied.*

Proof. By the definition of the tree of strategies, we can choose a node $\sigma \subset TP$ assigned to $\mathcal{S}_{\Phi,D,\Omega}$ of maximal length. By Lemma 4.2, we fix a stage s_0 such that σ is not initialized after stage s_0. Now consider the following two cases:

$\sigma^\frown fin \subset TP$: Then fix a stage $s_1 > s_0$ such that σ does not take an outcome to the left of *fin* after stage s_1. Then we never see another σ-expansionary stage, and so $\mathcal{S}_{\Phi,D,\Omega}$ is satisfied vacuously.

$\sigma^\frown d \subset TP$: Then fix a stage $s_1 > s_0$ such that σ first takes outcome d at a σ-stage $\geq s_1$. Assume this was due to cycle x, so by case σ1.3 of the construction, we have that $D(x)[s_1] \neq \Gamma^{L(D)}(x)[s_1]$. We first argue that $x \in D[s_1]$, since otherwise, $\Gamma^{L(D)}(x)[s_1] = 1$ with use $\gamma(x) \geq s^D(x)$, but after $\Gamma^{L(D)}(x)$ was defined, x has left D and so $s^D(x)$ has entered $L(D)$, destroying the computation $\Gamma^{L(D)}(x)$. This allows us to redefine Λ as described in σ1.3.

Now consider the number z which caused σ to proceed to case σ1.1; it must be of the form $z = s^D(x')$ for some x' which had left D already at a stage $s' < s_1$, say, causing z to enter $L(D)$ while $\Delta(z) = 0$. But just before stage s', we had $L(D)(z) = 0 = \Omega^E(z)$ since the σ-stage before stage s' was σ-expansionary; and since E was not allowed to change until stage s_1, we will have $\Omega^E(z)[s_1] = 0$ while $z \in L(D)$. By initialization, we then have that $\Omega^E(z) = 0$ is preserved, so $\mathcal{S}_{\Phi,D,\Omega}$ is satisfied. □

Lemma 4.8. *If a $\mathcal{T}_{\Phi,D,\Psi}$-strategy of maximal length along TP has outcome Γ, then it correctly builds a Γ-functional.*

Proof. By the definition of the tree of strategies, we can choose a node $\tau \subset TP$ assigned to $\mathcal{T}_{\Phi,D,\Psi}$ of maximal length. By Lemma 4.2, we fix a stage s_0 such that τ is not initialized after stage s_0. Now we prove that $\Gamma^{L(D)} = D$. It is clear by the construction (case τ1.2) that for any x, there are infinitely many stages $s \geq s_0$ at which we have $\Gamma^{L(D)}(x)[s] \downarrow= D(x)[s]$. It remains to show that $\gamma(x)$ is bounded. So fix x and assume that $\gamma(x)[s] = s$ is defined at stage s (via case τ1.2). Since we traveled the link, the numbers below outcome *fin* will be chosen big, in particular, any witness a chosen below it is bigger than the use $\varphi(x)$ (namely, from now on, it is bigger than $s^-[s+1] > \varphi(x)[s]$, where $s^-[s+1] = s$ is that ρ-expansionary stage). Hence, only numbers of $\mathcal{P}$-strategies below the Γ-outcome can change $D \upharpoonright (x+1)$ (and so change $L(D) \upharpoonright (\gamma(x)+1)$). Note, however, that the enumeration of each such number a initializes all lower-priority strategies, so we will have fewer and fewer such numbers a, and so $\gamma(x)$ will be increased only finitely often. Hence, $\Gamma^{L(D)} = D$. □

Lemma 4.9. *If an $\mathcal{S}_{\Phi,D,\Omega}$-strategy of maximal length along TP has outcome Γ or Δ, then it correctly builds a Γ- or Δ-functional.*

Proof. By the definition of the tree of strategies, we can choose a node $\tau \subset TP$ assigned to $\mathcal{S}_{\Phi,D,\Omega}$ of maximal length. By Lemma 4.2, we fix a stage s_0 such that σ is not initialized after stage s_0. If $\sigma^\frown\Gamma \subset TP$, then the proof is similar to the proof in Lemma 4.8. So let $\sigma^\frown\Delta \subset TP$. Then, after some stage $s_1 > s_0$, we never go to the left of outcome Δ. By the construction, this means that we can only have cases σ1.2, σ3, and σ4; and cases σ4 and then σ1.2 must occur infinitely often. Hence, $\Delta = L(D)$, and the lemma is proved. □

Lemma 4.10. *Each $\mathcal{R}$-requirement is satisfied by a node along the true path TP.*

Proof. Consider a requirement $\mathcal{R}_{\Phi,D}$. By the definition of the tree of strategies, we can choose a node $\rho \subset TP$ assigned to $\mathcal{R}_{\Phi,D}$ of maximal length. Clearly, ρ cannot be strictly between two fixed nodes forming a link created and canceled infinitely often (otherwise, ρ would not have maximal length). By Lemma 4.2, fix a stage s_0 such that ρ is not initialized after stage s_0. If ρ has finitely many ρ-expansionary stages, then $\mathcal{R}$ is satisfied vacuously; otherwise, assume that ρ has final versions of its set $E = E_\rho$ and functional $\Lambda = \Lambda_\rho$.

We prove that Λ^D is total and correctly computes E. Fix a natural number y and suppose inductively that $\Lambda^D(z) \downarrow= E(z)$ for all $z < y$ at all stages s starting at some stage $s_1 \geq s_0$. By Lemma 4.4, there are infinitely many stages at which case $\rho 2.1$ is executed. At such stages $t > s_1$, the strategy ρ ensures that $\Lambda^D(y) = E(y)$ with use y. As D is d.c.e, it follows that $D \upharpoonright (y+1)$ will eventually stop changing, after stage s_2, say, and hence $\Lambda^D(y)$ is defined. On the other hand, $E(y)$ can change at most twice: $E(y) = 0$ holds at all stages unless $y = y_x$ is a number used by a specific $\mathcal{T}$-strategy τ in relation to some number $x < y$. In this case, τ is the only strategy that can enumerate y into E, and this happens under case $\tau 1.1$, when $D(x)$ also changes and τ is declared satisfied. The change in D at x allows ρ to correct $\Lambda^D(y)$. The strategy τ then has outcome d until (if ever) it is initialized and so it will never deal with the number y again. After that, $E(y)$ can change only once, if it is extracted from E by ρ under case $\rho 2.1$. This happens if $D \upharpoonright (y+1)$ has reverted to an old state, and so x was previously enumerated into D when y was enumerated in E, but now $D(x) = 0$ again. Note that $D(x) = 0$ at all future stages, and so $\Lambda^D(y) = E(y)$ will remain true at all future stages.

If all substrategies of ρ along the true path have finite outcomes, then it follows from Lemma 4.6 and Lemma 4.7 that E is Turing incomparable to $L(D)$. Otherwise, it follows from Lemma 4.8 and Lemma 4.9 that either $L(D)$ is computable or $L(D) \equiv_T D$. In both cases, $\mathcal{R}$ is satisfied. □

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