KALIMULLIN PAIRS OF Σ_2^0 ω-ENUMERATION DEGREES

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Abstract. We study the notion of $\mathcal{K}$-pairs in the local structure of the ω-enumeration degrees. We introduce the notion of super almost zero sequences and investigate their structural properties.

The study of degree structures has been one of the central themes in computability theory. Although the main focus has been on the structure of the Turing degrees and its local substructure, of the degrees below the first jump of the least degree, significant work has been done to examine the properties of an extension of the Turing degrees, the structure of the enumeration degrees.

Enumeration reducibility introduced by Friedberg and Rogers [3] arises as a way to compare the computational strength of the positive information contained in sets of natural numbers. A set A is enumeration reducible to a set B if given any enumeration of the set B, one can effectively compute an enumeration of the set A. The induced structure of the enumeration degrees $\mathcal{D}_e$ is an upper semilattice with least element and jump operation. As we mention above, this structure can be viewed as an extension of the structure of the Turing degrees, due to an embedding $\iota : \mathcal{D}_T \to \mathcal{D}_e$ which preserves the order, the least upper bound and the jump operation. The local structure of the enumeration degrees, consisting of all degrees below the first enumeration jump of the least enumeration degree, $\mathcal{G}_e$, can therefore in turn be seen as an extension of the local structure of the Turing degrees.

The two structures, $\mathcal{D}_T$ and $\mathcal{D}_e$, as well as their local substructures, are closely related in algebraic properties, definability strength and in the techniques and methods used to study them. Results proved in one of the structures reveal properties of the other, methods used to study one of the structures suggest similar methods for the other and vice versa.

A proof technique which arises from the study of the structure of the enumeration degrees is the use of the following notion.

0.1. Definition.[Kalimullin] A pair of sets of natural numbers A and B is a $\mathcal{K}$-*pair* over a set U if there is a set $W \leq_e U$ such that:

$$A \times B \subseteq W \ \& \ \overline{A} \times \overline{B} \subseteq \overline{W}.$$

The notion of a $\mathcal{K}$-pair over U, originally known as a U-e-ideal, was introduced and used by Kalimullin to prove the definability of the jump operation in the global structure $\mathcal{D}_e$. In [7] Kalimullin proves that the property of being a $\mathcal{K}$-pair over U is degree theoretic and first order definable in the global structure $\mathcal{D}_e$. A pair of sets A and B form a $\mathcal{K}$-pair over a set U if and only if their degrees $\mathbf{a} = d_e(A)$ and

This research was supported by an BNSF Grant No. D002-258/18.12.08 and by a Marie Curie European Reintegration Grant No. 239193 within the 7th European Community Framework Programme.

$\mathbf{b} = d_e(B)$ and $\mathbf{u} = d_e(U)$ satisfy the property:

$$\mathcal{K}(\mathbf{a}, \mathbf{b}, \mathbf{u}) \rightleftharpoons \forall \mathbf{x}[(\mathbf{a} \vee \mathbf{x} \vee \mathbf{u}) \wedge (\mathbf{b} \vee \mathbf{x} \vee \mathbf{u}) = \mathbf{x} \vee \mathbf{u}].$$

We will therefore call a pair of enumeration degrees a $\mathcal{K}$-pair over a degree $\mathbf{u}$ if they contain representatives which form a $\mathcal{K}$-pair over a representative of $\mathbf{u}$ in the sense of Definition 0.1.

$\mathcal{K}$-pairs over $\mathbf{0}_e$ have been proven useful for coding structures in the local structure of the enumeration degrees, $\mathcal{G}_e$. It has been shown [6] for instance, that using countable $\mathcal{K}$-systems, systems of nonzero e-degrees such that every pair of distinct degrees forms a $\mathcal{K}$-pair, that every countable distributive semi-lattice can be embedded below every nonzero Δ^0_2 enumeration degree. In [5], Ganchev and Soskova show that the theory of $\mathcal{G}_e$ is computably isomorphic to first order arithmetic, by using $\mathcal{K}$-systems to code standard models of arithmetic.

In the last few years Soskov [10] has initiated the study of a further extension of the enumeration degrees: the ω-enumeration degrees, $\mathcal{D}_\omega$. This structure is an upper semi-lattice with jump operation, where the building blocks of the degrees are of a higher type - sequences of sets of natural numbers. The structure of the enumeration degrees has a definable copy in the ω-enumeration degrees, hence a similar relationship can be seen between $\mathcal{D}_e$ and $\mathcal{D}_\omega$, as the one described between $\mathcal{D}_T$ and $\mathcal{D}_e$. Here as well we can define a local structure, $\mathcal{G}_\omega$, consisting of the degrees bounded by the first ω-enumeration jump of the least degree. The structure of the ω-enumeration is based on ω-enumeration reducibility, a reducibility which combines two notions: enumeration reducibility and uniformity. A unique phenomenon to ω-enumeration reducibility is the existence of the so called *almost zero* sequences, sequences whose every member when considered in isolation is equivalent to the corresponding member of the least ω-sequence, $\emptyset_\omega$, but whose ω-enumeration degrees (called the *almost zero* or *a.z.* degrees) are nonzero. The class of *a.z.* degrees can be viewed as representing purely the notion of nonuniformity on the one hand, and as a class representing a new type of "lowness" property, which is not as usually defined by domination or the strength of their image under the jump operator.

In this article we study the notion of $\mathcal{K}$-pairs of ω-enumeration degrees in $\mathcal{G}_\omega$, inspired by Kallumilin's $\mathcal{K}$-pairs of enumeration degrees. $\mathcal{K}$-pairs of ω-enumeration degrees are defined by Ganchev and Soskova in [4] in terms of their structural properties rather than their set-theoretic properties. There they show that one can distinguish between two types of $\mathcal{K}$-pairs, ones that can be constructed by extending a $\mathcal{K}$-pair over $\emptyset^n$ with respect to enumeration reducibility to a sequence, and a second type which consists of *almost zero* sequence. The first type is then used in [4] to prove that the classes of high-low jump hierarchy are first order definable in $\mathcal{G}_\omega$, to prove that there is a first order definable copy of $\mathcal{G}_e$ in $\mathcal{G}_\omega$ and that the first order theory of true arithmetic is interpretable in $\mathcal{G}_\omega$. Here we concentrate on the second type of $\mathcal{K}$-pairs, the ones that consist of almost zero sequences. Our investigations lead us to the discovery of a sub-class of the *a.z.* degrees, called the *super a.z.* degrees. This class turns out to be a nontrivial proper sub-ideal of the *a.z.* degrees. For the class of $\mathcal{K}$-pairs of super a.z. degrees we give a set theoretic characterization, in the spirit of the original definition given by Kalimullin. Our first application of this characterization is that there exists an independent family of super almost zero sequences of sets $\mathcal{A}_i$ such that if $i \neq j$ then $d_\omega(\mathcal{A}_i)$ and $d_\omega(\mathcal{A}_j)$ are a $\mathcal{K}$-pair. As a corollary we obtain that every countable distributive lattice is embeddable in the a.z. degrees. Our second application is the proof of the fact that every nonzero super *a.z.* degree bounds a $\mathcal{K}$-pair of nonzero degrees. An easy modification of this proof shows that independent families of super almost zero sequences of sets exist in every nonempty interval of super almost zero sequences, revealing the high structural complexity of this class.

1. Preliminaries

1.1. **The structure of the enumeration degrees.** We assume that the reader is familiar with the notion of enumeration reducibility, and refer to Cooper [2] for a survey of basic results on the structure of the enumeration degrees and to Sorbi [8] for a survey of basic results on the local structure $\mathcal{G}_e$. For completeness we will nevertheless outline here basic definitions and properties of the enumeration degrees used in this article.

1.1. Definition. A set A is *enumeration reducible* ($\leq_e$) to a set B if there is a c.e. set Φ such that:

$$A = \Phi(B) = \{n \mid \exists u(\langle n, u\rangle \in \Phi \ \&\ D_u \subseteq B)\},$$

where D_u denotes the finite set with code u under the standard coding of finite sets. We will refer to the c.e. set Φ as an *enumeration operator* and its elements will be called *axioms*.

A set A is *enumeration equivalent* ($\equiv_e$) to a set B if $A \leq_e B$ and $B \leq_e A$. The equivalence class of A under the relation $\equiv_e$ is the enumeration degree $d_e(A)$ of A. The structure of the enumeration degrees $\langle \mathcal{D}_e, \leq\rangle$ is the class of all enumeration degrees with relation $\leq$ defined by $d_e(A) \leq d_e(B)$ if and only if $A \leq_e B$. It has a least element $\mathbf{0}_e = d_e(\emptyset)$, the set of all c.e. sets. We can define a least upper bound operation, by setting $d_e(A) \vee d_e(B) = d_e(A \oplus B)$ and a jump operator $d_e(A)' = d_e(J_e(A))$. The enumeration jump of a set A, denoted by $J_e(A)$ is defined by Cooper [1] as $\overline{L_A} \oplus L_A$, where $L_A = \{\ n |\ n \in \Phi_n(A)\}$.

Before we move on to the ω-enumeration degrees, we introduce one more piece of notation:

1.2. Definition. Let A be a set of natural numbers and i be a natural number:

(1) $A^{[i]} = \{\langle i, x\rangle \mid \langle i, x\rangle \in A\}$;
(2) For $R \in \{\leq, <, \geq, >\}$ we set $A^{[Ri]} = \{\langle j, x\rangle \mid \langle j, x\rangle \in A \ \wedge\ (jRi)\}$.
(3) $A[i] = \{x \mid \langle i, x\rangle \in A\}$.

The definition of the join operation can be extended to the following:

1.3. Definition. Let C be a computable set and $\{A_i\}_{i\in C}$ be a class of sets of natural numbers. Then $\bigoplus_{i\in C} A_i = \{\langle i, x\rangle \mid i \in C \ \wedge\ x \in A_i\}$.

1.2. **The ω-enumeration degrees.** Soskov [10] introduces a reducibility, $\leq_\omega$, between sequences of sets of natural numbers. The original definition involves the so called *jump set* of a sequence and can be found in [10]. We use an equivalent definition in terms of operators which is more approachable, as it resembles the definition of e-reducibility. Before we define ω-reducibility we will need to introduce two more bits of notations. Let $\mathcal{S}$ denote the class of all sequences of sets of natural numbers of length ω. With every member $\mathcal{A} \in \mathcal{S}$ we associated a jump sequence $P(\mathcal{A})$.

1.4. Definition. Let $\mathcal{A} = \{A_n\}_{n<\omega} \in \mathcal{S}$. The *jump sequence* of the sequence $\mathcal{A}$, denoted by $P(\mathcal{A})$ is the sequence $\{P_n(\mathcal{A})\}_{n<\omega}$ defined inductively as follows:

- $P_0(\mathcal{A}) = A_0$.
- $P_{n+1}(\mathcal{A}) = P'_n(\mathcal{A}) \oplus A_{n+1}$, where $P'_n(\mathcal{A})$ denotes the enumeration jump of the set $P_n(\mathcal{A})$.

The jump sequence $P(\mathcal{A})$ transforms a sequence $\mathcal{A}$ into a monotone sequence of sets of natural numbers with respect to $\leq_e$. Every member of the jump sequence contains full information on previous members.

Next we extend the notion of an e-operator so that it can be applied to members of $\mathcal{S}$.

1.5. Definition. Let $\mathcal{A} = \{A_n\}_{n<\omega}$ be a sequence of sets natural numbers and V be an e-operator. The result of applying the enumeration operator V to the sequence $\mathcal{A}$, denoted by $V(\mathcal{A})$, is the sequence $\{V[n](A_n)\}_{n<\omega}$. We say that $V(\mathcal{A})$ is enumeration reducible ($\leq_e$) to the sequence $\mathcal{A}$.

The motivation behind the definition of ω-reducibility is an attempt to capture the information content of a set of natural numbers together with all of its enumeration jumps. It turns out that e-reducibility between sequences of sets is too strong for this purpose.

1.6. Definition. Let $\mathcal{A}, \mathcal{B} \in \mathcal{S}$. We shall say that $\mathcal{B}$ is ω-enumeration reducible to $\mathcal{A}$, denoted by $\mathcal{B} \leq_\omega \mathcal{A}$, if $\mathcal{B} \leq_e P(\mathcal{A})$.

This definition, and the induced equivalence relation $\equiv_\omega$, where $\mathcal{A} \equiv_\omega \mathcal{B}$ if and only if $\mathcal{A} \leq_\omega \mathcal{B}$ and $\mathcal{B} \leq_\omega \mathcal{A}$, allows us to immediately state the following two properties:

(1) For every sequence $\mathcal{A}$, $\mathcal{A} \equiv_\omega P(\mathcal{A})$. Furthermore for every sequence $\mathcal{A}$, $P(\mathcal{A}) \equiv_e P(P(\mathcal{A}))$. Thus every ω-enumeration degree contains a member $\mathcal{X}$, such that $\mathcal{X} \equiv_e P(\mathcal{X})$.

(2) For every pair of sequences $\mathcal{A}$ and $\mathcal{B}$, $\mathcal{B} \leq_\omega \mathcal{A}$ if and only if $P(\mathcal{B}) \leq_e P(\mathcal{A})$.

Thus ω-reducibility can be thought of as e-reducibility between sequences of the form $P(\mathcal{A})$.

Clearly "$\leq_\omega$" is a reflexive and transitive relation and defines a preorder on $\mathcal{S}$. It induces in the usual way a degree structure, the structure of the ω-enumeration degrees, $\mathcal{D}_\omega = \langle \{d_\omega(\mathcal{A}) \mid \mathcal{A} \in \mathcal{S}\}, \leq_\omega \rangle$. This is a partial ordering with least element $\mathbf{0}_\omega$ the degree of the sequence $\emptyset_\omega$, where all members of the sequence $\emptyset_\omega$ are equal to $\emptyset$ or equivalently the degree of the sequence $\{\emptyset^n\}_{n<\omega}$.

Given two sequences $\mathcal{A} = \{A_n\}_{n<\omega}$ and $\mathcal{B} = \{B_n\}_{n<\omega}$ let $\mathcal{A} \oplus \mathcal{B} = \{A_n \oplus B_n\}_{n<\omega}$. Is it easy to see that $d_\omega(\mathcal{A} \oplus \mathcal{B})$ is the least upper bound of $d_\omega(\mathcal{A})$ and $d_\omega(\mathcal{B})$ and hence $\mathcal{D}_\omega$ is an upper semi-lattice. The operation $\oplus$ can be extended to any computable class of sequences of sets as follows.

1.7. Definition.

Let C be a computable set and $\{\mathcal{A}_i\}_{i \in C}$ be a class of sequences of sets, where for every $i \in C$, $\mathcal{A}_i = \{A_{i,n}\}_{n<\omega}$. Then

$$\bigoplus_{i \in C} \mathcal{A}_i = \{\bigoplus_{i \in C} A_{i,n}\}_{n<\omega}.$$

We define a jump operation: for every sequence $\mathcal{A}$, let $d_\omega(\mathcal{A})' = d_\omega(\mathcal{A}')$, where $\mathcal{A}' = \{P_{n+1}(\mathcal{A})\}_{n<\omega}$. We can iterate this definition to obtain the n-th jump of $\mathbf{a}$ for every natural number n. Namely we set $\mathbf{a}^0 = \mathbf{a}$ and for every $n \geq 0$, $\mathbf{a}^{(n+1)} = (\mathbf{a}^n)'$. Note that in contrast to other degree structures, here we can easily calculate a representative of the degree $d_\omega(\mathcal{A})^n$, namely the sequence $\{P_{n+k}(\mathcal{A})\}_{k<\omega}$.

The structure of the ω-enumeration degrees as an upper semilattice with jump operation, $\langle \mathcal{D}_\omega, \leq_\omega, \vee, ' \rangle$, can be seen as an extension of the structure of the enumeration degrees $\langle \mathcal{D}_e, \leq_e, \vee, ' \rangle$. Let A be any set of natural numbers and let $\mathcal{A} = \{A_n\}_{n<\omega}$ be the sequence defined by $A_0 = A$ and $A_{n+1} = \emptyset$. Then define

$$\kappa(d_e(A)) = \mathcal{A}.$$

The embedding κ preserves the order, the least upper bound and the jump operation. Thus as noted in the introduction, the images of the enumeration degrees under the embedding κ forms a substructure of the ω-enumeration degrees, which is furthermore first order definable in $\mathcal{D}_\omega$, see [11].

The jump operation gives rise to the local structure of the ω-enumeration degrees $\mathcal{G}_\omega$, consisting of all ω-enumeration degrees below the first jump of the least degree. This substructure is of high complexity. In [9] it is shown that every countable partial ordering can be embedded in any nonempty interval in $\mathcal{G}_\omega$. We will call these degrees Σ^0_2 ω-enumeration degrees. It is not difficult to check that every degree $\mathbf{a} \leq \mathbf{0}'_\omega$ contains a member $\mathcal{A} = \{A_n\}_{n<\omega}$, such that for every n the set A_n is $\Sigma^{\emptyset^n}_2$.

We extend this analogy further. For every sequence $\mathcal{A} = \{A_n\}_{n<\omega}$, we define the sequence $\bar{\mathcal{A}} = \{\bar{A}_n\}_{n<\omega}$. Then a sequence $\mathcal{A}$ will be called Δ^0_2 if both $\mathcal{A} \leq_\omega \emptyset'_\omega$ and $\bar{\mathcal{A}} \leq_\omega \emptyset'_\omega$. A sequence $\mathcal{A}$ will be called Σ^0_1 if $\mathcal{A} \leq_\omega \emptyset_\omega$. An alternative definition of ω-reducibility, which will be used in this article, is given by the following proposition, a proof of which can be found in [10]:

1.8. Proposition.

Let $\mathcal{A}, \mathcal{B} \in \mathcal{S}$. Then $\mathcal{A} \leq_\omega \mathcal{B}$ if and only if there is a Σ^0_1 sequence U, such that for every n:

$$A_n = U_n(P_n(\mathcal{B})) = \{x \mid \exists D[\langle x, D\rangle \in U_n \ \&\ D \subseteq P_n(\mathcal{B})]\}.$$

1.3. **Approximations and Σ^0_2 sequences.** The first step in defining nice approximations to sequences in $\mathcal{G}_\omega$ is to relativize the usual notions of a Σ^0_2, Δ^0_2 and Σ^0_1 approximation to a set with respect to $\emptyset^n$ for every n. We can define a $\Sigma^0_2(\emptyset^n)$ approximation to a set A to be a uniformly computable from $\emptyset^n$ sequence of finite sets $\{A^{\{s\}}\}_{s<\omega}$ such that $n \in A$ if and only if $(\exists s)(\forall t > s)(n \in A^{\{s\}})$. A $\Sigma^0_1(\emptyset^n)$ approximation to a set A is a $\Sigma^0_2(\emptyset^n)$ approximation $\{A^{\{s\}}\}_{s<\omega}$ with the additional property that for every s the $A^{\{s\}} \subseteq A^{\{s+1\}}$. A $\Delta^0_2(\emptyset^n)$ approximation to a set A is a $\Sigma^0_2(\emptyset^n)$ approximation $\{A^{\{s\}}\}_{s<\omega}$ with the additional property that for every n the limit $\lim_s A^{\{s\}}(n)$ exists. These are natural definitions motivated by the fact that a set A is $\Sigma^0_2(\emptyset^n)$ ($\Sigma^0_1(\emptyset^n)$ or $\Delta^0_2(\emptyset^n)$) if and only if it has a $\Sigma^0_2(\emptyset^n)$ ($\Sigma^0_1(\emptyset^n)$ or $\Delta^0_2(\emptyset^n)$) approximation.

1.9. Definition. Let $\mathcal{A} \in \mathcal{S}$ and $\{A^{\{s\}}_n\}_{s,n<\omega}$ be a sequence of finite sets such that for every n, the sequence $\{A^{\{s\}}_n\}_{s<\omega}$ is uniformly computable from $\emptyset^n$.

1. If for every n the sequence $\{A^{\{s\}}_n\}_{s<\omega}$ is a Σ^0_2 approximation to A_n, then $\{A^{\{s\}}_n\}_{s,n<\omega}$ is a Σ^0_2 approximation to the sequence $\mathcal{A}$.

(2) If for every n the sequence $\{A_n^{\{s\}}\}_{s<\omega}$ is a Σ_1^0 approximation to A_n, then $\{A_n^{\{s\}}\}_{s,n<\omega}$ is a Σ_1^0 approximation to the sequence $\mathcal{A}$.
(3) If for every n the sequence $\{A_n^{\{s\}}\}_{s<\omega}$ is a Δ_2^0 approximation to A_n, then $\{A_n^{\{s\}}\}_{s,n<\omega}$ is a Δ_2^0 approximation to the sequence $\mathcal{A}$.

Now it is not difficult to see that a sequence $\mathcal{A}$ is Σ_2^0 (Σ_1^0 or Δ_2^0) if and only if it has a Σ_2^0 (Σ_1^0 or Δ_2^0) approximation.

Recall that the length of agreement function between two sets of natural numbers, measured at stage s is defined as:

$$l(A, B, s) = \max\{k \leq s \mid A \upharpoonright k = B \upharpoonright k\}.$$

1.10. Definition. Let $\{A_n^{\{s\}}\}_{s,n<\omega}$ and $\{B_n^{\{s\}}\}_{s,n<\omega}$ be two sequences of sequences of sets.

(1) A stage s is n-expansionary if:
$$l(A_n^{\{s\}}, B_n^{\{s\}}, s) > \max_{t<s} l(A_n^{\{t\}}, B_n^{\{t\}}, t).$$
(2) A stage s is strongly n-expansionary if $s > n$, s is n-expansionary and
$$\forall k < n \exists t [n < t \leq s \;\&\; t \text{ is } k\text{-expansionary}].$$

The definition of strongly n-expansionary stages is designed so that if there are finitely many n-expansionary stages for some n, then there are finitely many strongly m-expansionary stages for every $m > n$.

1.11. Proposition. *Let $\{A_n^{\{s\}}\}_{s,n<\omega}$ and $\{B_n^{\{s\}}\}_{s,n<\omega}$ be Δ_2^0 approximations to sequences $\mathcal{A}$ and $\mathcal{B}$. Then the following assertions hold:*

(1) *If $\mathcal{A} = \mathcal{B}$ then for every n there are infinitely many strongly n-expansionary stages.*
(2) *If $\mathcal{A} \neq \mathcal{B}$ then there exists an index n_0, such that for all $k > n_0$ there are no strongly k-expansionary stages.*

Proof. (1) Suppose that $\mathcal{A} = \mathcal{B}$. First we note that for every n $\{A_n^{\{s\}}\}_{s<\emptyset}$ and $\{B_n^{\{s\}}\}_{s<\emptyset}$ are $\Delta_2^0(\emptyset^n)$ approximations to the equal sets A_n and B_n. From the properties of a $\Delta_2^0(\emptyset^n)$ approximation it follows that there are infinitely many n-expansionary stages and in particular there are infinitely many n-expansionary stages $s > n$.

Now fix n and for every $k < n$ let s_k be a k-expansionary stage, such that $s_k > n$. Then every n-expansionary stage $s > \max_{k<n} s_k$ will be strongly n-expansionary.

(2) Suppose that $\mathcal{A} \neq \mathcal{B}$. Let n be such that $A_n \neq B_n$. Then by the properties of a $\Delta_2^0(\emptyset^n)$ approximation it follows that there are finitely many n-expansionary stages. Let $n_0 = \max_s(s \text{ is } n\text{-expansionary})$. Then for all $k > n_0$ there are no n-expansionary stages $t > k$ and hence no strongly k-expansionary stages.

□

1.4. A.z. degrees and $\mathcal{K}$-pairs in the Σ_2^0 ω-enumeration degrees.

As noted above a new type of "lowness" property, given by the class of the almost zero ω-enumeration degrees, was defined and studied by Soskov and Ganchev in [11].

1.12. Definition. We shall say that the sequence $\mathcal{A} = \{A_n\}$ is *almost zero* (a.z.) if for every n we have that $A_n \leq_e \mathbf{0}^n$, or equivalently that $P_n(\mathcal{A}) \equiv_e \mathbf{0}^n$. A degree is almost zero, if it contains an almost zero member.

Even though this new lowness property is not defined as usual in terms of domination, or the strength of the jump operator applied to members which possess it, there are some unexpected connections. The class of a.z. degrees has an important role in the definability of the jump classes in $\mathcal{G}_\omega$, the low, the intermediate and the high degrees.

1.13. Definition. Let $\mathbf{a} \leq_\omega \mathbf{0}'_\omega$. Then:

(1) $\mathbf{a} \in L_n \iff \mathbf{a}^{(n)} = \mathbf{0}_\omega^{(n)}$.
(2) $\mathbf{a} \in H_n \iff \mathbf{a}^{(n)} = \mathbf{0}_\omega^{(n+1)}$.
(3) $H = \bigcup H_n; L = \bigcup L_n$.
(4) $\mathbf{a} \in I \iff \mathbf{a} \notin H \cup L$.

The first connection is given by Ganchev and Soskov in [11].

1.14. Theorem. *Let* $\mathbf{a} \leq_\omega \mathbf{0}'_\omega$. *Then*

(1) $\mathbf{a} \in H \iff (\forall a.z.\, \mathbf{x})(\mathbf{x} \leq_\omega \mathbf{a})$.
(2) $\mathbf{a} \in L \iff (\forall a.z.\, \mathbf{x})(\mathbf{x} \leq_\omega \mathbf{a} \Rightarrow \mathbf{x} = \mathbf{0}_\omega)$.

Later on in [4] a further definability result is shown:

1.15. Theorem. *For every natural number n the classes H_n and L_n are first order definable in the local theory of the ω-enumeration degrees.*

A special tool in the proof of this result is the use of the notion of a $\mathcal{K}$-pair of ω-enumeration degrees.

Recall that by Definition 0.1 a pair of sets A and B is a $\mathcal{K}$-pair over a set U, if there is a set of natural numbers $W \leq_e U$, such that $A \times B \subseteq W$ and $\bar{A} \times \bar{B} \subseteq \bar{W}$. Call this the $\mathcal{K}$-set property. Kalimullin has proved that this property is degree theoretic and first order definable in $\mathcal{D}_e$:

1.16. Theorem.*[Kalimullin] A pair of sets A, and B form a $\mathcal{K}$-pair over U if and only if their degrees $\mathbf{a} = d_e(A)$, $\mathbf{b} = d_e(B)$ and $\mathbf{u} = d_e(U)$ satisfy the formula:*

$$\mathcal{K}(\mathbf{a}, \mathbf{b}, \mathbf{u}) \rightleftharpoons \forall \mathbf{x}[(\mathbf{a} \vee \mathbf{x} \vee \mathbf{u}) \wedge (\mathbf{b} \vee \mathbf{x} \vee \mathbf{u}) = \mathbf{x} \vee \mathbf{u}].$$

Furthermore, if A, and B form a $\mathcal{K}$-pair over U and $U <_e A, B \leq_e U'$ then $A' \equiv_e B' \equiv_e U'$.

We can consider elements satisfying this property in any upper semi-lattice. The following proposition exhibits a structural property of $\mathcal{K}$-pairs, which makes them useful for coding structures.

1.17. Proposition. *Let $\mathcal{D} = (\mathbf{D}, \leq, \vee, \mathbf{0})$ be any upper semilattice with least element $\mathbf{0}$. Let $\mathbf{a}$ and $\mathbf{u}$ be elements in $\mathbf{D}$. The set of all $\mathbf{b}$ such that $\mathcal{D} \models \mathcal{K}(\mathbf{a}, \mathbf{b}, \mathbf{u})$ is an ideal. Furthermore, if $\mathbf{b} \in \mathbf{D}$ is such that $\mathcal{D} \models \mathcal{K}(\mathbf{a}, \mathbf{b}, \mathbf{u})$ then $\mathbf{a} \vee \mathbf{u}$ and $\mathbf{b} \vee \mathbf{u}$ form a minimal pair over $\mathbf{u}$.*

Proof. Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{u}$ be such that $\mathcal{D} \models \mathcal{K}(\mathbf{a}, \mathbf{b}, \mathbf{u})$. First suppose that $\mathbf{c} \leq \mathbf{b}$. Then for every $\mathbf{x} \in \mathcal{D}$ we have:

$$\mathbf{x} \leq (\mathbf{a} \vee \mathbf{x} \vee \mathbf{u}) \wedge (\mathbf{c} \vee \mathbf{x} \vee \mathbf{u}) \leq (\mathbf{a} \vee \mathbf{x} \vee \mathbf{u}) \wedge (\mathbf{b} \vee \mathbf{x} \vee \mathbf{u}) = \mathbf{x}.$$

Hence $\mathcal{D} \models \mathcal{K}(\mathbf{a}, \mathbf{c}, \mathbf{u})$.

On the other hand if $\mathbf{c}$ is such that $\mathcal{D} \models \mathcal{K}(\mathbf{a}, \mathbf{c}, \mathbf{u})$, then to prove that $\mathcal{D} \models \mathcal{K}(\mathbf{a}, \mathbf{b} \vee \mathbf{c}, \mathbf{u})$, fix $\mathbf{x} \in \mathbf{D}$ and let $\mathbf{y} \in \mathbf{D}$ be such that:

$$\mathbf{y} \leq (\mathbf{a} \vee \mathbf{x} \vee \mathbf{u}); \quad \mathbf{y} \leq (\mathbf{b} \vee \mathbf{c} \vee \mathbf{x} \vee \mathbf{u}).$$

Then from $\mathcal{D} \models \mathcal{K}(\mathbf{a}, \mathbf{b}, \mathbf{u})$ it follows that:

$$\mathbf{y} \leq (\mathbf{a} \vee \mathbf{c} \vee \mathbf{x} \vee \mathbf{u}) \wedge (\mathbf{b} \vee \mathbf{c} \vee \mathbf{x} \vee \mathbf{u}) = \mathbf{c} \vee \mathbf{x} \vee \mathbf{u}.$$

Now applying $\mathcal{D} \models \mathcal{K}(\mathbf{a}, \mathbf{c}, \mathbf{u})$ we get that:

$$\mathbf{y} \leq (\mathbf{a} \vee \mathbf{x} \vee \mathbf{u}) \wedge (\mathbf{c} \vee \mathbf{x} \vee \mathbf{u}) = \mathbf{x} \vee \mathbf{u}.$$

Finally $\mathbf{a} \vee \mathbf{u}$ and $\mathbf{b} \vee \mathbf{u}$ form a minimal pair over $\mathbf{u}$ as:

$$(\mathbf{a} \vee \mathbf{u}) \wedge (\mathbf{b} \vee \mathbf{u}) = (\mathbf{a} \vee \mathbf{0} \vee \mathbf{u}) \wedge (\mathbf{b} \vee \mathbf{0} \vee \mathbf{u}) = \mathbf{0} \vee \mathbf{u} = \mathbf{u}.$$

□

A completely different question is of course whether or not there are pairs of elements satisfying the $\mathcal{K}$-pair formula in a given upper semi-lattice. In the enumeration degrees such pairs exist. In fact consider the following notion:

1.18. Definition. A sequence $\{A_i\}$ of sets of natural numbers is called independent $\mathcal{K}$-family if the following two conditions are satisfied:

(1) For each i, $A_i \not\leq_e \bigoplus_{j \neq i} A_j$.
(2) If R_1 and R_2 are disjoint recursive sets then $\bigoplus_{i \in R_1} A_i$ and $\bigoplus_{j \in R_2} A_j$ is a $\mathcal{K}$-pair over $\emptyset$.

A detailed proof of the following result can be found in [6].

1.19. Theorem. *Every nonzero Δ^0_2 set uniformly bounds an independent $\mathcal{K}$-family.*

We will examine the analog of $\mathcal{K}$-pairs in the structure of the ω-enumeration degrees. Between the set theoretic and degree theoretic definition of $\mathcal{K}$-pairs, the latter is more useful. We shall therefore define $\mathcal{K}$-pairs of sequences, so that they satisfy the $\mathcal{K}$-pair degree theoretic property. For simplicity we shall restrict ourselves to $\mathcal{K}$-pairs over over the least degree $\mathbf{0}_\omega$.

Given $\mathcal{A} \in \mathcal{S}$, let

$$(\mathcal{A}) = \{\mathcal{B} : \mathcal{B} \in \mathcal{S} \;\&\; \mathcal{B} \leq_\omega \mathcal{A}\}.$$

1.20. Definition. Two sequences $\mathcal{A}$ and $\mathcal{B}$ of sets of natural numbers form a $\mathcal{K}$-pair if for every sequence $\mathcal{X}$, $(\mathcal{A} \oplus \mathcal{X}) \cap (\mathcal{B} \oplus \mathcal{X}) = (\mathcal{X})$.

The first step in characterizing the $\mathcal{K}$-pairs in the ω-enumeration degrees is the following result.

1.21. Theorem. *Let $\mathcal{A} \not\equiv_\omega \emptyset_\omega$ and $\mathcal{B} \not\equiv_\omega \emptyset_\omega$ be two Σ^0_2 sequences, which form and a $\mathcal{K}$-pair. Then both $\mathcal{A}$ and $\mathcal{B}$ are a.z. or for some n there exists a $\mathcal{K}$-pair A, B over $\emptyset^n$ such that $\emptyset^n <_e A, B \leq \emptyset^{n+1}$ and*

$$\mathcal{A} \equiv_\omega \{\underbrace{\emptyset, \dots, \emptyset}_{n}, A, \emptyset, \dots, \emptyset, \dots\} \text{ and}$$

$$\mathcal{B} \equiv_\omega \{\underbrace{\emptyset, \dots, \emptyset}_{n}, B, \emptyset, \dots, \emptyset, \dots\}.$$

Results on $\mathcal{K}$-pairs in the enumeration degrees give an abundance of instances of $\mathcal{K}$-pairs of the second type. Here we shall investigate $\mathcal{K}$-pairs of *a.z.* degrees.

2. Super a.z. sequences

The search for natural examples of $\mathcal{K}$-pairs in the Σ^0_2 ω-enumeration degree, leads us to a new class of ω-enumeration degrees, the super a.z. degrees. In this section we define this class, show that it is a proper sub-ideal of the a.z. degrees and describe some of its special properties in relation to the notion of $\mathcal{K}$-pair.

2.1. Definition. A sequence $\mathcal{A}$ is called *super* a.z. if there exists a sequence $\mathcal{I} = \{I_n\}_{n<\omega}$, such that $\mathcal{I} \leq_\omega \emptyset'_\omega$ and

(1) $(\forall n)(I_n \neq \emptyset)$.
(2) $(\forall n)(\forall a \in I_n)(A_n = W_a(\emptyset^n))$.

$\mathcal{I}$ is called an index sequence for $\mathcal{A}$.

The second part of the definition above ensures that every super a.z. sequence is a.z. We show furthermore that every such sequence and its complement is ω enumeration reducible to $\mathbf{0}'_\omega$ and hence is Δ^0_2.

2.2. Proposition. *If $\mathcal{A}$ is super a.z. then $\mathcal{A} \leq_\omega \emptyset'_\omega$ and $\bar{\mathcal{A}} \leq_\omega \emptyset'_\omega$.*

Proof. Let $\mathcal{A}$ be a super a.z. sequence and let $\mathcal{I} = \{I_n\}_{n<\omega} \leq_\omega \mathbf{0}'_\omega$ be an index sequence for $\mathcal{A}$. There is an enumeration operator Γ such that $I_n = \Gamma[n](\emptyset^{n+1})$. Then $x \in A_n$ if and only if there is an index $a \in I_n = \Gamma[n](\emptyset^{n+1})$, such that $x \in W_a(\emptyset^n)$. Now, $W_a(\emptyset^n)$ is uniformly in a enumeration reducible to $\emptyset^{n+1}$, i.e. there is a computable function r such that $W_a(\emptyset^n) = W_{r(a)}(\emptyset^{n+1})$. Define the enumeration operator Δ by setting:

$$\Delta[n] = \left\{ \langle x, D\rangle \mid \exists a \exists D_1 \exists D_2 (D = D_1 \cup D_2 \ \&\ \langle a, D_1\rangle \in \Gamma[n] \ \&\ \langle x, D_2\rangle \in W_{r(a)} \right\}$$

Then obviously $A_n = \Delta[n](\emptyset^{n+1})$ and hence $\mathcal{A} \leq \emptyset'_\omega$.

On the other hand $x \notin A_n$ if and only if there is an index $a \in I_n = \Gamma[n](\emptyset^{n+1})$, such that $x \in \bar{W}_a(\emptyset^n)$. Now, $\bar{W}_a(\emptyset^n)$ is uniformly in a enumeration reducible to $\emptyset^{n+1}$, i.e. there is a computable function $\bar{r}$ such that $\bar{W}_a(\emptyset^n) = W_{\bar{r}(a)}(\emptyset^{n+1})$.

Define the enumeration operator Λ by setting:

$$\Lambda[n] = \left\{ \langle x, D\rangle \mid \exists a \exists D_1 \exists D_2 (D = D_1 \cup D_2 \ \&\ \langle a, D_1\rangle \in \Gamma[n] \ \&\ \langle x, D_2\rangle \in W_{\bar{r}(a)} \right\}$$

Then $\bar{A}_n = \Lambda[n](\emptyset^{n+1})$ and hence $\bar{\mathcal{A}} \leq \emptyset'_\omega$ as well. □

The next property shows that the notion of a super a.z. sequence is degree theoretic. We can therefore define a super a.z. degree to be an ω-enumeration degree which consists of super a.z. sequences. We shall see that these form an ideal.

2.3. Proposition.

(1) *If $\mathcal{A}$ is super a.z. and $\mathcal{B} \leq_\omega \mathcal{A}$, then $\mathcal{B}$ is super a.z.*
(2) *The ω-enumeration degrees containing super a.z. sequences form an ideal.*

Proof. (1) First we note that if $\mathcal{A}$ is super a.z. the $P(\mathcal{A})$ is also super a.z. This follows from the fact that the monotonicity of the enumeration jump and the join operation are uniform. There are computable function g and j, such that if $A_n = W_a(\emptyset^n)$ then $A'_n = W_{g(a)}(\emptyset^{n+1})$ and if $A_{n+1} = W_e(\emptyset^{n+1})$ then $P_{n+1}(\mathcal{A}) = W_{j(g(a),e)}(\emptyset^{n+1})$. So if Is is the index sequence for $\mathcal{A}$ then $\mathcal{PI}$ is an index sequence for $P(\mathcal{A})$, where $PI_0 = I_0$ and for all n:

$$PI_{n+1} = \{j(g(a), e) \mid a \in PI_n \;\; \& \; e \in PI_{n+1}\}\,.$$

Obviously if $\mathcal{I} \leq_\omega \emptyset'_\omega$ then $\mathcal{PI} \leq_\omega \emptyset'_\omega$.

Let $\mathcal{A} = \{A_n\}$ be super a.z. with index sequence $\mathcal{I} = \{I_n\}_{n<\omega}$. Without loss of generality we may assume that $\mathcal{A} \equiv_e P(\mathcal{A})$. Let $\mathcal{B} = \{B_n\}_{n<\omega} \leq_\omega \mathcal{A}$. To show that $\mathcal{B}$ is super a.z., we need to construct a Σ^0_2 index sequence $\mathcal{J}$ for $\mathcal{B}$. Let Γ be the enumeration operator which reduces $\mathcal{B}$ to $\mathcal{A}$. Fix n. Then $x \in B_n$ if and only if $x \in \Gamma[n](A_n)$, if and only if there is an index $a \in I_n$ such that $x \in \Gamma[n](W_a^{\emptyset^n})$. The set $\Gamma[n](W_a\emptyset^n)$ is obviously a c.e. in $\emptyset^n$ set, for which we can obtain an index uniformly in n and a, by a computable function say m. Now we can define $J_n = \{m(n, a) \mid a \in I_n\}$. Then $\mathcal{J} = \{J_n\}_{n<\omega}$ is a Σ^0_2 index sequence for $\mathcal{B}$.

(2) By the Part 1. of the proof of this proposition, it follows that the super a.z. sequences are downwards closed. To show that the ω-enumeration degrees containing super a.z. sequences form an ideal, it remains to be proved that the least upper bound of two super a.z. degrees is super a.z. So let $\mathcal{A}$ and $\mathcal{B}$ be super a.z. sequences with index sequences $\mathcal{I}$ and $\mathcal{J}$ respectively. Using the computable function j as in (1) we get $W_a^{\emptyset^n} \oplus W_b^{\emptyset^n} = W_{j(a,b)}^{\emptyset^n}$, so an index sequence $\mathcal{IJ}$ for $\mathcal{A} \oplus \mathcal{B}$ can be obtained by setting $IJ_n = \{j(a, b) \mid a \in I_n \;\&\; b \in J_n\}$.

□

The existence of nonzero super a.z. sequences will follow from Theorem 4.1 below. Here we shall first prove that the class of super a.z. degrees does not exhaust all a.z. degrees below $\mathbf{0}'_\omega$ and is hence a proper sub-ideal of the Σ^0_2 a.z. degrees.

2.4. Proposition. *Not all a.z.* $\mathcal{A} \leq_\omega \emptyset'_\omega$ *are super a.z.*

Proof. The proof of this fact is done by a diagonalization construction. Let $\{W_e\}$ be some effective listing of all enumeration operators. We shall construct an a.z. sequence $\mathcal{A}$ whose n-th member A_n for every n will be either $\emptyset$ or $\{1\}$, selected to ensure that $W_n(\emptyset'_\omega)$ is not an index sequence for $\mathcal{A}$.

Fix $n > 0$. There are three possibilities for $W_n[n](\emptyset^{n+1})$:

(1) $W_n[n](\emptyset^{n+1})$ is empty. In this case $W_n(\emptyset'_\omega)$ is not an index sequence at all. We set $A_n = \emptyset$.
(2) $W_n[n](\emptyset^{n+1})$ contains an index a, such that $W_a^{\emptyset^n}$ contains an odd element. In this case again we set $A_n = \emptyset$. Then $P_n(\mathcal{A}) = (P_{n-1}(\mathcal{A}))' \oplus A_n$ contains no odd elements and hence again $W_n(\emptyset'_\omega)$ is not an index sequence for $\mathcal{A}$.
(3) Finally if $W_n[n](\emptyset^{n+1})$ contains an index a, such that $W_a^{\emptyset^n}$ does not contain any odd element, then we set $A_n = \{1\}$, ensuring that $P_n(\mathcal{A})$ contains at least one odd element and hence again we have diagonalized against $W_n(\emptyset'_\omega$.

In the case $n = 0$ the argument above will work again, even though $P_0(\mathcal{A}) = A_0$ does not have the form $X \oplus Y$. To avoid dealing with this special case, however we can simply note that by the padding lemma W_0 will appear infinitely many times in the effective listing of all c.e. sets and hence we can simply set $A_0 = \emptyset$.

The constructed sequence is obviously a.z. as all of its members are finite sets.

We give the formal definition of an enumeration operator V, so that $\mathcal{A} = V(\emptyset'_\omega)$ is constructed following the intuitive description above. Let r be again the computable function satisfying $W_a^{\emptyset^n} = W_{r(a)}(\emptyset^{n+1})$ for every a. Now note that the set E_n consisting of the indices a of sets $W_a^{\emptyset^n}$ which do not contain any odd elements is $\Pi^0_1(\emptyset^n)$ and hence enumeration reducible to $\emptyset^{n+1}$ uniformly in n, i.e. there is an enumeration operator Γ such that $E_n = \Gamma[n](\emptyset^{n+1})$.

Set $V[0] = \emptyset$ and for every $n > 0$, set:

$$V[n] = \{\langle 1, D\rangle \mid \exists a \exists D_1 \exists D_2 [\langle a, D_2\rangle \in W_n[n] \ \& \ \langle a, D_2\rangle \in \Gamma[n] \ \& \ D = D_1 \cup D_2]\}$$

We prove that $\mathcal{A} = V(\emptyset'_\omega)$ is an a.z. sequence which is not super a.z. Towards a contradiction assume that $\mathcal{A}$ is super a.z. and $W_n(\emptyset'_\omega)$ is an index sequence for $\mathcal{A}$. By the padding lemma we may assume that $n > 0$. Then $W_n[n](\emptyset^n)$ is nonempty and consists of indices for $P_n(\mathcal{A})$. Now $x \in A_n = V[n](\emptyset^{n+1})$ if and only if $x = 1$ and $W_n[n](\emptyset^{n+1})$ contains an index a of a c.e in $\emptyset^n$ set which has no odd elements. In other words $P_n(\mathcal{A}) = (P_{n-1}(\mathcal{A}))' \oplus A_n$ contains an odd element if and only if $W_n[n](\emptyset^{n+1})$ contains an index a of a set with no odd elements, which gives the anticipated contradiction. $\square$

The following propositions describes the main property of the super a.z. sequences:

2.5. Proposition. *Let $\mathcal{A}$ be super a.z. Then for every sequence $\mathcal{X}$,*

$$P(\mathcal{A} \oplus \mathcal{X}) \equiv_e \mathcal{A} \oplus P(\mathcal{X}).$$

Proof. It is easy to see that for every two sequences $\mathcal{A}$ and $\mathcal{X}$ of sets of natural numbers,

$$\mathcal{A} \oplus P(\mathcal{X}) \leq_e P(\mathcal{A} \oplus \mathcal{X}).$$

Let $\mathcal{I} = \{I_n\}_{n<\omega}$ be an index sequence for the super a.z. $\mathcal{A} = \{A_n\}_{n<\omega}$ below $\emptyset'_\omega$. Fix an arbitrary sequence $\mathcal{X} = \{X_n\}_{n<\omega}$.

Let g be a recursive function such that for all e and all $X \subseteq \mathbb{N}$, $W_e(X)' = W_{g(e)}(X')$.

First we shall show that there exists a recursive function $\rho(n)$ such that for all n,

$$(A_n \oplus P_n(\mathcal{X}))' = W_{\rho(n)}(P_n(\mathcal{X})').$$

Fix $x \in \mathbb{N}$ and $a \in I_n$. Then

$$x \in A_n \oplus P_n(\mathcal{X}) \iff x \in W_a(\emptyset^{(n)}) \oplus P_n(\mathcal{X}).$$

Clearly there exists a recursive function $\lambda(n,a)$ such that

$$W_a(\emptyset^{(n)}) \oplus P_n(\mathcal{X}) = W_{\lambda(n,a)}(P_n(\mathcal{X})).$$

Then

$$x \in (A_n \oplus P_n(\mathcal{X}))' \iff x \in W_{\lambda(n,a)}(P_n(\mathcal{X}))' \iff x \in W_{g(\lambda(n,a))}(P_n(\mathcal{X})').$$

Since $I_n \neq \emptyset$, we get from here that

$$(A_n \oplus P_n(\mathcal{X}))' = \{x : (\exists a \in I_n)(x \in W_{g(\lambda(n,a))}(P_n(\mathcal{X})'))\}.$$

Since $\mathcal{I} \leq_e \emptyset'_\omega$ and $\emptyset'_\omega \leq_e \{P_n(\mathcal{X})'\}$, there exists a recursive function ι such that for all n, $I_n = W_{\iota(n)}(P_n(\mathcal{X})')$. Then

$$(P_n(\mathcal{X}) \oplus A_n)' = \{x : (\exists a \in W_{\iota(n)}(P_n(\mathcal{X})'))(x \in W_{g(\lambda(n,a))}(P_n(\mathcal{X})'))\}.$$

The last equality implies immediately the existence of the desired recursive function ρ.

Now we are ready to define a recursive function $\mu(n)$ such that for all n,

$$P_n(\mathcal{A}\oplus\mathcal{X}) = W_{\mu(n)}(A_n \oplus P_n(\mathcal{X})).$$

Let a_0 be an index of the identity enumeration operator. Set $\mu(0) = a_0$. Then

$$P_0(\mathcal{A}\oplus\mathcal{X}) = A_0 \oplus X_0 = A_0 \oplus P_0(\mathcal{X}) = W_{\mu(0)}(A_0 \oplus P_0(\mathcal{X})).$$

Suppose that $\mu(n)$ is defined so that $P_n(\mathcal{A}\oplus\mathcal{X}) = W_{\mu(n)}(A_n \oplus P_n(\mathcal{X}))$. Then

$$P_n(\mathcal{A}\oplus\mathcal{X})' = W_{\mu(n)}(A_n \oplus P_n(\mathcal{X}))' = W_{g(\mu(n))}((A_n \oplus P_n(\mathcal{X}))').$$

Hence, by the properties of the recursive function ρ defined above,

$$P_n(\mathcal{A}\oplus\mathcal{X})' = W_{g(\mu(n))}(W_{\rho(n)}(P_n(\mathcal{X})')).$$

From here, since $P_{n+1}(\mathcal{X}) = P_n(\mathcal{X})' \oplus X_{n+1}$ and $P_{n+1}(\mathcal{A}\oplus\mathcal{X}) = P_n(\mathcal{A}\oplus\mathcal{X})' \oplus A_{n+1} \oplus X_{n+1}$, one can define effectively $\mu(n+1)$ so that

$$P_{n+1}(\mathcal{A}\oplus\mathcal{X}) = W_{\mu(n+1)}(A_{n+1} \oplus P_{n+1}(\mathcal{X})).$$

□

3. Characterizing super a.z. $\mathcal{K}$-pairs in $\mathcal{G}_\omega$

Let $\mathcal{A}$ and $\mathcal{B}$ be a pair of Σ^0_2 sequences of sets. We would like to find a condition that the sequences must satisfy, in the spirit of the original definition of $\mathcal{K}$-pairs in the enumeration degrees, in order for their degrees to form a $\mathcal{K}$-pair. The most natural property to require is the following:

3.1. Definition. We shall say that $\mathcal{A}$ and $\mathcal{B}$ satisfy the $\mathcal{K}$-sequence property if there exists a sequence $\mathcal{R} \leq_\omega \emptyset_\omega$, such that for all n, we have that $(A_n \times B_n \subseteq R_n \;\&\; \bar{A}_n \times \bar{B}_n \subseteq \bar{R}_n)$.

This property turns out to be however, not a sufficiently strong requirement. Indeed let A be such that $A' \equiv_e \emptyset'$ and B be any set. Consider the following two sequences $\mathcal{A} = (A, \emptyset, \emptyset, \emptyset, \dots)$ and $\mathcal{B} = (\emptyset, B, \emptyset, \emptyset, \dots)$. Obviously they satisfy the $\mathcal{K}$-sequence property with the sequence $\mathcal{R} = \emptyset_\omega$. In fact as $\{P_{n+1}(\mathcal{A})\}_{n<\omega} \leq_\omega \emptyset'_\omega$, it follows that $P(\mathcal{A})$ and $P(\mathcal{B})$ satisfy the $\mathcal{K}$-sequence property with $(\emptyset, P_1(\mathcal{A}) \times \mathbb{N}, P_2(\mathcal{A}) \times \mathbb{N}, \dots)$. By the characterization of Σ^0_2 $\mathcal{K}$-pairs given in Theorem 1.21, however their degrees do not form a $\mathcal{K}$-pair. Thus strengthening the requirement by asking that $P(\mathcal{A})$ and $P(\mathcal{B})$ satisfy the $\mathcal{K}$-sequence property would still not suffice.

This condition turns out to be necessary, as is proved below.

3.2. Proposition. *Let $\mathcal{A}$ and $\mathcal{B}$ be Σ^0_2 sequences of sets, which do not satisfy the $\mathcal{K}$-sequence property. Then $\mathcal{A}$ and $\mathcal{B}$ do not form a $\mathcal{K}$-pair.*

Proof. Let $\mathcal{A}$ and $\mathcal{B}$ be Σ^0_2 sequences of sets, which do not satisfy the $\mathcal{K}$-sequence property.

Suppose first that $\mathcal{A}$ and $\mathcal{B}$ are not a.z. and form a nontrivial $\mathcal{K}$-pair then by Theorem 1.21 for some n there exists a $\mathcal{K}$-pair A, B over $\emptyset^n$ such that $\emptyset^n <_e A, B \leq \emptyset^{n+1}$ and

$$\mathcal{A} \equiv_\omega \{\underbrace{\emptyset, \dots, \emptyset}_{n}, A, \emptyset, \dots, \emptyset, \dots\} \text{ and}$$

$$\mathcal{B} \equiv_\omega \{\underbrace{\emptyset, \dots, \emptyset}_{n}, B, \emptyset, \dots, \emptyset, \dots\}.$$

It follows by Theorem 1.16 that $A' \equiv_e B' \equiv_e \emptyset^{n+1}$ and hence there are enumeration operators V and U such that

$$\mathcal{A} = V(\{\underbrace{\emptyset, \dots, \emptyset^{n-1}}_{n}, \emptyset^n \oplus A, \emptyset^{n+1}, \dots, \}) \text{ and}$$
$$\mathcal{B} = U(\{\underbrace{\emptyset, \dots, \emptyset^{n-1}}_{n}, \emptyset^n \oplus B, \emptyset^{n+1}, \dots, \}).$$

Furthermore by the ideal property of $\mathcal{K}$-pairs, it follows that A_n and B_n form a $\mathcal{K}$-pair over $\emptyset^n$ as well. So there is a set $W \leq_e \emptyset^n$, such that $A_n \times B_n \subseteq W$ and $\bar{A}_n \times \bar{B}_n \subseteq \bar{W}$. Then consider the sequence $\mathcal{R}$ defined by $R_k = U[k](\emptyset^k) \times V[k](\emptyset^k)$, for $k \neq n$ and $R_n = W$. Then $\mathcal{R} \leq_\omega \emptyset_\omega$, $\mathcal{A} \times \mathcal{B} \subseteq \mathcal{R}$ and $\bar{\mathcal{A}} \times \bar{\mathcal{B}} \subseteq \bar{\mathcal{R}}$, contradicting the assumptions on $\mathcal{A}$ and $\mathcal{B}$.

Now suppose that $\mathcal{A}$ and $\mathcal{B}$ are a.z. sequences. Without loss of generality we may assume that all members of these sequences are infinite sets. We shall construct a pair of sequences $\mathcal{X}$ and $\mathcal{Y}$ so that $\mathcal{Y} \leq_\omega d_\omega(\mathcal{X}) \vee d_\omega(\mathcal{A}))$ and $\mathcal{Y} \leq_\omega d_\omega(\mathcal{X}) \vee d_\omega(\mathcal{B}))$, but $\mathcal{Y} \not\leq_\omega \mathcal{X}$.

Fix a computable enumeration $\{(D_x, E_x)\}_{x<\omega}$ of all pairs of finite sets and a computable listing $\{W_e\}_{e<\omega}$ of all enumeration operators.

For every n we shall set $Y_n = \Gamma(X_n \oplus A_n) = \Lambda(X_n \oplus B_n)$, where

$$\Gamma = \{\langle \langle a, b\rangle, \{\langle a, b\rangle\} \oplus \{a\}\rangle \mid \langle a, b\rangle < \omega\},$$
$$\Lambda = \{\langle \langle a, b\rangle, \{\langle a, b\rangle\} \oplus \{b\}\rangle \mid \langle a, b\rangle < \omega\}.$$

This will ensure that $\mathcal{Y} \leq_e \mathcal{X} \oplus \mathcal{A}$ and $\mathcal{Y} \leq_e \mathcal{X} \oplus \mathcal{B}$, which is even stronger than $\mathcal{Y} \leq_\omega \mathcal{X} \oplus \mathcal{A}$ and $\mathcal{Y} \leq_\omega \mathcal{X} \oplus \mathcal{B}$. We will construct $\mathcal{X}$ as an a.z. sequence, so that for every e we have that $\mathcal{Y} \neq W_e(P(\mathcal{X}))$. This will ensure that $\mathcal{Y} \not\leq_\omega \mathcal{X}$. The construction will be in stages. At every stage e we will ensure that for some m, $Y_m \neq W_e[m](X_m)$ and extend the definition of the sequence $\mathcal{X}$ up to X_m.

Construction

Stage (e): Suppose that we have constructed finite sets X_k, for all $k \leq n$. We will define a sequence $\mathcal{R}$ and a sequence $\{P_k\}_{k<\omega}$ by induction on k:

1. If $k \leq n$ then $R_k = A_k \times B_k$ and $P_k = P_k(\mathcal{X})$. (Note that $P_k(\mathcal{X})$ uses only the members of the sequence $\mathcal{X}$, indexed by numbers less than or equal to k, so at stage e we can compute $P_k(\mathcal{X})$ even though $\mathcal{X}$ is not fully defined.)
2. If $k > n$ then $R_k = \big\{\langle a, b\rangle \mid \langle a, b\rangle \in W_e[k](P'_{k-1} \oplus \{\langle a, b\rangle\}))\big\}$ and $P_k = P'_{k-1} \oplus \emptyset$.

Now as $\mathcal{X}$ is constructed as an a.z. sequence, it follows that for all $k \leq n$, $P_k \leq_e \emptyset^k$. The construction of P_k for $k > n$ is uniform hence $\{P_k\}_{k<\omega} \leq_\omega \emptyset_\omega$. Now since $\mathcal{A}$ and $\mathcal{B}$ are a.z. sequences, it follows that the sequence $\mathcal{R} \leq_\omega \emptyset_\omega$ as well. By our assumption on $\mathcal{A}$ and $\mathcal{B}$ we have that there is an m, such that $A_m \times B_m \not\subseteq R_m$ or $\bar{A}_m \times \bar{B}_m \not\subseteq \bar{R}_m$. Fix the least such m.

First we extend the definition of $\mathcal{X}$ by setting $X_k = \emptyset$ for all k, such that $n < k < m$. Note that by this we ensure that for all $k < m$, $P_k = P_k(\mathcal{X})$. Furthermore we ensure that for all k in the interval $n < k < m$, $\emptyset = Y_k = \Gamma(X_k \oplus A_k) = \Lambda_k(X_k \oplus B_k)$.

We claim that there is an element $\langle a, b\rangle$, such that at least one of the following two conditions is true:

1. $a \in A_m$ and $b \in B_m$ and $\langle a, b\rangle \notin W_e[m](P'_{m-1}(\mathcal{X}) \oplus \{\langle a, b\rangle\})$.
2. $a \notin A_m$ and $b \notin B_m$ and $\langle a, b\rangle \in W_e[m](P'_{m-1}(\mathcal{X}) \oplus \{\langle a, b\rangle\})$.

Indeed suppose that condition 1 is not true, i.e. for all $\langle a,b\rangle$, if $a \in A_m$ and $b \in B_m$ then $\langle a,b\rangle \in W_e[m](P'_{m-1}(\mathcal{X}) \oplus \{\langle a,b\rangle\})$. Then $A_m \times B_m \subseteq R_m$, hence there is an element $\langle a,b\rangle \in \bar{A}_m \times \bar{B}_m \setminus \bar{R}_m$. For this element $\langle a,b\rangle$ we have that $a \notin A_m$, $b \notin B_m$ and $\langle a,b\rangle \in W_e[m](P'_{m-1}(\mathcal{X}) \oplus \{\langle a,b\rangle\})$, i.e. condition 2 is true.

Now let $\langle a,b\rangle$ be the least element which satisfies condition 1 or condition 2 and set $X_m = \{\langle a,b\rangle\}$. If condition 1 is true for $\langle a,b\rangle$ then $\{\langle a,b\rangle\} = Y_m = \Gamma[m](X_m \oplus A_m) = \Lambda[m](X_m \oplus B_m)$ and $\langle a,b\rangle \notin W_e[m](P_m(\mathcal{X}))$. If condition 2 is true then $\emptyset = Y_m = \Gamma[m](X_m \oplus A_m) = \Lambda[m](X_m \oplus B_m)$ and $\langle a,b\rangle \in W_e[m](P_m(\mathcal{X}))$. In both cases $Y_m \neq W_e[m](P_m(\mathcal{X}))$. □

For the class of super a.z. degrees this condition turns out to be also sufficient as is proved below.

3.3. Theorem. *Let $\mathcal{A}$ and $\mathcal{B}$ be super a.z. sequences. $\mathcal{A}$ and $\mathcal{B}$ form a $\mathcal{K}$-pair if and only if they satisfy the $\mathcal{K}$-sequence property.*

Proof. Fix $\mathcal{A}$ and $\mathcal{B}$ to be super a.z. sequences. The left to right direction follows from Proposition 3.2. For the right to left direction suppose that $\mathcal{R} \leq_\omega \emptyset_\omega$ is such that for all n, $(A_n \times B_n \subseteq R_n \ \&\ \bar{A}_n \times \bar{B}_n \subseteq \bar{R}_n)$. Let $\mathcal{X}$ be any sequence and let $\mathcal{Y}$ be a sequence, such that $\mathcal{Y} \leq_\omega \mathcal{A} \oplus \mathcal{X}$ and $\mathcal{Y} \leq_\omega \mathcal{B} \oplus \mathcal{A}$. We shall prove that $\mathcal{Y} \leq_\omega \mathcal{X}$.

By Proposition 2.5 it follows that $\mathcal{Y} \leq_\omega \mathcal{A} \oplus \mathcal{X}$ and $\mathcal{Y} \leq_\omega \mathcal{B} \oplus \mathcal{X}$ is equivalent to the existence of two enumeration operators Γ and Λ such that for every n, we have $Y_n = \Gamma[n](A_n \oplus P_n(\mathcal{X})) = \Lambda[n](B_n \oplus P_n(\mathcal{X}))$. We construct a new operator Δ as follows. For every n we set:

$$\Delta[n] = \{\langle x, D_1 \cup D_2\rangle \mid \langle x, F_1 \oplus D_1\rangle \in \Gamma[n] \ \&\ \langle x, F_2 \oplus D_2\rangle \in \Lambda[n] \ \&\ F_1 \times F_2 \subseteq R_n\}$$

We claim that $\mathcal{Y} = \Delta(P(\mathcal{X}))$. Indeed, fix n. Suppose that $x \in Y_n$. Then $x \in \Gamma[n](A_n \oplus P_n(\mathcal{X}))$ and $x \in \Lambda[n](B_n \oplus P_n(\mathcal{X}))$. There are axioms $\langle x, F_1 \oplus D_1\rangle \in \Gamma[n]$ and $\langle x, F_2 \oplus D_2\rangle \in \Lambda[n]$ such that $F_1 \subseteq A_n$, $D_1 \subseteq P_n(\mathcal{X})$ and $F_2 \subseteq B_n$, $D_2 \subseteq P_n(\mathcal{X})$. It follows that $F_1 \times F_2 \subseteq A_n \times B_n \subseteq R_n$ and $D_1 \cup D_2 \subseteq P_n(\mathcal{X})$, so $x \in \Delta[n](P(\mathcal{X}))$.

Now suppose that $x \in \Delta[n](P_n(\mathcal{X}))$. Let $\langle x, D_1 \cup D_2\rangle$ be the valid axiom for x in $\Delta[n]$, i.e. such that $D_1 \cup D_2 \subseteq P_n(\mathcal{X})$. By the definition of $\Delta[n]$ it follows that there are finite sets F_1 and F_2 and axioms $\langle x, F_1 \oplus D_1\rangle \in \Gamma[n]$ and $\langle x, F_2 \oplus D_2\rangle \in \Lambda[n]$ such that $F_1 \times F_2 \subseteq R_n$. If $F_1 \subseteq A_n$ then $x \in \Gamma[n](A_n \oplus P_n(\mathcal{X})) = Y_n$. Assume that $F_1 \nsubseteq A_n$ and let f_1 be an element such that $f_1 \in F_1 \setminus A_n$. If $F_2 \subseteq B_n$ then $x \in \Lambda[n](B_n \oplus P_n(\mathcal{X})) = Y_n$. Assume that $F_2 \nsubseteq B_n$ and let f_2 be an element such that $f_2 \in F_2 \setminus B_n$. Now we reach a contradiction, since $\langle f_1, f_2\rangle \in F_1 \times F_2 \subseteq R_n$ and at the same time $\langle f_1, f_2\rangle \in \bar{A}_n \times \bar{B}_n \subseteq \bar{R}_n$. It follows that $x \in Y_n$ and so $Y_n = \Delta[n](P_n(\mathcal{X}))$. We note that it furthermore follows from this proof that an index of the operator Δ can be obtained effectively from indices of the operators Γ and Λ. □

4. Embedding countable distributive lattices in the super a.z. degrees

Now that we have established a necessary and sufficient condition for two super a.z. sequences to form a $\mathcal{K}$-pair, we can finally prove the existence of nontrivial super a.z. $\mathcal{K}$-pairs. First we give a relatively simple construction of an infinite independent $\mathcal{K}$-system.

4.1. Theorem. *There exists a sequence of super a.z. sequences $\mathcal{A}_i$ such that*

(1) $(\forall i)(\mathcal{A}_i \not\leq_\omega \bigoplus_{j\neq i} \mathcal{A}_j)$.
(2) *If R_1 and R_2 are disjoint recursive sets then $\bigoplus_{i\in R_1} \mathcal{A}_i$ and $\bigoplus_{i\in R_2} \mathcal{A}_i$ are a K-pair.*

A sequence $\mathcal{A}_i$ which satisfies the above conditions is called independent $\mathcal{K}$-system.

Proof. We shall construct inductively a sequence $\{A_s\}_{s<\omega}$ of sets of natural numbers and set $A_{i,s} = \{x : \langle i, x\rangle \in A_s\}$. During the construction we shall ensure that at most one of the sets $A_{i,s}$ is not empty. Hence if R_1 and R_2 are disjoint recursive sets then at least one of the sets $X_s = \bigoplus_{i\in R_1} A_{i,s}$ and $Y_s = \bigoplus_{i\in R_2} A_{i,s}$ is empty. Let $W_s = \emptyset$ for all s. Clearly $\{W_s\} \leq_\omega \emptyset_\omega$ and for all s, $X_s \times Y_s \subseteq W_s$ and $\bar{X}_s \times \bar{Y}_s \subseteq \bar{W}_s$. Thus the $\mathcal{K}$-sequence property will hold for $\bigoplus_{i\in R_1} \mathcal{A}_i$ and $\bigoplus_{i\in R_2} \mathcal{A}_i$.

To ensure that the sequences $\bigoplus_{i\in R_1} \mathcal{A}_i$ and $\bigoplus_{i\in R_2} \mathcal{A}_i$ are super a.z. we shall define a procedure which will produce uniformly in s using oracle $\emptyset^{s+1}$ an index a_s such that $A_s = W_{a_s}(\emptyset)$.

So, by the theorem above we shall have that the condition (2) is satisfied.

Set $A_0 = \emptyset$ and let a_0 be a Σ_1 index of $\emptyset$. Suppose that the sets A_t, $t \leq s$, and the respective indices a_t, $t \leq s$, are defined. Let $s = \langle i, e\rangle$. We shall define A_{s+1} so that $A_{i,s+1} \neq W_e[s+1](P_{s+1}(\bigoplus_{j\neq i} \mathcal{A}_j))$. This part of the construction will ensure the satisfaction of the condition (1) of the theorem.

For every $t \leq s$ set $A^*_t = \{\langle j, x\rangle : j \neq i \ \&\ \langle j, x\rangle \in A_t\}$. Clearly one can find effectively in $\emptyset^{s+1}$ indices $b_0, \dots .b_s$ such that for $t \leq s$, $A^*_t = W_{b_t}(\emptyset^t)$. Let $P_0 = A^*_0$ and $P_{t+1} = P'_t \oplus A^*_{t+1}$, $t < s$.

Notice that $P_s = P_s(\bigoplus_{j\neq i} \mathcal{A}_j)$. We shall define A_{s+1} so that $\bigoplus_{j\neq i} A_{j,s+1} = \emptyset$ and hence $P_{s+1}(\bigoplus_{j\neq i} \mathcal{A}_j) = P'_s \oplus \emptyset$.

Our next goal is to define effectively in $\emptyset^{s+1}$ a function h such that for all $t \leq s$, $P_t = W_{h(t)}(\emptyset^t)$.

Let us fix recursive functions g and λ such that for all $X \subseteq \mathbb{N}$ and $a, b \in \mathbb{N}$, $W_a(X)' = W_{g(a)}(X')$ and $W_{\lambda(a,b)}(X) = W_a(X) \oplus W_b(X)$.

Let $h(0) = b_0$. Clearly $P_0 = W_{b_0}(\emptyset)$. Suppose that for some $s < t$, $h(t)$ is defined and $P_t = W_{h(t)}(\emptyset^t)$. Then

$$P_{t+1} = W_{h(t)}(\emptyset^t)' \oplus W_{b_{t+1}}(\emptyset^{t+1}) = W_{g(h(t))}(\emptyset^{t+1}) \oplus W_{b_{t+1}}(\emptyset^{t+1}).$$

Set $h(t+1) = \lambda(g(h(t)), b_{t+1})$. Then $P_{t+1} = W_{h(t+1)}(\emptyset^{t+1})$.

Set $P_{s+1} = P'_s \oplus \emptyset = W_{h(s)}(\emptyset^s)' \oplus \emptyset = W_{g(h(s))}(\emptyset^{s+1}) \oplus \emptyset$. Clearly, one can find effectively in $\emptyset^{s+1}$ an index p_{s+1} such that $P_{s+1} = W_{p_{s+1}}(\emptyset^{s+1})$.

Now, let $A_{s+1} = \emptyset$ if $0 \in W_e[s+1](P_{s+1})$ and let $A_{s+1} = \{\langle i, 0\rangle\}$, otherwise.

Since we can decide in $\emptyset^{s+2}$ whether $0 \in W_e[s+1](W_{p_{s+1}}(\emptyset^{s+1}))$ or not, we can compute in $\emptyset^{s+2}$ an index a_{s+1} such that $A_{s+1} = W_{a_{s+1}}(\emptyset)$. □

The theorem above establishes among other things that there are nonzero super a.z. degrees. Another important application of this results is the following:

4.2. Theorem. *Every countable distributive lattice is embeddable in the super a.z. degrees preserving the least element.*

Proof. Since every countable distributive lattice is embeddable in the lattice of the computable sets $\mathcal{R}$ preserving least and greatest elements, it is enough to prove the theorem for $\mathcal{R}$. Fix an independent $\mathcal{K}$-system of super a.z sequences $\{\mathcal{A}_i\}_{i<\omega}$.

Given a computable set R, let $\varphi(R) = d_\omega(\bigoplus_{i\in R}\mathcal{A}_i)$. Evidently φ is injective since if R_1 and R_2 are computable sets and $i \in R_1 \setminus R_2$ then $d_\omega(\mathcal{A}_i) \leq_\omega \varphi(R_1)$ and $d_\omega(\mathcal{A}_i) \not\leq_\omega \varphi(R_2)$.

It is easily seen that for every pair R_1 and R_2 of computable sets, $\bigoplus_{i\in R_1}\mathcal{A}_i \oplus \bigoplus_{i\in R_2}\mathcal{A}_i \equiv_e \bigoplus_{i\in R_1\cup R_2}\mathcal{A}_i$. Hence

$$\varphi(R_1 \cup R_2) = d_\omega(\bigoplus_{i\in R_1\cup R_2}\mathcal{A}_i) = d_\omega(\bigoplus_{i\in R_1}\mathcal{A}_i \oplus \bigoplus_{i\in R_2}\mathcal{A}_i) = \varphi(R_1) \vee \varphi(R_2).$$

Let $C = R_1 \cap R_2$, $A = R_1 \setminus C$ and $B = R_2 \setminus C$. Set $\mathbf{a} = d_\omega(\bigoplus_{i\in A}\mathcal{A}_i)$, $\mathbf{b} = d_\omega(\bigoplus_{i\in B}\mathcal{A}_i)$ and $\mathbf{c} = \bigoplus_{i\in C}\mathcal{A}_i$. Clearly $\mathbf{a}$ and $\mathbf{b}$ are a K-pair over $\mathbf{0}_\omega$ and hence

$$\varphi(R_1 \cap R_2) = \mathbf{c} = (\mathbf{c} \vee \mathbf{a}) \wedge (\mathbf{c} \vee \mathbf{b}) = \varphi(C \cup A) \wedge \varphi(C \cup B) = \varphi(R_1) \wedge \varphi(R_2).$$

□

5. Bounding $\mathcal{K}$-pairs of super a.z. degrees

We show that every nonzero super a.z. sequence bounds a nontrivial $\mathcal{K}$-pair of super a.z. sequences. First we shall need to establish a dynamic characterization of $\mathcal{K}$-pairs of super a.z. degrees.

5.1. Proposition. *Suppose $\{B_n^{\{s\}}\}_{n,s<\omega}$ and $\{C_n^{\{s\}}\}_{s,n<\omega}$ are Δ^0_2 approximations to sequences $\mathcal{B}$ and $\mathcal{C}$. Suppose that for every n, s and k the following is true:*

$$(1) : x \in B_n^{\{s\}} \setminus B_n^{\{s+1\}} \cap \omega^{[k]} \Rightarrow \omega^{[\geq k]} \upharpoonright s \subseteq C_n$$

and symmetrically:

$$(2) : x \in C_n^{\{s\}} \setminus C_n^{\{s+1\}} \cap \omega^{[k]} \Rightarrow \omega^{[\geq k]} \upharpoonright s \subseteq B_n.$$

Then $\mathcal{C}$ and $\mathcal{B}$ satisfy the $\mathcal{K}$-sequence property.

Proof. Consider the sequence $\mathcal{R}$, defined by:

$$R_n = \bigcup_s B_n^{\{s\}} \times C_n^{\{s\}}.$$

Then from the definition of a Δ^0_2 approximation to a sequence it follows that $\mathcal{R} \leq_\omega \emptyset_\omega$.

Suppose $\langle b, c\rangle \in B_n \times C_n$. Then there is a stage s_b such that $\forall t > s_b\ b \in B_n^{\{t\}}$ and there is a stage s_c such that $\forall t > s_c\ c \in C_n^{\{t\}}$. Then $\langle b, c\rangle \in B_n^{\{s\}} \times C_n^{\{s\}}$, where $s = \max(s_b, s_c)$, hence $B_n \times C_n \subseteq R_n$.

Now let $\langle \bar{b}, \bar{c}\rangle \in \bar{B}_n \times \bar{C}_n$. Assume towards a contradiction that $\langle \bar{b}, \bar{c}\rangle \in R_n$. Then there is a stage s such that $\bar{b} < s$, $\bar{c} < s$ and $\bar{b} \in B_n^{\{s\}}$ and $\bar{c} \in C_n^{\{s\}}$. Let $\bar{b} = \langle k, x\rangle$ and $\bar{c} = \langle m, x\rangle$. Suppose $k \leq m$ and let t be the least stage $t \geq s$ such that $\bar{b} \notin B_n^{\{t+1\}}$. Such a stage exist by the properties of a Δ^0_2 approximation and the fact that $\bar{b} \notin B_n$. Now by property (1) we have that $\bar{c} \in C_n$, contradicting our choice of $\bar{c}$. If $m < k$ we obtain a contradiction with (2) is a similar way. □

5.2. Theorem. *Let $\mathcal{A}$ be a super a.z. sequence, such that $\mathcal{A} \not\leq_\omega \emptyset_\omega$. There exist sequences $\mathcal{B}$ and $\mathcal{C}$ such that:*

1. *$\mathcal{B} \leq_\omega \mathcal{A}$ and $\mathcal{C} \leq_\omega \mathcal{A}$.*
2. *$\mathcal{B} \not\leq_\omega \emptyset_\omega$ and $\mathcal{C} \not\leq_\omega \emptyset_\omega$.*

(3) *$\mathcal{B}$ and $\mathcal{C}$ form a $\mathcal{K}$-pair.*

Proof. Let $\mathcal{A}$ be a super a.z. sequence. Let $\mathcal{I} \leq_\omega \emptyset'_\omega$ be an index sequence for $\mathcal{A}$.

Let $\{I_n^{\{s\}}\}_{s,n<\omega}$ be a Σ^0_2 approximation to $\mathcal{I}$. We will use the notion of the age of an element x at level n at stage s in the approximation $\{I_n^{\{s\}}\}_{s,n<\omega}$, defined as:

$$age(x,n,s) = \begin{cases} \mu t \leq s[\forall r \in [t,s](x \in I_n^{\{r\}})], & \text{if } x \in I_n^{\{s\}} \\ s+1, & \text{if } x \notin I_n^{\{s\}} \end{cases}$$

We approximate $\mathcal{A}$ by the sequence of sequences $\{A_n^{\{s\}}\}$, defined as follows. Fix n and s. If $I_n^{\{s\}} = \emptyset$ then $A_n^{\{s\}} = \emptyset$. Otherwise let a be the least oldest element in $I_n^{\{s\}}$, i.e. $a \in I_n^{\{s\}}$ and for all $b \in I_n^{\{s\}}$, $age(a,n,s) < age(b,n,s)$ or $age(a,n,s) = age(b,n,s)$ and $a < b$. Set $A_n^{\{s\}} = W_a(\emptyset^n)^{\{s\}}$.

Note that for every n the approximation $\{A_n^{\{s\}}\}_{s<\omega}$ is $\Delta^0_2(\emptyset^n)$. This follows from the fact that that for every n, I_n is not empty and $\{I_n^{\{s\}}\}_{s<\omega}$ is a $\Sigma^0_2(\emptyset^n)$ approximation. Hence after finitely many wrong guesses there will be a stage s, such that $\forall t > s$ the least oldest element in $I_n^{\{t\}}$ remains the same and hence $\{A_n^{\{t\}}\}_{s<t<\omega}$ is in fact a $\Sigma^0_1(\emptyset^n)$ approximation to A_n.

We will construct Σ^0_1 approximations $\{U_n^{\{s\}}\}_{s,n<\omega}$ and $\{V_n^{\{s\}}\}_{s,n<\omega}$ to two enumeration operator U and V, so that $\mathcal{B} = U(\mathcal{A})$ and $\mathcal{C} = V(\mathcal{A})$ satisfy the statement of the theorem. For every n, s we introduce the following notation: $B_n^{\{s\}} = U_n^{\{s\}}(A_n^{\{s\}})$ and $C_n^{\{s\}} = V_n^{\{s\}}(A_n^{\{s\}})$. Note that these are as well Δ^0_2 approximations, in fact for every n if we choose s as in the paragraph above, as the stage from which on we have correctly guessed the least oldest element in the approximation to I_n, then both $\{B_n^{\{t\}}\}_{s<t<\omega}$ and $\{C_n^{\{t\}}\}_{s<t<\omega}$ are $\Sigma^0_1(\emptyset^n)$ approximations.

Fix an effective enumeration of all enumeration operators $\{W_e\}_{e<\omega}$. The construction will ensure that the following requirements are met:

(1) To ensure that $\mathcal{B}$ and $\mathcal{C}$ have the $\mathcal{K}$-sequence property by Proposition 5.1 it is enough to as that for every n:

$$\mathcal{K}(\mathcal{B},n) : x \in B_n^{\{s\}} \setminus B_n^{\{s+1\}} \cap \omega^{[k]} \Rightarrow \omega^{[\geq k]} \upharpoonright s \subseteq C_n$$

$$\mathcal{K}(\mathcal{C},n) : x \in C_n^{\{s\}} \setminus C_n^{\{s+1\}} \cap \omega^{[k]} \Rightarrow \omega^{[\geq k]} \upharpoonright s \subseteq B_n.$$

(2) To ensure that $\mathcal{B}$ and $\mathcal{C}$ are of nonzero degree, we ask that for every e the following two requirements are satisfied.

$$\mathcal{N}(\mathcal{B},e) : \exists n(B_n \neq W_e[n](\emptyset^n)).$$

$$\mathcal{N}(\mathcal{C},e) : \exists n(C_n \neq W_e[n](\emptyset^n)).$$

The construction will run in stages. The strategy to satisfy requirements of type $\mathcal{K}$ is straightforward. Whenever we see that there is an element x, such that $x \in B_n^{\{s\}} \setminus B_n^{\{s+1\}} \cap \omega^{[k]}$ for example, we enumerate permanently all elements in $y \in \omega^{[\geq k]} \upharpoonright s$ in C_n, by adding the axiom $\langle y, \emptyset\rangle$ to the operator V_n.

To ensure that the $\mathcal{N}$-type requirements are satisfied we will firstly order them linearly:

$$\mathcal{R}_0 = \mathcal{N}(\mathcal{B},0) < \mathcal{R}_1 = \mathcal{N}(\mathcal{C},0) < \dots < \mathcal{R}_{2i} = \mathcal{N}(\mathcal{B},i) < \mathcal{R}_{2i+1} = \mathcal{N}(\mathcal{C},i) < \dots$$

and say that requirements, which appear earlier in this ordering, are of higher priority. Next to every requirement, we connect for every n and s the n-th length of agreement measured at stage s:

$$l(2i,n,s) = l(B_n^{\{s\}}, W_i^{\{s\}}[n](\emptyset^n), s);$$
$$l(2i+1,n,s) = l(C_n^{\{s\}}, W_i^{\{s\}}[n](\emptyset^n), s).$$

A stage will be called strongly n-expansionary for $\mathcal{R}_{2i}$ if it is strongly n-expansionary for the approximations $\{B_n^{\{s\}}\}_{s,n<\omega}$ and $\{W_i^{\{s\}}[n](\emptyset^n)\}_{s,n<\omega}$. A stage will be called strongly n-expansionary for $\mathcal{R}_{2i+1}$ if it is strongly n-expansionary for the approximations $\{C_n^{\{s\}}\}_{s,n<\omega}$ and $\{W_i^{\{s\}}[n](\emptyset^n)\}_{s,n<\omega}$.

Construction:

Set $U_n^{\{s\}} = V_n^{\{s\}} = \emptyset$ for every n.

At stage $s+1$ we construct $U_n^{\{s+1\}}$ and $V_n^{\{s+1\}}$ for every $n < s$, by adding axioms to the sets $U_n^{\{s\}}$ and $V_n^{\{s\}}$, constructed at the previous stage s.

Step 1: The $\mathcal{K}$-requirements:

For every $n < s$ we do the following: Let $D_n^{\{s+1\}} = B_n^{\{s\}} \setminus U_n^{\{s\}}(A_n^{\{s+1\}}) \cup C_n^{\{s\}} \setminus V_n^{\{s\}}(A_n^{\{s+1\}})$. If $D_n = \emptyset$ then move on to Step 2 of the construction. Otherwise let k be the least element such that $D_n^{\{s+1\}} \cap \omega^{[k]} \neq \emptyset$ and x be the least element in $D_n^{\{s+1\}} \cap \omega^{[k]}$.

If $x \in B_n^{\{s\}}$ then enumerate the axiom $\langle y, \emptyset\rangle$ in $V_n^{\{s+1\}}$ for every $y \in \omega^{[\geq k]} \restriction s$.

If $x \in C_n^{\{s\}}$ then enumerate the axiom $\langle y, \emptyset\rangle$ in $U_n^{\{s+1\}}$ for every $y \in \omega^{[\geq k]} \restriction s$.

Step 2: The $\mathcal{N}$-requirements:

For every $n < s$ we do the following let $\mathcal{R}_e$ be the highest priority requirement such that the stage s is strongly n-expansionary for $\mathcal{R}_e$.

If $e = 2i$ then for all $y < l(e,n,s)$, such that $y \in A_n^{\{s\}}$ enumerate the axiom $\langle\langle y, e\rangle, \{y\}\rangle$ in $U_n^{\{s+1\}}$.

If $e = 2i+1$ then for all $y < l(e,n,s)$, such that $y \in A_n^{\{s\}}$ enumerate the axiom $\langle\langle y, e\rangle, \{y\}\rangle$ in $V_n^{\{s+1\}}$.

End of Construction:

Now we will verify that the construction produces the required enumeration operators.

5.3. Proposition. *For every n and every i at every stage s, $\mathcal{D}_n^{\{s+1\}} \cap \omega^{[2i]} \subseteq B_n^{\{s\}} \setminus C_n^{\{s\}}$ and $\mathcal{D}_n^{\{s+1\}} \cap \omega^{[2i+1]} \subseteq C_n^{\{s\}} \setminus B_n^{\{s\}}$*

Proof. Suppose that $\mathcal{D}_n^{\{s+1\}} \cap \omega^{[2i]} \neq \emptyset$ and let $x \in \mathcal{D}_n^{\{s+1\}} \cap \omega^{[2i]}$. Then $x \in B_n^{\{s\}} \cup C_n^{\{s\}}$ so there is an axiom $\langle x, F\rangle$ in $U_n^{\{s\}}$ or $V_n^{\{s\}}$. Furthermore this axiom is not valid at stage $s+1$, i.e. $F \not\subseteq A^{\{s+1\}}$, hence $F \neq \emptyset$. It follows that this axiom is enumerated by the requirement $\mathcal{R}_{2i}$, which enumerates axioms only in the operator U_n. Hence $x \in B_n^{\{s\}} \setminus C_n^{\{s\}}$.

That $\mathcal{D}_n^{\{s+1\}} \cap \omega^{[2i+1]} \subseteq C_n^{\{s\}}$ is proved similarly. □

5.4. Lemma. *$\mathcal{B} = U(\mathcal{A})$ and $\mathcal{C} = V(\mathcal{A})$ form a $\mathcal{K}$-pairs.*

Proof. Fix n and consider the approximations $\{B_n^{\{s\}}\}_{s<\omega}$ and $\{C_n^{\{s\}}\}_{s<\omega}$. These are Δ_2^0 approximations to the the sequences $\mathcal{B} = U(\mathcal{A})$ and $\mathcal{C} = V(\mathcal{A})$. Since $\mathcal{A}$

is super a.z., it follows that $\mathcal{B}$ and $\mathcal{C}$ are super a.z. and hence by Proposition 3.3 it follows that if $\mathcal{B}$ and $\mathcal{C}$ have the $\mathcal{K}$-sequence property then they form a $\mathcal{K}$-pair. By Proposition 5.1 it is enough to ensure that their approximations satisfy the following two statements:

$$(1) : x \in B_n^{\{s\}} \setminus B_n^{\{s+1\}} \cap \omega^{[k]} \Rightarrow \omega^{[\geq k]} \restriction s \subseteq C_n$$

$$(2) : x \in C_n^{\{s\}} \setminus C_n^{\{s+1\}} \cap \omega^{[k]} \Rightarrow \omega^{[\geq k]} \restriction s \subseteq B_n.$$

Suppose that, say, (1) is not true, i.e. suppose that there is an element $x \in B_n^{\{s\}} \setminus B_n^{\{s+1\}} \cap \omega^{[k]}$ but $\omega^{[\geq k]} \restriction s \nsubseteq C_n$. Let us consider the construction at stage $s+1$. Since $U_n^{\{s\}} \subseteq U_n^{\{s+1\}}$ it follows that $U_n^{\{s\}}(A_n^{\{s+1\}}) \subseteq B_n^{\{s+1\}}$, hence $x \in B_n^{\{s\}} \setminus U_n^{\{s\}}(A_n^{\{s+1\}}) \subseteq D_n^{\{s+1\}}$. Then $D_n^{\{s+1\}}$ is not empty and by Proposition 5.3 k is even. We have two cases:

Case 1: $2l \leq k$ is the least m such that $D_n^{\{s+1\}} \cap \omega^{[m]} \neq \emptyset$. But then by construction the axiom $\langle y, \emptyset\rangle$ is enumerated in $V_n^{\{s+1\}}$ for every $y \in \omega^{[\geq 2l]} \restriction s \subseteq \omega^{[\geq k]} \restriction s$, hence $\omega^{[\geq k]} \restriction s \subseteq C_n$.

Case 2: $2l+1 < k$ is the least m such that $D_n^{\{s+1\}} \cap \omega^{[m]} \neq \emptyset$. But then by construction the axiom $\langle y, \emptyset\rangle$ is enumerated in $U_n^{\{s+1\}}$ for every $y \in \omega^{[\geq m]} \restriction s$. In particular the axiom $\langle x, \emptyset\rangle$ is enumerated in $U_n^{\{s+1\}}$, so $x \in B_n^{\{s+1\}}$, contradicting our choice of x.

That (2) is true is proved similarly. □

5.5. Proposition. *$\mathcal{B}$ and $\mathcal{C}$ form a nontrivial $\mathcal{K}$-pair, i.e. $d_\omega(\mathcal{B}) \neq \emptyset_\omega$ and $d_\omega(\mathcal{C}) \neq \emptyset_\omega$.*

Proof. We prove by induction on e the following two statements:

(1) There is a natural number n such that $\mathcal{R}_e$ does not add any elements to U_m or V_m for every $m \geq n$.
(2) $\mathcal{R}_e$ is satisfied.

Suppose that the statement is true for all $j < e$.

Suppose that $e = 2i$, i.e $\mathcal{R}_e = \mathcal{N}(\mathcal{B}, i)$. If there is an n such that there are finitely many strongly n-expansionary stages for $\mathcal{R}_e$ then the two statements are true for $\mathcal{R}_e$. Indeed, as we are working with Δ^0_2 approximations to to B_n and $W_i[n](\emptyset^n)$ we can apply Proposition 1.11. By Part (1) of Proposition 1.11 $\mathcal{B} \neq W_i(\emptyset_\omega)$, hence there is an n such that $B_n \neq W_i[n](\emptyset^n)$ and $\mathcal{R}_e$ is satisfied. By Part (2) it follows that there is an index n_0 such that for all $m > n_0$ there are no strongly m-expansionary stages for $\mathcal{R}_e$. Then according to Step 2 of the construction $\mathcal{R}_e$ never enumerates any axioms in either U_m or V_m for any $m > n_0$.

So assume towards a contradiction that for every n there are infinitely many strongly n-expansionary stages for $\mathcal{R}_e$. Then by Proposition 1.11 it follows that $\mathcal{B} = W_i(\emptyset_\omega)$. We prove that in this case $\mathcal{A} \equiv_e \mathcal{B} = W_i(\emptyset^n_\omega)$, contradicting the assumption that $\mathcal{A} \nleq_\omega \emptyset_\omega$.

Let n_0 be a number such that all $\mathcal{R}_j$, $j < e$, do not enumerate axioms in any U_m or V_m, where $m \geq n_0$. As $\mathcal{R}_e$ is the only $\mathcal{R}$-requirement which enumerates axioms for elements of the form $\langle e, y\rangle$ in either operator and $\mathcal{R}_e$ enumerates such axioms only in U, it follows that for every $m \geq n_0$, there are no axioms in V_m for the elements $x \in \omega^{[\leq e]}$. Hence for all $m \geq n_0$ the $\mathcal{K}$-requirements do not enumerate axioms for elements $x \in \omega^{[e]}$ in U_m.

Fix $m \geq n_0$. If $\langle y, e\rangle \in U_m(A_m)$ then there is a strongly m-expansionary stage s for $\mathcal{R}_e$ at which $y \in A_m^{\{s\}}$ and the axiom $\langle\langle y, e\rangle, \{y\}\rangle$ is enumerated in $U_m^{\{s\}}$ and this axiom is valid, hence $y \in A_m$. On the other hand if $y \in A_m$ then let s_y be a stage such that for all $t \geq s_y$, we have $y \in A_m^{\{t\}}$. Such a stage exists by the properties of the Δ^0_2 approximation to $\mathcal{A}$. Let s'_y be the least strongly m-expansionary stage, such that $s'_y > \max(s_y)$ and $l(e, m, s'_y) > y$. Note that by definition $s'_y > m$. Then at stage s'_y Step 2 of the construction will enumerate $\langle\langle y, e\rangle, \{y\}\rangle$, hence $\langle y, e\rangle \in B_m = U_m(A_m)$.

Fix $a_0, \dots a_{n_0}$ as indices such that $A_k = W_{a_k}(\emptyset^k)$ for all $k < n_0$. Then define Δ by $\Delta[k] = \left\{\langle x, \emptyset\rangle \mid x \in W_{a_k}(\emptyset^k)\right\}$ for $k < n_0$ and $\Delta[k] = \{\langle y, \{\langle e, y\rangle\}\rangle \mid y \in \omega\}$. It follows that for every n, $A_n = \Delta[n](B_n)$, proving as claimed that $\mathcal{A} \equiv_e \mathcal{B}$. □ □

The given proof of Theorem 5.2 can be easily extended in a couple of ways. First of all we can modify the proof to show that every nontrivial super a.z. sequence $\mathcal{A}$ bounds an independent $\mathcal{K}$-system $\{\mathcal{A}_i\}_{i<\omega}$ of super a.z. degrees. Secondly we can strengthen the $\mathcal{R}$-requirements to ensure that for every i, $\mathcal{A}_i \not\leq_\omega \mathcal{B}$, where $\mathcal{B}$ is a super a.z. sequence such that $\mathcal{A} \not\leq_\omega \mathcal{B}$. As a result we obtain an even stronger embedding result:

5.6. Corollary. *Every countable distributive lattice is embeddable in every nonempty interval of super a.z. degrees.*

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