A STRUCTURAL DICHOTOMY IN THE ENUMERATION DEGREES

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Abstract. We give several new characterizations of the continuous enumeration degrees. The main one proves that an enumeration degree is continuous if and only if it is not half of a nontrivial relativized $\mathcal{K}$-pair. This leads to a structural dichotomy in the enumeration degrees.

1. Introduction

The Turing degrees, $\mathcal{D}_T$, measure the computability-theoretic complexity of sets of natural numbers. By coding, they can be used to measure the complexity of other mathematical objects. For example, a real number r in the unit interval can be coded by a function $n_r : \mathbb{Q}^+ \to \mathbb{Q}$ that takes as input a positive ε and outputs a rational number q within ε of r. We call n_r a *name* for r. A real number is thus associated with a set of names, which are discrete objects and hence have Turing degree. It is not difficult to see that every real number has a name of least Turing degree: the degree of its binary expansion. In this way, we can associate a Turing degree to every real number r. In many cases, however, the Turing degrees are not sufficient to measure the compexity of objects studied in effective mathematics. An early example of this phenomenon was given by Richter [19], who proved that we cannot associate a Turing degree to every countable linear ordering. In fact, the only countable linear orderings that have a Turing degree are the ones with computable presentations. In search for an answer to a similar question—"Does every continuous function on the unit interval have a name of least Turing degree?"—Miller [15] introduced the *continuous degrees* to measure of the complexity of continuous functions, and, more generally, points in computable metric spaces. He proved that the Turing degrees properly embed into the continuous degrees, and that the continuous degrees, in turn, properly embed into the *enumeration degrees*.

Enumeration reducibility, $\leqslant_e$, and the enumeration degrees, $\mathcal{D}_e$, were introduced by Friedberg and Rogers [5]. They form a natural extension of the Turing degrees: by mapping the Turing degree of a set A to the enumeration degree of $A \oplus \overline{A}$ we get an embedding ι of $\mathcal{D}_T$ into $\mathcal{D}_e$. The image of a Turing degree is called a *total* enumeration degree. So, the enumeration degrees turn out to be sufficient to capture the effective content of a continuous function on the unit interval: there is

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a least enumeration degree such that the total degrees bounding it are exactly (the images of) the Turing degrees of names of the continuous function. The enumeration degrees of continuous functions give us a proper subclass of the enumeration degrees: the *continuous enumeration degrees*.

The study of the continuous enumeration degrees has revealed an important connection between degree theory and topology. All proofs that *nontotal* continuous enumeration degrees exists—in other words, that the continuous degrees are a *proper* extension of the Turing degrees—have used nontrivial topological theorems. Miller's original proof [15] uses a variant of Brouwer's fixed point theorem for multivalued functions on the Hilbert cube. Levin's construction of a neutral measure [13] uses Sperner's Lemma and was shown by Day and Miller [4] to also produce a nontotal continuous degree. More recently, Kihara and Pauly [12] and independently Hoyrup (unpublished) use facts from topological dimension theory to prove the existence of a nontotal continuous enumeration degree. The connection can be followed in the reverse direction as well. For example, a structural property of the continuous enumeration degrees was the main tool in Kihara and Pauly's [12] solution to the Second level Borel isomorphism problem; they constructed an uncountable Polish space which is neither second-level Borel isomorphic to the unit interval nor to the Hilbert cube.

Andrews, Igusa, Miller, and Soskova [2] have recently given several characterizations of the continuous enumeration degrees as a subclass of the enumeration degrees, including a chracterization via a simple structural property: an enumeration degree $\mathbf{a}$ is *almost total* if and only if for every total enumeration degree $\mathbf{x} \nleqslant \mathbf{a}$ we have that $\mathbf{x} \vee \mathbf{a}$ is a total degree. An enumeration degree is continuous if and only if it is almost total. The total enumeration degrees were shown to be definable by Cai, Ganchev, Lempp, Miller, and Soskova [3], and so the continuous enumeration degrees also form a definable subclass of the enumeration degrees.

Another class of enumeration degrees that has been studied extensively is the class of *Kalimullin pairs* or $\mathcal{K}$-pairs[1], introduced by Kalimullin [10]. A pair of sets $\{A, B\}$ form a $\mathcal{K}$-pair relative to U if and only if there is some set $W \leqslant_e U$ such that $A \times B \subseteq W$ and $\overline{A} \times \overline{B} \subseteq \overline{W}$. $\mathcal{K}$-pairs lie at the heart of most natural definability results in the enumeration degrees. Kalimulin [10] proved that they have a natural structural definition—they are the degrees $\{\mathbf{a}, \mathbf{b}\}$ that form a *robust minimal pair* relative to a degree $\mathbf{u}$: for all $\mathbf{x} \geqslant \mathbf{u}$ we have that $\mathbf{x} = (\mathbf{a} \vee \mathbf{x}) \wedge (\mathbf{b} \vee \mathbf{x})$. He then used them to define the enumeration jump operator. Ganchev and Soskova [6] proved that $\mathcal{K}$-pairs are definable in $\mathcal{D}(\leqslant \mathbf{0}_e')$, the substructure of the Σ^0_2 enumeration degrees, and used them to prove the first order definability of a series of subclasses of $\mathcal{D}(\leqslant \mathbf{0}_e')$, including the total Δ^0_2 enumeration degrees [7], the downwards properly Σ^0_2 enumeration degrees, the upwards properly Σ^0_2 enumeration degrees [6], and the Low_n and High_n enumeration degrees, for all $n \in \omega$ [8]. A special type of $\mathcal{K}$-pairs—the maximal $\mathcal{K}$-pairs—were used by Cai, Ganchev, Lempp, Miller, and Soskova [3] to define the total enumeration degrees.

In this paper we study the relationship between the continuous degrees and $\mathcal{K}$-pairs more closely. We give several new characterizations of continuous degrees, leading up to our main new characterization: an enumeration degree is continuous if and only if it is not half a nontrivial $\mathcal{K}$-pair relative to any enumeration degree.

[1]$\mathcal{K}$-pairs were called *e-ideal pairs* in Kalimullin's paper. Ganchev and Soskova established the new name.

This new characterization gives a simpler first order definition of the continuous enumeration degrees in terms of quantifier complexity. In the language with $\leqslant$ and $\vee$, the definition via almost total degrees is Π_3, while the definition via $\mathcal{K}$-pairs is Π_2. It also allows for an interesting structural dichotomy in the enumeration degrees. Recall that an enumeration degree $\mathbf{a}$ is a *strong quasiminimal cover* of $\mathbf{b}$ if and only if $\mathbf{a} > \mathbf{b}$ and for all total enumeration degrees $\mathbf{x}$, if $\mathbf{x} \leqslant \mathbf{a}$ then $\mathbf{x} \leqslant \mathbf{b}$. The characterization of the continuous degrees as almost total, along with properties of nontrivial $\mathcal{K}$-pairs allow us to derive the following:

Theorem 1.1. *For every enumeration degree* $\mathbf{a}$*, exactly one of the following two properties holds:*

(1) *The degree* $\mathbf{x}$ *is continuous, so for every total enumeration degree* $\mathbf{x} \nleqslant \mathbf{a}$*,* $\mathbf{a} \vee \mathbf{x}$ *is total.*
(2) *There is a total enumeration degree* $\mathbf{x} \nleqslant \mathbf{a}$ *such that* $\mathbf{a} \vee \mathbf{x}$ *is a strong quasiminimal cover of* $\mathbf{x}$*.*

A subclass of the enumeration degrees that is larger than the continuous enumeration degree is the *cototal degrees.* A degree is cototal if it contains a *cototal set*, i.e., a set A, such that $A \leqslant_e \overline{A}$. This class arises naturally in many areas of effective mathematics, including graph theory [1], symbolic dynamics [14] and computable structure theory [14]. The cototal enumeration degrees also reveal a topological connection: Kihara et al. [11] showed that the cototal enumeration degrees are the degrees of points in computably G_δ topological spaces. Miller and Soskova [17] prove that they form a dense substructure of the enumeration degrees, viewed as an upper-semilaltice with jump operation. We study the connection between cototal sets and $\mathcal{K}$-pairs. Our investigation leads us to a conjecture that, if true, would yield the first order definability of the cototal enumeration degrees within the structure of the enumeration degrees.

We end with an alternative characterization of the continuous degrees in terms of the relation "PA above" extended to enumeration oracles, introduced by Miller and Soskova [16]. This new characterization opens up many questions for future work on this topic.

2. Preliminaries

We start by giving formal definitions of standard notions used throughout this paper. For a more thorough expositions on degree theory, we refer the reader to Odifreddi [18].

2.1. Enumeration reducibility and the enumeration degrees.

Definition 2.1 (Friedberg, Rogers [5])**.** $A \leqslant_e B$ if and only if there is a c.e. set W such that

$$A = W(A) = \{x \colon (\exists v)\, \langle x, v\rangle \in W \;\&\; D_v \subseteq B\}.$$

Here D_v denotes the finite set with code v in the standard coding of finite sets. The set W is called an *enumeration operator* and the pair $\langle x, v\rangle$ is called an *axiom* for x in W.

Enumeration reducibility gives rise to a degree structure as usual: A is *enumeration equivalent*, $\equiv_e$, to B if $A \leqslant_e B$ and $B \leqslant_e A$; the *enumeration degree* of a set A is $d_e(A) = \{B \colon A \equiv_e B\}$. The preorder $\leqslant_e$ on sets induces a partial order on the enumeration degrees, $\mathcal{D}_e$. The least enumeration degree $\mathbf{0}_e$ consists of all c.e. sets. We can supply $\mathcal{D}_e$ with a least upper bound operation by setting $d_e(A) \vee d_e(B) = d_e(A \oplus B)$, where $A \oplus B = \{2n : n \in A\} \cup \{2n+1 : n \in B\}$. We can also define a jump operator: $d_e(A)' = d_e(K_A \oplus \overline{K_A})$, where $K_A = \bigoplus_{e \in \omega} W_e(A)$ and $\{W_e\}_{e \in \omega}$ is some fixed computable enumeration of all c.e. sets, or equivalently enumeration operators.

As mentioned above, the Turing degrees can be embedded into the enumeration degrees. The reason now should be clear: it follows easily from the definition of enumeration reducibility that $A \leqslant_T B$ if and only if $A \oplus \overline{A} \leqslant_e B \oplus \overline{B}$.

Definition 2.2. A set A is *total* if $\overline{A} \leqslant_e A$ (i.e., if $A \oplus \overline{A} \equiv_e A$). An enumeration degree is *total* if it contains a total set.

The set of total enumeration degrees as an upper semi-lattice with jump operation is an isomorphic copy of the Turing degrees.

Selman [20] provided a useful alternative way to think about enumeration reducibility. An *enumeration* of a set A is a total function $f : \omega \to \omega$ with range equal to A. The definition of enumeration reducibility given above can be restated as follows: A is *enumeration reducible* to B if there is a uniform way to compute an enumeration of A from an enumeration of B. Selman proved that the uniformity condition is not necessary.

Theorem 2.3 (Selman [20])**.** *$A \leqslant_e B$ if and only if every enumeration of B computes an enumeration of A*

2.2. **The continuous degrees.** A computable presentation of a metric space $\mathcal{M}$ consists of a fixed dense sequence $Q^{\mathcal{M}} = \{q_n\}_{n \in \omega}$ on which the metric is computable as a function on indices. Metric spaces with computable presentations include Cantor space 2^ω, Baire space ω^ω, the continuous functions on the unit interval $\mathcal{C}[0,1]$, the Hilbert cube $[0,1]^\omega$, and many others. For a computable presentation of $\mathcal{C}[0,1]$, for example, we fix an effective enumeration of the polygonal functions having segments with rational endpoints. A *name* for a point x in a computable metric space is a function $n_x : \mathbb{Q}_{>0} \to \omega$ that gives a way to approximate x via the sequence $Q^{\mathcal{M}}$: it takes a rational number $\varepsilon > 0$ as input and produces an index $n_x(\varepsilon)$ such that $d_{\mathcal{M}}(x, q_{n_x(\varepsilon)}) < \varepsilon$. Such names can easily be coded as elements of Baire space. For points x, y in (possibly different computably presented metric spaces), we say that x reduces to y if every name for y uniformly computes a name for x. This reducibility induces a degree structure, the *continuous degrees.* Miller [15] proves that there are universal computably presented metric spaces: every continuous degree contains an element of $\mathcal{C}[0,1]$ and, more importantly for our purposes, also an element of $[0,1]^\omega$.

In order to understand the embedding of the continuous degrees into the enumeration degrees, we use the fact that the Hilbert cube $[0,1]^\omega$ is universal. We take the usual metric on the Hilbert cube: $d(\alpha, \beta) = \sum_{n \in \omega} 2^{-n}|\alpha(n) - \beta(n)|$. A dense set witnessing that $[0,1]^\omega$ is computable is, for example, a computable enumeration of the rational sequences with finite support. Given $\alpha \in [0,1]^\omega$, consider the set

$$C_\alpha = \bigoplus_{n \in \omega} \{q \in \mathbb{Q} : q < \alpha(n)\} \oplus \{q \in \mathbb{Q} : q > \alpha(n)\}.$$

Miller [15] proved that enumerating C_α is just as difficult as computing a name for α. Thus the map $\alpha \mapsto C_\alpha$ induces an embedding of the continuous degrees into the enumeration degrees.

Definition 2.4. An enumeration degree $\mathbf{a}$ is *continuous* if it contains a set of the form C_α, where $\alpha \in [0,1]^\omega$.

Note that the i-th component of C_α is very close to a total set in structure: $\{q \in \mathbb{Q}\colon q <_{\mathbb{Q}} \alpha(i)\} \oplus \{q \in \mathbb{Q}\colon q >_{\mathbb{Q}} \alpha(i)\} = X \oplus \overline{X}$, unless $\alpha(i)$ is rational. The nonuniformity introduced by the rational components of some $\alpha \in [0,1]^\omega$ suffice to produce nontotal enumeration degrees:

Theorem 2.5 (Miller [15])**.** *There are nontotal continuous degrees.*

We will use two of the three characterizations of continuous enumeration degrees proved in [2]. We restate the definition of an almost total degree for convenience:

Definition 2.6. We say that an enumeration degree $\mathbf{a}$ is *almost total* if whenever $\mathbf{b} \nleqslant \mathbf{a}$ is total, $\mathbf{a} \vee \mathbf{b}$ is also total.

The second characterization relies on an extension of the notion of a Π^0_1 class to an enumeration oracle. We will use $\langle A\rangle$ to indicate that we are treating A as an enumeration oracle rather than a Turing oracle.

Definition 2.7. Let $A \subseteq \omega$. Call $U \subseteq 2^\omega$ a $\Sigma^0_1\langle A\rangle$ *class* if there is a set $W \leqslant_e A$, such that $U = [W]^\prec = \{X \in 2^\omega\colon (\exists \sigma \in W)\ X \succeq \sigma\}$. A $\Pi^0_1\langle A\rangle$ *class* is the complement of a $\Sigma^0_1\langle A\rangle$ class.

Note that a $\Pi^0_1\langle A \oplus \overline{A}\rangle$ class is just a $\Pi^0_1[A]$ class in the usual sense. Further, note that the elements of a $\Pi^0_1\langle A\rangle$ class are infinite binary sequences, hence total objects.

Definition 2.8. A set $A \subseteq \omega$ is *codable* if there is a nonempty $\Pi^0_1\langle A\rangle$ class $P \subseteq 2^\omega$ such that for every $X \in P$, A is c.e. relative to X.

Theorem 2.9 (Andrews, Igusa, Miller, Soskova [2])**.** *Fix $A \subseteq \omega$. The following are equivalent*

1. *The enumeration degree of A is almost total;*
2. *A is codable;*
3. *A has continuous enumeration degree.*

2.3. **Kalimullin pairs.** Consider once again the definition of a relativized Kalimullin pair.

Definition 2.10 (Kalimullin [10])**.** A pair of sets of natural numbers $\{A, B\}$ is a *Kalimullin pair ($\mathcal{K}$-pair) relative to a set* U if there is a set $W \leqslant_e U$ such that $A \times B \subseteq W$ and $\overline{A} \times \overline{B} \subseteq \overline{W}$.

It is very easy to come up with an example of a $\mathcal{K}$-pair relative to any U: if $B \leqslant_e U$, then for every set A we have that $\{A, B\}$ is a $\mathcal{K}$-pair relative to U as witnessed by $\omega \times B$. Similarly, if $A \leqslant_e U$. $\mathcal{K}$-pairs of this sort are not interesting; we call the *trivial.* Nontrivial $\mathcal{K}$-pairs exist: a standard example of a $\mathcal{K}$-pair relative to $\varnothing$ is given by a the pair $\{A, \overline{A}\}$, where A is any *semi-computable set.* Semi-computable sets were introduced and studied by Jockusch [9]. He showed that A is semi-computable if and only if A is a left cut in some computable linear ordering on ω. He also showed that every nonzero Turing degree contains a semi-computable set that is neither c.e. nor co-c.e.

$\mathcal{K}$-pairs have many interesting properties. For example, if we fix A and consider the set $\mathcal{K}(A) = \{B\colon \{A, B\}$ is a $\mathcal{K}$-pair$\}$, then $\mathcal{K}(A)$ is an ideal with respect to

$\leqslant_e$. In particular, being half of a $\mathcal{K}$-pair is a degree notion. We summarize the properties that we will use in the following theorem.

Theorem 2.11 (Kalimullin [10]). *Let $\{A, B\}$ be a nontrivial $\mathcal{K}$-pair relative to U as witnessed by W.*

(1) $A \leqslant_e \overline{B} \oplus W$ *and* $B \leqslant_e \overline{A} \oplus W$.
(2) $\overline{A} \leqslant_e B \oplus \overline{W}$ *and* $\overline{B} \leqslant_e A \oplus \overline{W}$.
(3) $d_e(A \oplus U)$ *and* $d_e(B \oplus U)$ *are strong quasiminimal covers of* $d_e(U)$

We will also need the structural definition of $\mathcal{K}$-pairs as *robust minimal pairs.*

Theorem 2.12 (Kalimullin [10]). *A pair of sets $\{A, B\}$ are a $\mathcal{K}$-pair relative to U if and only if their enumeration degrees* $\mathbf{a}$, $\mathbf{b}$ *and* $\mathbf{u}$ *satisfy:*

$$(\forall \mathbf{x} \geqslant \mathbf{u})[(\mathbf{a} \vee \mathbf{x}) \wedge (\mathbf{b} \vee \mathbf{x}) = \mathbf{x}].$$

3. Functions that are codable by extensions

Our first characterization of the continuous enumeration degrees, and the one that motivated this work, concerns the notion of a function codable by its extensions.

Definition 3.1. A function $f : \omega \to \omega$ is *codable by extensions* if for every extensions $h \supseteq f$ we have that $G_f \leqslant_e G_h$, where G_x denotes the graph of x.

Clearly, every total function is codable by extensions, because its only extension is the function itself. The notion becomes interesting when one considers graphs of nontotal functions. We prove below that the enumeration degrees of the graphs of function that are codable by extensions are exactly the continuous enumeration degrees.

Theorem 3.2. *An enumeration degree is continuous if and only if it contains the graph of a function that is codable by extensions.*

Proof. Suppose that $\mathbf{a}$ is continuous and fix an element of Hilbert cube $\alpha \in [0, 1]^\omega$ so that $\mathbf{a} = d_e(C_\alpha)$, where

$$C_\alpha = \bigoplus_{n \in \omega} \{q \in \mathbb{Q} \colon q < \alpha(n)\} \oplus \{q \in \mathbb{Q} \colon q > \alpha(n)\}.$$

Consider the function $f : \omega \to \omega$ such that

$$f(\langle n, q \rangle) = \begin{cases} 0, & \text{if } q < \alpha(n); \\ 1, & \text{if } \alpha(n) > q. \end{cases}$$

Note, that if $\alpha(n) = q$ then $f(\langle n, q \rangle)$ is undefined. Clearly, $C_\alpha \equiv_e G_f$. Furthermore, for any extension h of f, we have that

$$C_\alpha = \bigoplus_{n \in \omega} \{q \in \mathbb{Q} \colon (\exists r > q)\ h(\langle n, r \rangle) = 0\} \oplus \{q \in \mathbb{Q} \colon (\exists r < q)\ h(\langle n, r \rangle) = 1\}.$$

Hence $G_f \equiv_e C_\alpha \leqslant_e G_h$, and so f is codable by extensions.

For the reverse direction, let f be a function that is codable by extensions. Consider the set P of the graphs of all extensions of f. Then P is a $\Pi^0_1\langle G_f \rangle$ class, as $P = 2^\omega \setminus S$, where S is the set of finite binary strings that are not initial segments of the characteristic function of the graph of some extension of f. In other words, $\sigma \in S$ if and only if

$$(\exists x, y, z)\,[\sigma(\langle x, y\rangle) = \sigma(\langle x, z\rangle) = 1 \;\&\; y \neq z]$$
$$\text{or } (\exists x, y)\,[\langle x, y\rangle \in G_f \;\&\; \sigma(\langle x, y\rangle) = 0]\,.$$

Every member of the class P can enumerate G_f, hence G_f is codable and so G_f has continuous degree by Theorem 2.9. □

The characterizations of the continuous degrees via functions that are codable by extensions leads to a connection with the notion of a nontrivial $\mathcal{K}$-pair, or rather the opposite: a continuous degree can never be a part of a nontrivial $\mathcal{K}$-pair relative to any set U.

Proposition 3.3. *If f is codable by extensions, then G_f is not half of any relativized nontrivial $\mathcal{K}$-pair.*

Proof. Suppose towards a contradiction that G_f is codable by extensions and the pair $\{G_f, B\}$ is a nontrivial $\mathcal{K}$-pair relative to some set U. Let $W \leqslant_e U$ witness this. If for some b we find that $\langle\langle x, y\rangle, b\rangle \in W$ and $\langle\langle x, z\rangle, b\rangle \in W$, where $y \neq z$, then $\overline{G_f} \times \overline{B} \subseteq \overline{W}$ ensures that $b \in B$. Since $B \nleqslant_e W$, there must be some $b \in B$ for which the above is not true and hence $\{\langle x, y\rangle \colon \langle\langle x, y\rangle, b\rangle \in W\}$ is the graph of a function h, and $G_h \leqslant_e W$. As $G_f \times B \subseteq W$, it follows that $f \subseteq h$, so $G_f \leqslant_e G_h \leqslant_e U$, contrary to our assumption that $\{G_f, B\}$ is nontrivial relative to U. □

A natural questions arises: does this property characterize the continuous degrees? This will be proved in Section 5.

4. Array-avoiding sets

Towards a positive answer to the question posed at the end of the previous section, we explore a combinatorial characterization of the continuous degrees.

Definition 4.1. We say that A is *array-avoiding* if $A \neq \omega$ and for every computable sequence of finite sets $\{D_n\}_{n\in\omega}$ such that for every n we have that $D_n \nsubseteq A$, there is some $C \supseteq A$ such that for every n we still have $D_n \nsubseteq C$, but also $A \nleqslant_e C$.

The property above is trivially satisfied by ω, just because there cannot be a sequence of finite sets $\{D_n\}_{n\in\omega}$ such that for every n we have that $D_n \nsubseteq \omega$; for that reason we exclude it from the definition. Array-avoiding sets capture the non-continuous degrees.

Theorem 4.2. *The enumeration degree* $\mathbf{a}$ *is continuous if and only if some set in* $\mathbf{a}$ *is not array-avoiding, if and only if no set in* $\mathbf{a}$ *is array-avoiding.*

Proof. Suppose first that $\mathbf{a}$ is continuous and fix a partial function f such that $G_f \in \mathbf{a}$ and f is codable by extensions. We will build a computable sequence of finite sets $\{D_n\}_{n\in\omega}$ that ensures G_f is not array-avoiding. The sequence is quite simple: it is just a computable listing of all pairs $\{\langle x, y\rangle, \langle x, z\rangle\}$ where $y \neq z$. Any $C \supseteq G_f$ that retains the property that $D_n \nsubseteq C$ for every n is the graph of some extension of f, and hence $G_f \leqslant_e C$.

We can extend this idea to every set in the degree $\mathbf{a}$. Fix $A \in \mathbf{a}$ and let Γ be such that $G_f = \Gamma(A)$. Consider the sequence $\{D_n\}_{n\in\omega}$ that lists finite sets $F \cup E$, such that $\langle\langle x, y\rangle, F\rangle \in \Gamma$ and $\langle\langle x, z\rangle, E\rangle \in \Gamma$ for some $y \neq z$. (Note that, assuming

$A \neq \omega$, we can ensure that this is a nonempty sequence by finitely modifying Γ.) Clearly, $D_n \nsubseteq A$ for every n. On the other hand, if $C \supseteq A$ and C still has the property that $D_n \nsubseteq A$ for all n, then $\Gamma(C)$ is the graph of a function $h \supseteq g$ and hence $A \leqslant_e G_f \leqslant_e G_h \leqslant_e C$.

For the reverse direction, suppose that A is not array-avoiding. Let $\{D_n\}_{n \in \omega}$ be a computable sequence of sets that witnesses this. In other words, if $C \supseteq A$ has the property that $D_n \nsubseteq C$ for all n, then $A \leqslant_e C$. Then A is easily seen to be codable because the set of all supersets $C \supseteq A$ that satisfy $D_n \nsubseteq C$ for all n is a $\Pi^0_1\langle A\rangle$ class $P = 2^\omega \smallsetminus [S]^\prec$, where $\sigma \in S$ if and only if $\sigma(x) = 0$ for some $x \in A$ or $D_n \subseteq \{x : \sigma(x) = 1\}$ for some n. It is nonempty as $A \in P$, and by assumption, every member of P enumerates A. □

We will not give a direct proof that being array-avoiding implies being half of a nontrivial $\mathcal{K}$-pair, although it will follow from the work in the next section. For now, as a warm-up, we will prove that an apparent strengthening of array-avoiding is equivalent to being half of a nontrivial $\mathcal{K}$-pair.

Definition 4.3. We say that A is *uniformly array-avoiding* if $A \neq \omega$ and there is a Z such that $A \nleqslant_e Z$ and for every computable sequence of finite sets $\{D_n\}_{n\in\omega}$ such that $D_n \nsubseteq A$ for every n, there is a $C \supseteq A$ such that we still have $D_n \nsubseteq C$ for every n, but also $C \leqslant_e Z$.

The proof of the nontrivial direction in the theorem below outlines the main ideas that ultimately lead to the full characterization of non-continuous enumeration degrees as halves of nontrivial $\mathcal{K}$-pairs.

Theorem 4.4. *A is half of a nontrivial $\mathcal{K}$-pair if and only if A is uniformly array-avoiding.*

Proof. Suppose first that $\{A, B\}$ is a nontrivial $\mathcal{K}$-pair relative to some set U as witnessed by W. We will show that A is uniformly array-avoiding with $Z = W$. Nontriviality ensures that $A \nleqslant_e W$. Now let $\{D_n\}_{n\in\omega}$ be a computable sequence of finite sets such that $D_n \nsubseteq A$ for all n. Consider the set

$$B_0 = \{b : (\exists n)\ D_n \times \{b\} \subseteq W\}.$$

Then $B_0 \subseteq B$ and $B_0 \leqslant_e W$. But $B \nleqslant_e W$, so we can pick some element $b \in B \smallsetminus B_0$. Now, consider the set $C = \{a : \langle a, b\rangle \in W\}$. The set C extends A and satisfies the property that for every n the set $D_n \nsubseteq C$ (or else $b \in B_0$). On the other hand, $C \leqslant_e W$. Hence A is uniformly array-avoiding.

Now let A be uniformly array-avoiding as witnessed by Z. We will build sets B and W such that $A, B \nleqslant_e W$, $A \times B \subseteq W$, and $\overline{A} \times \overline{B} \subseteq \overline{W}$. The construction proceeds by stages. At stage n, we build finite sets B_n, B_n^-, and W_n satisfying the following four conditions:

(1) $B_n \subseteq B_{n+1}$, $B_n^- \subseteq B_{n+1}^-$, $W_n \subseteq W_{n+1}$;
(2) $B_n \cap B_n^- = \varnothing$;
(3) $A \upharpoonright n \times B_n \subseteq W_n$;
(4) $\langle a, b\rangle \in W_n \Rightarrow (a \in A \vee b \in B_n)$.

We let $B = \bigcup_n B_n$ and $W = \bigcup_n W_n$. Properties (1), (3), and (4) ensure that $\{A, B\}$ is a $\mathcal{K}$-pair relative to W. What remains is to ensure that $A, B \nleqslant_e W$. In order to do this, to every element $x \in B_n$ we will associate a superset $C_x \supseteq A$ such that $C_x \leqslant_e Z$. Unless otherwise stated $C_x = \omega$. The set C_x may be shrunk infinitely many times, but will always be a superset of A. We will also associate to every $y \in B_n^-$ a finite set $T_y \subseteq A$. Once defined, T_y will not be changed. (In fact, T_y will be the yth column of W. We will not use this fact here, but it will be needed in Proposition 6.3.)

We start the construction by setting $B_0 = B_0^- = W_0 = \varnothing$. Suppose we have constructed B_n, B_n^-, and W_n and consider the set

$$X_n = \bigcup_{x \in B_n} C_x \times \{x\} \cup \bigcup_{y \in B_n^-} T_y \times \{y\} \cup \bigcup_{z \notin B_n \cup B_n^-} \omega \times \{z\}.$$

By property (1), it follows that for every n we have that $X_{n+1} \subseteq X_n$. We will ensure that $W_{n+1} \subseteq X_n$, and so $W \subseteq X_n$ for every n. We have two cases, depending on whether we are at an even or an odd stage.

Suppose that $\mathbf{n = 2k}$. We ensure that $A \neq \Gamma_k(W)$, where $\{\Gamma_k\}_{k\in\omega}$ is some effective listing of all enumeration operators. Note that $X_n \leqslant_e \bigoplus_{x \in B_n} C_x \leqslant_e Z$, so $A \neq \Gamma_k(X_n)$. Fix an element a that witnesses this difference.
Case 1. If $a \in A$, then we set $W_n^* = W_n$. Note, that $W \subseteq X_n$ ensures that we have satisfied our requirement.
Case 2. If $a \in \Gamma_k(X_n)$, fix an axiom $\langle a, D\rangle \in \Gamma_k$ such that $D \subseteq X_n$. We set $W_n^* = W_n \cup D$. We will ensure that $W_n^* \subseteq W_{n+1}$, hence once again we will have satisfied our requirement.

We set $B_{n+1} = B_n \cup \{x \colon (\exists a \in \overline{A})\ \langle a, x\rangle \in W_n^* \smallsetminus W_n\}$. Note that B_{n+1} does not contain any element from B_n^-, because if $\langle a, y\rangle \in X_n$ and $y \in B_n^-$ then $a \in T_y \subseteq A$. We set $B_{n+1}^- = B_n^-$ and set $W_{n+1} = W_n^* \cup A \restriction (n+1) \times B_{n+1}$. It is straightforward to check that properties (1)–(4) still hold.

Suppose that $\mathbf{n = 2k+1}$. In this case, we would like to ensure that $B \neq \Gamma_k(W)$. If $\Gamma_k(X_n)$ is finite, then we do not need to do anything; by a close inspection of the even stages, B will be infinite. So suppose that $\Gamma_k(X_n)$ is infinite and fix $z \in \Gamma_k(X_n) \smallsetminus B_n \cup B_n^-$. We will use this z to create a difference. Pick an axiom $\langle z, D\rangle \in \Gamma_k$ such that $D \subseteq X_n$, and enumerate z into B_{n+1}^- and D into W_{n+1}. This might be in conflict with our desire to preserve properties (2) and (4), namely it could be that $\langle a, z\rangle \in D$ for some $a \in \overline{A}$. So, once again we have two cases:
Case 1. There is an axiom $\langle z, D\rangle \in \Gamma_k$ such that for all $\langle a, x\rangle \in D$:

(1) if $x \in B_n \cup \{z\}$, then $\langle a, x\rangle \in W_n$ or $a \in A$;
(2) if $x \in B_n^-$, then $a \in T_x$.

Note that these conditions ensure that $D \subseteq X_n$. In this case, we can proceed with our original plan: we set $W_n^* = W_n \cup D$ and $B_{n+1} = B_n \cup \{b \colon (\exists \langle a, b\rangle \in D)\ a \notin A\}$. We set $B_{n+1}^- = B_n^- \cup \{z\}$ and $T_z = \{a \colon \langle a, z\rangle \in W_n^*\}$. Note that $T_z \subseteq A$ by our choice of D. Finally, we set $W_{n+1} = W_n^* \cup A \restriction (n+1) \times B_{n+1}$. Once again it is easy to see that properties (1)–(4) still hold.
Case 2. Every axiom $\langle z, D\rangle \in \Gamma_k$ such that

$$(\forall x \in B_n^-) \text{ if } \langle a, x\rangle \in D, \text{ then } a \in T_x,$$

has the property that $\{a \colon (\exists x \in B_n \cup \{z\})\ \langle a, x\rangle \in D \smallsetminus W_n\} \nsubseteq A$. In that case, the sequence $\{F_m\}_{m\in\omega}$ listing all such sets—which is nonempty as $z \in \Gamma_k(X_n)$—is a computable sequence of finite sets such that for all m, we have that $F_m \nsubseteq A$. By the uniform array-avoidance of A, there is a $C \supseteq A$ such that $C \leqslant_e Z$ and C still has the property that $F_m \nsubseteq C$ for all m. We set $C_z = C$ and for every $x \in B_n$ we give the parameter C_x a new value namely $(C_x \cap C) \cup \{a \colon \langle a, x\rangle \in W_n\}$ and we set

$B_{n+1} = B_n \cup \{z\}$. This ensures that $z \notin \Gamma_k(X_{n+1})$, as every axiom for z in Γ_k that satisfies the restriction imposed by B_n^- on X_{n+1} contains an element $\langle a, x\rangle$ such that $x \in B_{n+1}$ and $\langle a, x\rangle \notin W_n \cup C$. We have thus satisfied our requirement. We set $B_{n+1}^- = B_n^-$ and $W_{n+1} = W_n \cup A\restriction(n+1) \times B_{n+1}$. □

5. Forcing with $\Pi_1^0\langle A\rangle$ classes

We use the main ideas from the proof of Theorem 4.4 to show the link between $\mathcal{K}$-pairs and the *non*continuous degrees.

Theorem 5.1. *If A does not have continuous degree, then A is half of a nontrivial relativized $\mathcal{K}$-pair.*

Proof. We will use a forcing notion $\mathcal{F}$ to construct B and W, so that $B \times A \subseteq W$, $\overline{B} \times \overline{A} \subseteq \overline{W}$, and $A, B \nleqslant_e W$. A forcing condition p has the form $\langle \beta, \{\sigma_i\}_{i\in\omega}, P\rangle$, where $\beta \in 2^{<\omega}$, $\{\sigma_i\}_{i\in\omega}$ is a sequence of finite binary strings such that for all $i \geqslant |\beta|$, $\sigma_i = \varnothing$, and P is a nonempty $\Pi_1^0\langle A\rangle$ class, satisfying a certain list of properties that we describe below. We think of P as subset of $(2^\omega)^\omega$, i.e., every element $X \in P$ codes a sequence of sets $\{X_k\}_{k\in\omega}$ (in fact, $X = \bigoplus_k X_k$). We let P_i consist of the i-th projection of the elements in P, i.e.,

$$P_i = \{X_i \colon (\exists X_0, \dots, X_{i-1}, X_{i+1}, \dots) \bigoplus_{k\in\omega} X_k \in P\}.$$

It is not difficult to see that each P_i is also a $\Pi_1^0\langle A\rangle$ class, although that will not be relevant for the construction. We think of each element $X \in P$ as providing a bound on W, in the sense that $W \subseteq X$ for some $X \in P$. We think of β as an initial segment of the set B. Every σ_i codes a finite set $D_i = \{x \colon \sigma_i(x) = 1\}$. We approximate W by $W_p = \bigcup_{i\in\omega}\{i\} \times D_i = \bigcup_{i<|\beta|}\{i\} \times D_i$. We ask that, in addition, forcing conditions satisfy the following properties:

(1) If $\beta(i) = 0$, then $D_i \subseteq A$ and $P_i = \{\sigma_i 0^\omega\}$.
(2) If $\beta(i) \neq 0$, then for every $X \in P_i$, we have that $\sigma_i \preceq X$ and $A \subseteq X$.
(3) If $X \in P$ and $Y \subseteq X$ is such that if we write Y as $\bigoplus_i Y_i$, then for every i we have that the above two conditions hold, i.e.,
  - if $\beta(i) = 0$, then $Y_i = \sigma_i 0^\omega$ and
  - if $\beta(i) \neq 0$, then $\sigma_i \preceq Y_i$ and $A \subseteq Y_i$;

  then $Y \in P$.

Note that the forcing condition ensures that $W_p \subseteq X$ for all $X \in P$.

We say that a condition $q = \langle \gamma, \{\tau_i\}_{i\in\omega}, Q\rangle$ extends a condition $p = \langle \beta, \{\sigma_i\}_{i\in\omega}, P\rangle$, written as $q \leqslant p$, if

- $\beta \preceq \gamma$;
- for all i, $\sigma_i \preceq \tau_i$;
- each $X \in Q$ is a subset of some $Y \in P$.

We build a sequence of conditions $p_0 \geqslant p_1 \geqslant p_2 \geqslant \cdots$. In the end, $B = \bigcup_{n\in\omega} \beta_{p_n}$ and $W = \bigcup_n W_{p_n}$ will be the required sets. Note that property (2) of a condition ensures that if $i \in B$, then σ_i is an initial segment of a superset of A. Hence if we ensure that σ_i grows unboundedly in length for all $i \in B$, then we automatically get that $B \times A \subseteq W$. On the other hand, property (1) ensures that $\overline{B} \times \overline{A} \subseteq \overline{W}$. We only need to further ensure that $A, B \nleqslant_e W$.

The initial condition is $p_0 = (\varnothing, (\varnothing, \varnothing, \dots), S))$, where S is the $\Pi_1^0\langle A\rangle$ class consisting of all supersets of A. Suppose that we have built $p_n = \langle \beta_n, \{\sigma_i\}_{i\in\omega}, P\rangle$.

We describe how to extend p_n to $p_{n+1} = \langle \beta_{n+1}, \{\tau_i\}_{i\in\omega}, Q\rangle$. We have two cases depending on the parity of n.

Suppose that $\mathbf{n = 2k}$. We ensure that $A \neq \Gamma_k(W)$. We first check if there is an $a \in A$ such that $Q_a = \{X \in P : a \notin \Gamma_k(X)\}$ is nonempty. If there is such an a then we let $\beta_{n+1} = \beta_n$, $\tau_i = \sigma_i$ for all i, and $Q = Q_a$. It is straightforward to see that p_{n+1} is a condition: Q is a $\Pi^0_1\langle A\rangle$ subclass of P so properties (1) and (2) are trivially satisfied. Property (3) is satisfied because if $a \notin \Gamma_k(X)$ and $Y \subseteq X$ then $a \notin \Gamma_k(Y)$. Furthermore, this condition forces $a \in A \setminus \Gamma(W)$.

If there is no $a \in A$ such that Q_a is nonempty, then A is a subset of $\Gamma_k(X)$ for every $X \in P$. Since A does not have continuous degree and hence by Theorem 2.9 is not codable, there must be some element $X \in P$ that does not enumerate A via Γ_k. So $A \subset \Gamma_k(X)$ and we can fix $a \in \Gamma_k(X) \setminus A$. Fix s such that $a \in \Gamma_k(X \upharpoonright s)$. We can think of $X \upharpoonright s$ as $\bigoplus_{i<m} \tau_i$, where $m > |\beta_n|$ and pick s large enough so that for every $i < |\beta_n|$ we have that $\sigma_i \prec \tau_i$. Let β_{n+1} be the string of length m obtained by appending 1's to β_n and let $\tau_i = \varnothing$ for $i \geqslant m$. Notice that here we are ensuring that $|\tau_i| > |\sigma_i|$. Since this case will definitely be the true case every time $\Gamma_k(X) = \omega$ for all X, we will ensure that $B \times A \subseteq W$ as discussed above. Finally we set Q to be the subclass of P, subject to the restraints in properties (1) and (2), namely $Q = \{Y \in P : X \upharpoonright s \preceq Y\}$. This is nonempty $\Pi^0_1\langle A\rangle$ class because $X \in Q$. The resulting p_{n+1} is easily seen to be a condition. Furthermore, for all $q \leqslant p_{n+1}$ we have that $a \in \Gamma_k(W_q)$, hence this ensures that $A \neq \Gamma_k(W)$, as promised.

Suppose that $\mathbf{n = 2k+1}$. We ensure that $B \neq \Gamma_k(W)$. We first check if there is a $b > |\beta_n|$ and $X \in P$ such that $b \in \Gamma_k(X)$. If not, we do not need to make any changes to p_n at this stage, so we set $p_{n+1} = p_n$. The even steps ensure that B is an infinite set, hence the requirement is automatically satisfied. So suppose that there is a $b > |\beta_n|$ such that $b \in \Gamma_k(X)$ for some $X \in P$. We would like to define β_{n+1} so that $\beta_{n+1}(b) = 0$ and Q so that every element in Q enumerates b via Γ_k. Just like in the proof of Theorem 2.9, this might not be possible because it could be that every axiom in Γ_k for b contains an element $\langle b, a\rangle$, where $a \in \overline{A}$, so we cannot build Q satisfying condition (1). This is why we consider two cases.

Case 1. There is an axiom $\langle b, D\rangle \in \Gamma_k$, such that $D \subseteq X$ for some X and for every pair $\langle i, a\rangle \in D$ we have that $\sigma_i(a) = 1$ or $a \in A$. (Note that if $\beta(i) = 0$ and $\langle i, a\rangle \in D$, then $\sigma_i(a) = 1$ because $D \subseteq X$.) In that case we can proceed with our original plan: first to ensure property (1) fix τ_b to be the initial segment of A covering all a such that $\langle b, a\rangle \in D$. Next we trim the elements of the $\Pi^0_1\langle A\rangle$ class P to get P' so that $P'_b = \{\tau_b 0^\omega\}$ and for all $i \neq b$ we have that $P'_i = P_i$. Note, that we still have $D \subseteq X'$, where X' is obtained by this trimming process from X, by our choice of τ_b. Furthermore, P' is a $\Pi^0_1\langle A\rangle$ class. To see this, write $P = 2^\omega \setminus [U]^\prec$ where $U \leqslant_e A$. We may assume that U is closed upward. Then $P' = 2^\omega \setminus [V]^\prec$ where $\rho \in V$ if when we write $\rho = \bigoplus_i \rho_i$, we have that either ρ_b is incompatible with τ_b, or if all strings ρ^* that we get by replacing ρ_b by strings of the same length are in U. It is straightforward to see that $P' \subseteq 2^\omega \setminus [V]^\prec$. On the other hand, if $X \notin P'$, but $X_b = \sigma_b 0^\omega$, then by compactness there must be some level s such that all possible strings ρ^* obtained as above from the string $\rho = X \upharpoonright s$ must be thrown into U and so $X \notin 2^\omega \setminus [V]^\prec$. The set V is clearly enumeration reducible to A.

We now fix s large enough so that $D \subseteq X' \upharpoonright s$ and $X' \upharpoonright s$ can be written as $\bigoplus_{i<m} \tau_i$, where $m > b$ and $\sigma_i \preceq \tau_i$ for all $i < |\beta_n|$ or $i = b$. We extend β_n to β_{n+1} of length m, so that $\beta_{n+1}(b) = 0$ and for all $i \neq b$ such that $|\beta_n| \leqslant i < m$, we set

$\beta_{n+1}(i) = 1$. We set $\tau_i = \varnothing$ if $i \geqslant m$. We set $Q = \{Y \in P' : X' \upharpoonright s \preceq Y\}$. Thus we have ensured that $b \notin B$ and $b \in \Gamma_k(W_q)$ for every $q \leqslant p_n$.

Case 2. For every $X \in P$, if $\langle b, D\rangle \in \Gamma_k$ and $D \subseteq X$, then there is a pair $\langle i, a\rangle \in D$ such that $\sigma_i(a) \neq 1$ (so it is either undefined or equals 0) and $a \notin A$. Consider the $\Pi^0_1\langle A\rangle$ class $Q = \{X \in P : b \notin \Gamma_k(X)\}$. This is a nonempty class because by property (3) of P the sequence $Y = \bigoplus Y_i$ where $Y_i = \sigma_i 0^\omega$ if $\beta_n(i) = 0$ and otherwise

$$Y_i(x) = \begin{cases} \sigma_i(x), & \text{if } x < |\sigma_i|; \\ A(x), & \text{if } x \geqslant |\sigma_i| \end{cases}$$

must be a member of Q. We set β_{n+1} to be the string of length $b + 1$ obtained by adding 1's to β_n and leave $\tau_i = \sigma_i$ for all i. Once again, since $Q \subseteq P$ and we have added no new 0's to β_{n+1}, it is easy to see that Q satisfies properties (1) and (2). To see that it satisfies (3), we again note that if $Y \subseteq X$ and $b \notin \Gamma_k(X)$ then $b \notin \Gamma_k(Y)$ and hence p_{n+1} is a condition. This condition forces $b \in B \smallsetminus \Gamma_k(W)$. □

Putting everything together, we get:

Theorem 5.2. *For a set $A \subseteq \omega$, the following are equivalent:*
1. *A has continuous enumeration degree.*
2. *The degree of A contains the graph of function that is codable by extensions.*
3. *A is not array-avoiding.*
4. *A is not uniformly array-avoiding.*
5. *A is not half of a nontrivial relativized $\mathcal{K}$-pair.*

Combing the characterization of the continuous degrees in terms of $\mathcal{K}$-pairs with the characterization as the *almost total* degrees from [2], we get the promised structural dichotomy in the enumeration degrees.

Theorem 1.1. *For every enumeration degree* $\mathbf{a}$*, exactly one of the following two properties holds:*
1. *The degree* $\mathbf{x}$ *is continuous, so for every total enumeration degree* $\mathbf{x} \nleqslant \mathbf{a}$*,* $\mathbf{a} \vee \mathbf{x}$ *is total.*
2. *There is a total enumeration degree* $\mathbf{x} \nleqslant \mathbf{a}$ *such that* $\mathbf{a} \vee \mathbf{x}$ *is a strong quasiminimal cover of* $\mathbf{x}$.

Proof. The first property is exactly the definition of almost totality. If $\mathbf{a}$ is not continuous, then by Theorem 5.2, we get that $\mathbf{a}$ is half of a nontrivial $\mathcal{K}$-pair; let $\mathbf{b}$ and $\mathbf{u}$ be such that $\{\mathbf{a}, \mathbf{b}\}$ is a nontrivial $\mathcal{K}$-pair relative to $\mathbf{u}$. Using forcing, it is not hard to build a total enumeration degree $\mathbf{x} \geqslant \mathbf{u}$ such that $\mathbf{a} \nleqslant \mathbf{x}$ and $\mathbf{b} \nleqslant \mathbf{x}$. (For example, this follows from a much more general theorem of Soskov [21] about jump inversion in $\mathcal{D}_e$.) Note that $\{\mathbf{a}, \mathbf{b}\}$ is a nontrivial $\mathcal{K}$-pair relative to $\mathbf{x}$. By Theorem 2.11, $\mathbf{a} \vee \mathbf{x}$ is a strong quasiminimal cover of $\mathbf{x}$. □

6. Cototal sets and $\mathcal{K}$-pairs

In this section, we briefly examine the connection between cototal sets and $\mathcal{K}$-pairs, ending with a conjectured definition of cototality in the enumeration degrees.

Definition 6.1. A set A is *cototal* if $A \leqslant_e \overline{A}$. An enumeration degree is *cototal* if it contains a cototal set.

Andrews et al. [1] note that if A has cototal enumeration degree, then K_A is cototal representative of that degree. In fact, they show that the operator that maps $d_e(A)$ to $d_e(\overline{K_A})$ is degree invariant and call it the *skip operator*. The skip of $d_e(A)$ is denoted by $d_e(A)^\lozenge$, so we have that A is cototal if and only if $d_e(A) \leqslant_e d_e(A)^\lozenge$.

It is straightforward to see that every total enumeration degree is cototal, as $\overline{A \oplus \overline{A}} = \overline{A} \oplus A \equiv_e A \oplus \overline{A}$. More generally, every continuous degree is cototal. Recall that an enumeration degree is continuous if it contains a set of the form $C_\alpha = \bigoplus_{i\in\omega} \{q \in \mathbb{Q}\colon q <_{\mathbb{Q}} \alpha(i)\} \oplus \{q \in \mathbb{Q}\colon q >_{\mathbb{Q}} \alpha(i)\}$, for some $\alpha \in [0,1]^\omega$. It follows that every continuous degree is cototal as, $2\langle i,q\rangle \in C_\alpha$ if and only if there is an $r > q$ such that $2\langle i,r\rangle + 1 \in \overline{C}_\alpha$, and similarly, $2\langle i,q\rangle + 1 \in C_\alpha$ if and only if there is an $r < q$ such that $2\langle i,r\rangle \in \overline{C}_\alpha$.

The class of cototal enumeration degrees is strictly bigger than the continuous degrees. For example, cototal enumeration degrees can be halves of nontrivial $\mathcal{K}$-pairs. One way to see this is to note that every Σ^0_2 enumeration degree is cototal [1]. As we already saw, if A is semi-computable, then $\{A, \overline{A}\}$ is a $\mathcal{K}$-pair, and A can be chosen as a non-c.e. and non-co-c.e. member of any nonzero Turing degree. On the other hand, the kind of $\mathcal{K}$-pairs that cototal sets can be part of is restricted, as can be seen by the following result.

Proposition 6.2. *If A is of cototal enumeration degree and $\{A, B\}$ is a nontrivial $\mathcal{K}$-pair relative to U, then $A \leqslant_e U'$.*

Proof. Suppose that A has cototal enumeration degree and that $\{A, B\}$ is a nontrivial $\mathcal{K}$-pair relative to U. Then as $A \equiv_e K_A$, it follows that $\{K_A, B\}$ is a nontrivial $\mathcal{K}$-pair relative to U. Let $W \leqslant_e U$ be such that $K_A \times B \subseteq W$ and $\overline{K_A} \times \overline{B} \subseteq \overline{W}$. By the properties of $\mathcal{K}$-pairs outlined in Theorem 2.11, we have that $K_A \leqslant_e \overline{K_A} \leqslant_e B \oplus \overline{W}$. Of course $K_A \leqslant_e A \oplus \overline{W}$, so

$$d_e(K_A) \leqslant_e d_e(K_A \oplus \overline{W} \oplus U) \wedge d_e(B \oplus \overline{W} \oplus U) = d_e(\overline{W} \oplus U) \leqslant_e d_e(U)'.$$

Hence $A \leqslant_e U'$. □

Ideally, we would hope that the reverse statement is true as well: if A is not cototal then there are sets B and U such that $A \nleqslant_e U'$ and $\{A, B\}$ are a nontrivial $\mathcal{K}$-pair relative to U. Unfortunately, our current methods do not suffice to prove this statement. What we can show is much weaker.

Proposition 6.3. *If A is not cototal, then there are sets B and W such that $A \nleqslant_e \overline{W}$ and $\{A, B\}$ are a nontrivial $\mathcal{K}$-pair relative to W.*

Proof. Let $A \nleqslant_e \overline{A}$. Then A is not of continuous degree and hence by Theorem 5.2 it is uniformly array-avoiding. Let Z witness that. We will build sets B and W such that $A, B \nleqslant_e W$, $A \nleqslant_e \overline{W}$, $A \times B \subseteq W$, and $\overline{A} \times \overline{B} \subseteq \overline{W}$. The construction is a slight modification of the one in Theorem 4.4. At stage n we build finite sets B_n, B_n^-, and W_n satisfying the following four conditions:

(1) $B_n \subseteq B_{n+1}$, $B_n^- \subseteq B_{n+1}^-$, $W_n \subseteq W_{n+1}$;
(2) $B_n \cap B_n^- = \varnothing$;
(3) $A \restriction n \times B_n \subseteq W_n$;
(4) $\langle a, b\rangle \in W_n \Rightarrow (a \in A \vee b \in B_n)$.

We let $B = \bigcup_n B_n$ and $W = \bigcup_n W_n$. Properties (1), (3), and (4) ensure that $\{A, B\}$ is a $\mathcal{K}$-pair relative to W. What remains is to ensure that $A, B \nleqslant_e W$ and $A \nleqslant_e \overline{W}$. In order to do this, to every $x \in B_n$ we will associate a superset $C_x \supseteq A$

such that $C_x \leqslant_e Z$. Unless otherwise stated, $C_x = \omega$. The set C_x may be shrunk infinitely many times, but will always be a superset of A. We will also associate to every $y \in B_n^-$ a finite set $T_y \subseteq A$. Once defined, T_y will not be changed and will be the yth column of W.

We start the construction by setting $B_0 = B_0^- = W_0 = \varnothing$. Suppose we have constructed B_n, B_n^-, and W_n. As before, we ensure that $W_{n+1} \subseteq X_n$, where

$$X_n = \bigcup_{x \in B_n} C_x \times \{x\} \cup \bigcup_{y \in B_n^-} T_y \times \{y\} \cup \bigcup_{z \notin B_n \cup B_n^-} \omega \times \{z\}.$$

From what we have said so far, it follows that we will also ensure that that $\overline{W_{n+1}} \subseteq Y_n$, where

$$Y_n = \overline{W_n} \cap \left(\bigcup_{x \in B_n} \overline{A} \times \{x\} \cup \bigcup_{y \in B_n^-} \overline{T_y} \times \{y\} \cup \bigcup_{z \notin B_n \cup B_n^-} \omega \times \{z\} \right).$$

In this case as well we have that $Y_{n+1} \subseteq Y_n$, and so $\overline{W} \subseteq Y_n$ for all n.

Fix an effective listing of all enumeration operators $\{\Gamma_k\}_{k \in \omega}$. We have three cases depending on the stage.

If $\mathbf{n = 3k}$, then we ensure that $A \neq \Gamma_k(W)$ in exactly the same way as we did in Theorem 4.4. If $\mathbf{n = 3k + 1}$, then we ensure that $B \neq \Gamma_k(W)$, again using the same steps as in Theorem 4.4.

Suppose that $\mathbf{n = 3k + 2}$. We ensure that $A \neq \Gamma_k(\overline{W})$. Note that $Y_n \leqslant_e \overline{A}$, so $A \neq \Gamma_k(Y_n)$. Fix an a witnessing this difference.

Case 1. If $a \in A$, then we do not need to do anything, as $\overline{W} \subseteq Y_n$ ensures that we have satisfied our requirement. We just move on to the next stage.

Case 2. If $a \in \Gamma_k(Y_n)$, then fix an axiom $\langle a, D \rangle \in \Gamma_k$ such that $D \subseteq Y_n$. We would like to add D to $\overline{W}$. We do this by shrinking the sets C_x and by adding elements to B_{n+1}^-: For all $\langle b, x \rangle \in D$ such that $x \in B_n$, we have that $b \notin A$ and $\langle b, x \rangle \notin W_n$, so we can remove b from C_x (without interfering with the requirements that $A \subseteq C_x$ and $W_{n+1} \subseteq X_{n+1}$). For all $\langle b, y \rangle$ such that $y \in B_n^-$, we have that $b \in \overline{T_y}$ and since T_y does not change, we can be sure that $\langle b, y \rangle \in \overline{W}$. Finally, if $\langle b, z \rangle \in D$ and $z \notin B_n \cup B_n^-$, we enumerate $z \in B_{n+1}^-$ and set $T_z = \{c \colon \langle c, z \rangle \in W_n\}$, which is safe because $D \subseteq \overline{W_n}$. We set $B_{n+1} = B_n$ and $W_{n+1} = W_n$. It is straightforward to check that properties (1)–(4) still hold. □

There seem to be serious obstacles to modifying the construction above to get $A \nleqslant_e W' = K_W \oplus \overline{K_W}$. Nevertheless, we conjecture that the reverse is still true.

Conjecture 6.4. A degree $\mathbf{a}$ is cototal if and only if, whenever $\{\mathbf{a}, \mathbf{b}\}$ is a nontrivial $\mathcal{K}$-pair relative to $\mathbf{u}$ we have that $\mathbf{a} \leqslant \mathbf{u}'$

7. PA relative to an enumeration oracle

In this final section of our paper, we propose two more properties that relate to the continuous and to the cototal enumeration degree. Both properties rely on the extension of the relation "PA above" to enumeration oracle.

Definition 7.1 (Miller, Soskova [16]). $\langle B \rangle$ *is PA above* $\langle A \rangle$ if B enumerates a member of every $\Pi_1^0\langle A \rangle$ class.

Note that this relation is degree invariant. We write $d_e(A) \ll d_e(B)$ if $\langle B\rangle$ is PA above $\langle A\rangle$. Furthermore, it is an extension of the usual relation on Turing degrees, because if $\mathbf{x}$ and $\mathbf{y}$ are Turing degrees, then $\mathbf{x} \ll \mathbf{y}$ if and only if $\iota(\mathbf{x}) \ll \iota(\mathbf{y})$, i.e., the relation is preserved under the embedding $\iota\colon \mathcal{D}_T \hookrightarrow \mathcal{D}_e$. (Recall, that ι maps $d_T(A)$ to $d_e(A \oplus \overline{A})$.) On nontotal enumeration degrees, however, the relation "PA above" can behave strikingly differently.

Definition 7.2. A set A is *⟨self⟩-PA* if $\langle A\rangle$ is PA above $\langle A\rangle$.

Miller and Soskova [16] prove the existence of Δ^0_2 ⟨self⟩-PA sets. Furthermore, they show that the set of total degrees below a ⟨self⟩-PA set A forms a Scott set, i.e., an ideal closed with respect to the relation "PA above".

We consider the following two new properties.

Definition 7.3. Let $A \subseteq \omega$.

(1) Say that A is *PA bounded* if for every set B, if $\langle B\rangle$ is PA above $\langle A\rangle$, then $A \leqslant_e B$.
(2) Say that there is a *universal* $\Pi^0_1\langle A\rangle$ *class* if there is a $\Pi^0_1\langle A\rangle$ class U such that for every member $X \in U$ we have that $\left\langle X \oplus \overline{X}\right\rangle$ is PA above $\langle A\rangle$.

Both properties clearly hold for total enumeration degrees, and so they exhibit the "expected" behavior of sets with respect to the relation "PA above". We show that the two properties together characterize the continuous degrees, while the first property implies cototality.

Proposition 7.4. *Fix $A \subseteq \omega$.*

(1) *A is PA bounded and there is a universal $\Pi^0_1\langle A\rangle$ class if and only if A has continuous degree.*
(2) *If A is PA bounded, then A has cototal degree.*

Proof. (1) If A is continuous, then A is codable; fix a $\Pi^0_1\langle A\rangle$ class P such that every member in P enumerates A. To see that A is PA bounded, note that every set B such that $\langle B\rangle$ is PA above $\langle A\rangle$ enumerates a member of the $\Pi^0_1\langle A\rangle$ class P and hence enumerates A.

To see that there is a universal $\Pi^0_1\langle A\rangle$ class, we build a new class R by joining each $X \in P$ with DNC^X_2, the standard $\Pi^0_1[X]$ class consisting of all $\{0,1\}$-valued diagonally non-computable functions relative to X. Recall that a function f is *diagonally non-computable relative to* X if for every e, we have that $\varphi^X_e(e) \neq f(e)$. It is not hard to see that if f is $\{0,1\}$-valued, then it (is the characteristic function of a set that) is PA above X.

Fix $S \leqslant_e A$ such that $P = 2^\omega \smallsetminus [S]^\prec$. We let

$$R = \{\sigma \oplus \tau \colon \sigma \in S \vee \exists n\ (\tau(n) = \varphi^\sigma_{n,|\sigma|}(n))\}.$$

Then $U = 2^\omega \smallsetminus R$ is a $\Pi^0_1\langle A\rangle$ class and every member of this class has the form $Z = X \oplus Y$, where A is c.e. in X (equivalently $A \leqslant_e X \oplus \overline{X}$) and Y is PA above X (in the Turing sense). It follows that $\left\langle Z \oplus \overline{Z}\right\rangle$ is PA above $\langle A\rangle$ for every $Z \in U$, and so U is a universal $\Pi^0_1\langle A\rangle$ class.

For the reverse direction, suppose that A is PA bounded and that there is a universal $\Pi^0_1\langle A\rangle$ class U. Every member of U is PA relative to $\langle A\rangle$, and so by boundedness enumerates A. It follows that A is codable, hence by Theorem 2.9, it has continuous degrees.

(2) Suppose that A is PA bounded. Consider a total set $X \oplus \overline{X}$ above the skip of A, i.e., such that $\overline{K_A} \leqslant_e X \oplus \overline{X}$. We claim that $\langle X \oplus \overline{X} \rangle$ is PA above $\langle A \rangle$. To see this, consider a nonempty $\Pi^0_1\langle A\rangle$ class $P = 2^\omega \smallsetminus [S]^\prec$. We may assume that $S \leqslant_e A$ is closed upward. Consider the set $E = \{\sigma \colon \forall n \exists \tau \ (|\tau| = n \ \& \ \tau \supseteq \sigma \ \& \ \tau \notin S)\}$ of strings that can be extended to an element of P. Note that $E \leqslant_e \overline{K_A}$, and so it is c.e. in X. It follows that $X \oplus \overline{X}$ can enumerate an element in P, proving that $\langle X \oplus \overline{X} \rangle$ is PA above $\langle A \rangle$. By PA boundedness, $A \leqslant_e X \oplus \overline{X}$. This holds for any total set $X \oplus \overline{X}$ above $\overline{K_A}$, so by Selman's theorem, $A \leqslant_e \overline{K_A}$. Therefore, A has cototal degree. □

It is not clear that PA boundedness characterizes the cototal enumeration degrees. We do know, at least, that cototality does not imply the existence of a universal class. As noted previously, there are Δ^0_2 sets, hence sets of cototal degree, that are ⟨self⟩-PA. This combined with the following proposition yields the desired conclusion.

Proposition 7.5. *If A is ⟨self⟩-PA, then A does not have a universal $\Pi^0_1\langle A\rangle$ class.*

Proof. Fix a ⟨self⟩-PA set A. If there were a $\Pi^0_1\langle A\rangle$ class consisting of sets that are PA above $\langle A \rangle$, then A would enumerate a set $X \oplus \overline{X}$ such that $\langle X \oplus \overline{X} \rangle$ is PA above $\langle A \rangle$. In that case, X would be PA (in the Turing sense) above every Y such that $Y \oplus \overline{Y} \leqslant_e A$. On the other hand, as we noted earlier, the total sets enumeration below A form a Scott set. It follows that there is some $Y \oplus \overline{Y} \leqslant_e A$ such that Y is PA above X (again, in the Turing sense). But this is impossible, as the "PA above" relation is strict when restricted to Turing oracles. □

The statement above gives an alternative, though similar, proof that the degrees of ⟨self⟩-PA sets are disjoint from the continuous degrees. This was originally proved by Miller and Soskova [16], who show that there is a universal Martin-Löf test relative to *every* continuous degree, but not relative to *any* ⟨self⟩-PA degree.

We are left with the following questions:

Question 7.6. Are there cototal degrees that are not PA bounded?

Question 7.7. Are there PA bounded degrees that are not of continuous degree? In particular, can a ⟨self⟩-PA degree be PA bounded?

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