THE Δ^0_2 TURING DEGREES: AUTOMORPHISMS AND DEFINABILITY

THEODORE A. SLAMAN AND MARIYA I. SOSKOVA

ABSTRACT. We prove that the Δ^0_2 Turing degrees have a finite automorphism base. We apply this result to show that the automorphism group of $\mathcal{D}_T(\leq \mathbf{0}')$ is countable and that all its members have arithmetic presentations. We prove that every relation on $\mathcal{D}_T(\leq \mathbf{0}')$ induced by an arithmetically definable degree invariant relation is definable with finitely many Δ^0_2 parameters and show that rigidity for $\mathcal{D}_T(\leq \mathbf{0}')$ is equivalent to its biinterpretability with first order arithmetic.

1. Introduction

We wish to understand the notion of relative definability between sets of natural numbers and in particular its natural presentation as a degree structure. There are many ways in which one might approach this task. Here we consider the following three questions as guiding: What is the expressive power of the theory of the degree structure? What are the definable relations in the structure? How can we describe its automorphism group? For the structure of the Turing degrees $\mathcal{D}_T$ we have a complete answer to the first question and an appealing conjecture for the other two. Simpson [6] proved that the first order theory of the Turing degrees is computably isomorphic to the theory of second order arithmetic. In [7], Slaman and Woodin conjecture that the relationship between the structure of the Turing degrees and second order arithmetic is much stronger. If true their *biinterpretability conjecture* gives complete answers to the second and third questions: the definable relations in $\mathcal{D}_T$ are exactly the ones induced by degree invariant relations definable in second order arithmetic and the only automorphism of $\mathcal{D}_T$ is the identity. In other words the biinterpretability conjecture is that the structure of the Turing degrees is logically equivalent to second order arithmetic. Slaman and Woodin come to this conjecture through their work on the definability and automorphisms of the Turing degrees described in [7]. They establish that the structure of the Turing degrees has a finite automorphism base and as a consequence obtain that the biinterpretability conjecture is true if we allow the use of finitely many parameters, that the automorphism group of the Turing degrees is countable and that every member has an arithmetically definable presentation. Furthermore they show that every relation in $\mathcal{D}_T$ induced by degree invariant relations definable in second order arithmetic is first order definable with parameters in $\mathcal{D}_T$. Finally they establish that the *rigidity* of $\mathcal{D}_T$, the statement that $\mathcal{D}_T$ has no nontrivial automorphisms, is equivalent to its biinterpretability with second order arithmetic.

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In parallel to the study of the global structure of the Turing degrees, two substructures are investigated: the structure of the Turing degrees bounded by the degree of the halting problem $\mathbf{0}'$ and the substructure of the computably enumerable degrees, denoted respectively by $\mathcal{D}_T(\leq \mathbf{0}')$ and $\mathcal{R}$. The same three guiding questions can be addressed to understand these local structures. In this case as well we have complete answers to the first one. Both structures have theories that are computably isomorphic to first order arithmetic. Shore [4] proves this for $\mathcal{D}_T(\leq \mathbf{0}')$ and Slaman and Harrington (see [2]) prove it for $\mathcal{R}$. The second question is understood only to a certain extent. Nies, Shore and Slaman [2] prove that any relation in $\mathcal{R}$ that is induced by a definable relation in first order arithmetic, invariant not only under Turing reducibility, but also under double jump, is definable in $\mathcal{R}$. Later Shore [5] proves that this holds also for $\mathcal{D}_T(\leq \mathbf{0}')$. Even less is known regarding the third question.

In this article we focus on the structure $\mathcal{D}_T(\leq \mathbf{0}')$. We establish a relationship between the local structure $\mathcal{D}_T(\leq \mathbf{0}')$ and first order arithmetic, similar to the one proved by Slaman and Woodin for the global structure $\mathcal{D}_T$ and second order arithmetic. Our main result is that $\mathcal{D}_T(\leq \mathbf{0}')$ has a finite automorphism base and it is biinterpretable with first order arithmetic using the elements of this base as parameters. We further show that $Aut(\mathcal{D}_T(\leq \mathbf{0}'))$ is countable and its members have arithmetic presentations and that the structure $\mathcal{D}_T(\leq \mathbf{0}')$ is atomic. We answer Question 4.8 from [9]: every relation in $\mathcal{D}_T(\leq \mathbf{0}')$ induced by an arithmetically definable degree invariant relation is definable with finitely many parameters. Finally, we show that rigidity for $\mathcal{D}_T(\leq \mathbf{0}')$ is equivalent to its biinterpretability with first order arithmetic.

In a sequel to this paper, [8], we use similar methods to investigate the structure of the enumeration degrees. There we reveal a strong connection between the local and global structures: the natural presentation of the structure of the computably enumerable degrees is an automorphism base for the structure of the enumeration degrees.

2. The local coding theorem and its applications

One of the main tools in Slaman and Woodin's analysis of the automorphisms of $\mathcal{D}_T$ is their coding theorem, the statement that every countable relation in the Turing degrees is uniformly definable from parameters. For the local structure $\mathcal{D}_T(\leq \mathbf{0}')$ Slaman and Woodin give a more restricted version of the coding theorem. In order to state it we need the following definitions related to *lowness*. We use the notation $\{i\}^A$ to denote the i-th partial computable functional with oracle A in some standard effective listing of all partial computable functionals.

Definition 1. (1) *A set A is* low *if its jump A' is computable from the halting set. A Turing degree is* low *if it contains a low set.*

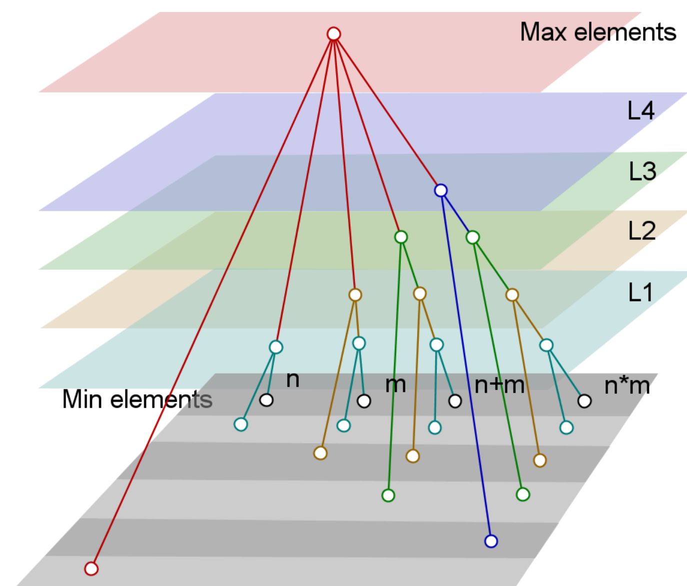
(2) *A sequence of sets $\{Z_i\}_{i<\omega}$ is* uniformly computable *from a set A if there is a computable function f, such that $Z_i = \{f(i)\}^A$.*

(3) *A sequence of sets $\{Z_i\}_{i<\omega}$ is* uniformly low *if there is a computable function g, such that $(\bigoplus_{j<i} Z_i)' = \{g(i)\}^{\emptyset'}$.*

(4) *A set of degrees $\mathcal{Z} \subseteq \mathcal{D}_T(\leq \mathbf{0}')$ is uniformly low if there is a uniformly low sequence $\{Z_i\}_{i<\omega}$, such that $\mathcal{Z} = \{d_T(Z_i) \mid i < \omega\}$. We call the sequence $\{Z_i\}_{i<\omega}$ a* presentation *of $\mathcal{Z}$.*

We give an example of a uniformly low set of degrees. Let $\{G_i\}_{i<\omega}$ be a sequence of sets such that $G = \bigoplus_{i<\omega} G_i$ is low. Then the set $\mathcal{G} = \{d_T(G_i) \mid i < \omega\}$ is uniformly low. In particular if G is any Δ^0_2 1-generic set then $\mathcal{G}$ is a uniformly low anti-chain. We refer to [7] for a review of properties of 1-generic sets, some of which will be used in this article.

The local coding theorem of Slaman and Woodin [9] is that every uniformly low set of degrees that is bounded by a low degree is uniformly definable from parameters in $\mathcal{D}_T(\leq \mathbf{0}')$. Among many other applications, it gives a method for coding a model of arithmetic in $\mathcal{D}_T(\leq \mathbf{0}')$. We can represent a model of arithmetic as a partial ordering as described in [2]. We start with a countable anti-chain of minimal elements $\{p_n\}_{n<\omega}$ which will represent the natural numbers. Then, for each $n, m \in \omega$ we add an element $c_{n,m}$ which represents the pair (p_n, p_m). Next we add ascending chains of lengths 2 and 3, respectively, from p_n to $c_{n,m}$ and from p_m to $c_{n,m}$. Finally, to code addition, we add a chain of length 4 from p_{n+m} to $c_{n,m}$ and for multiplication, a chain of length 5 from p_{nm} to $c_{n,m}$.



Now we fix a 1-generic set G and let $G = \bigoplus_{i<\omega} G_i$. Recall that $\{G_i\}_{i<\omega}$ has the following property: if F and E are finite sets then $\bigoplus_{i\in F} G_i \leq_T \bigoplus_{i\in E} G_i$ if and only if $F \subseteq E$. We use this fact to build six anti-chains that together form the partial order described above. The minimal elements form one anti chain, the maximal elements another and we have four intermediate anti-chains L_1, L_2, L_3 and L_4. We partition $\mathbb{N}$ into six computable infinite pieces D_i, $i = 1 \ldots 6$. To make notation easier we assume that $D_1 = 2\mathbb{N}$. We use G_{2n} to represent p_n. This will be the first anti-chain. To every pair of natural numbers (n, m) we computably assign four elements with indices in D_2, three elements with indices in D_3, two with indices in D_4 and one element with an index in each of D_5 and D_6. To construct the second anti-chain L_1 for every pair of natural numbers (n, m) we join G_{2n} with the first assigned element from D_2, G_{2m} with the second, $G_{2(n+m)}$ and G_{2n*m} with the third and fourth. Similarly L_2 is constructed by joining each of the elements above G_{2m}, $G_{2(n+m)}$ and G_{2n*m} from L_1 with the assigned elements from D_3, L_3 by joining the elements from D_4 to the L_2 elements above $G_{2(n+m)}$ and G_{2n*m}, L_4 by joining the element from D_5 to the L_3 element above G_{2n*m}. To get the final sixth anti-chain and the representatives for $c_{n,m}$ we join up all of the constructed element with the one assigned to the pair (n, m) from D_6. Each of the six anti-chains is a uniformly low sequence bounded by G. We have a presentation of a partial order $\mathcal{P}$ which by the Coding Theorem is uniformly definable from parameters $\vec{\mathbf{p}}$ in $\mathcal{D}_T(\leq \mathbf{0}')$.

We will denote by $\mathcal{M}(\vec{\mathbf{p}}) = (\mathbb{N}^{\mathcal{M}(\vec{\mathbf{p}})}, 0^{\mathcal{M}(\vec{\mathbf{p}})}, 1^{\mathcal{M}(\vec{\mathbf{p}})}, +^{\mathcal{M}(\vec{\mathbf{p}})}, *^{\mathcal{M}(\vec{\mathbf{p}})})$ the standard model of arithmetic coded in this way: $\mathbb{N}^{\mathcal{M}(\vec{\mathbf{p}})}$ is the set of minimal elements in $\mathcal{P}$, $0^{\mathcal{M}(\vec{\mathbf{p}})}$ and $1^{\mathcal{M}(\vec{\mathbf{p}})}$ are the elements representing p_0 and p_1 respectively, $\mathcal{M}(\vec{\mathbf{p}}) \models n + m = k$ and $\mathcal{M}(\vec{\mathbf{p}}) \models n * m = l$ if and only if there is maximal element in $\mathcal{P}$ at distance 1 from $n^{\mathcal{M}(\vec{\mathbf{p}})}$, 2 from $m^{\mathcal{M}(\vec{\mathbf{p}})}$, 3 from $k^{\mathcal{M}(\vec{\mathbf{p}})}$ and 4 from $l^{\mathcal{M}(\vec{\mathbf{p}})}$.

We extend these ideas a little further. Suppose that $\{Z_i\}_{i<\omega}$ is uniformly computable from a low set Z and represents a uniformly low set of degrees $\mathcal{Z}$. Let G be 1-generic relative to Z and computable from Z'. Let $\mathcal{M}(\vec{\mathbf{p}})$ be the standard model of arithmetic coded below G. Consider the sequence $\{Z_i \oplus G_{2i}\}_{i<\omega}$. This is an anti-chain, bounded by $G \oplus Z$. As $(G \oplus Z)' = G \oplus Z' \leq_T Z'$, it follows that $\{Z_i \oplus G_{2i}\}_{i<\omega}$ is a uniformly low anti-chain, bounded by a low degree and hence as well definable with parameters. It follows that the function mapping $i^{\mathcal{M}}$ to $d_T(Z_i)$ is also definable.

Definition 2. *Let $\mathcal{Z}$ be a set of degrees, represented by a sequence $\{Z_i\}_{i<\omega}$. We say that $\vec{\mathbf{p}}$ define an* indexing *of $\mathcal{Z}$ via the sequence $\{Z_i\}_{i<\omega}$ if they define a standard model of arithmetic $\mathcal{M}$ and a function $\varphi : \mathbb{N}^{\mathcal{M}} \to \mathcal{D}$ such that $\varphi(i^{\mathcal{M}}) = d_T(Z_i)$.*

The discussion above allows us to phrase the local coding theorem in a slightly stronger form.

Theorem 2.1 (Slaman and Woodin [9])**.** *For every uniformly low set of degrees $\mathcal{Z}$ and uniformly low sequence $\{Z_i\}_{i<\omega}$ representing $\mathcal{Z}$ there are finitely many parameters $\vec{\mathbf{p}} \in \mathcal{D}_T(\leq \mathbf{0}')$ that define an indexing of $\mathcal{Z}$ via $\{Z_i\}_{i<\omega}$.*

Slaman and Woodin [9] prove that the set of computably enumerable degrees $\mathcal{R}$ is definable with finitely many parameters from $\mathcal{D}_T(\leq \mathbf{0}')$. The proof uses an idea from Welch [10]. Let $\{W_e\}_{e<\omega}$ be the standard listing of all c.e. sets and let $K = \{\langle e, x\rangle \mid x \in W_e\}$. By the Sacks Splitting Theorem [3] there are two low sets A and B such that $A \cup B = K$. Let $A = \bigoplus_{e<\omega} A_e$ and $B = \bigoplus_{e<\omega} B_e$. Then $\mathcal{A} = \{d_T(A_e) \mid e < \omega\}$ and $\mathcal{B} = \{d_T(B_e) \mid e < \omega\}$ are uniformly low sets of degrees and for every e we have that $d_T(W_e) = d_T(A_e) \vee d_T(B_e)$. If $\vec{\mathbf{p}}$ and $\vec{\mathbf{q}}$ are parameters that code standard models of arithmetic $\mathcal{M}(\vec{\mathbf{p}})$ and $\mathcal{M}(\vec{\mathbf{q}})$ then the mapping $n^{\mathcal{M}(\vec{\mathbf{p}})} \to n^{\mathcal{M}(\vec{\mathbf{q}})}$ is definable from $\vec{\mathbf{p}}$ and $\vec{\mathbf{q}}$. Thus by extending the number of parameters we use, we can assume that any finite number of indexings use the same coded model of arithmetic. Thus we can find a finite list of parameters $\vec{\mathbf{p}}$ which code a model $\mathcal{M}$, an indexing of the sequence $\{A_e\}_{e<\omega}$ and an indexing of the sequence $\{B_e\}_{e<\omega}$. We combine these to obtain an indexing of the c.e. degrees via the sequence $\{W_e\}_{e<\omega}$. We will call this *an indexing of the c.e. degrees*. In line with Theorem 2.1 we restate the definability of the c.e. degrees from parameters as follows.

Theorem 2.2 (Slaman and Woodin [9])**.** *There are finitely many parameters in $\mathcal{D}_T(\leq \mathbf{0}')$ that define an indexing of the c.e. degrees.*

This statement is stronger than the definability with parameters of the c.e. degrees. Let A be a set of natural numbers with the following closure property: if $e \in A$ and $W_e \equiv_T W_j$ then $j \in A$. A induces a set of c.e. degrees $\mathcal{A} = \{d_T(W_e) \mid e \in A\}$. It follows from Theorem 2.2 that if A is arithmetical then $\mathcal{A}$ is definable in $\mathcal{D}_T(\leq \mathbf{0}')$ with parameters. To see this, let $\vec{\mathbf{p}}$ be parameters that define a model of arithmetic $\mathcal{M}$ and an indexing φ of the c.e. degrees. We use the arithmetic definition of A, effectively translated in the language of degree theory, to define the set $A^{\mathcal{M}}$ of the degrees $e^{\mathcal{M}}$ such that $e \in A$ and then observe that $\mathcal{A} = \varphi(A^{\mathcal{M}})$ is definable from $\vec{\mathbf{p}}$.

3. Biinterpretability with parameters

Consider the sequence $\{X_e\}_{e<\omega}$, where

- If $\{e\}^{\emptyset'}$ is a total $\{0,1\}$-valued function then X_e is the set with characteristic function $\{e\}^{\emptyset'}$.
- $X_e = \emptyset$ otherwise.

Then $\mathcal{X} = \{d_T(X_e) \mid e < \omega\}$ is the set of all Δ_2^0 degrees. An indexing of the Δ_2^0 degrees via the sequence $\{X_e\}_{e<\omega}$ will be called simply an *indexing of the Δ_2^0 degrees*.

We will say that $\mathcal{D}_T(\leq \mathbf{0}')$ is *biinterpretable* with first order arithmetic if there is a definable indexing of the Δ_2^0 degrees. We use ideas similar to the ones described in the previous section to show that there are finitely many parameters in $\mathcal{D}_T(\leq \mathbf{0}')$ that define such an indexing, i.e. that $\mathcal{D}_T(\leq \mathbf{0}')$ is biinterpretable with first order arithmetic modulo the use of finitely many parameters.

We will state two technical theorems and leave their proofs for the last two sections of this paper. The first one will allow us to reduce our task to finding an indexing of the low Δ_2^0 degrees.

Theorem 3.1. *Let $\mathbf{y} \leq \mathbf{0}'$. There are low degrees $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ and $\mathbf{x}_4$ such that*

$$\mathbf{y} = (\mathbf{x}_1 \vee \mathbf{x}_2) \wedge (\mathbf{x}_3 \vee \mathbf{x}_4).$$

The second technical result shows that there is a uniformly low set of degrees $\mathcal{Z}$, bounded by a low degree, such that every low Turing degree has a unique position with respect to the elements of $\mathcal{Z}$ and the c.e. degrees.

Theorem 3.2. *There exists a uniformly low set of Turing degrees $\mathcal{Z}$, bounded by a low degree $\mathbf{z} < \mathbf{0}'$, such that if $\mathbf{x}, \mathbf{y} < \mathbf{0}'$, $\mathbf{x}' = \mathbf{0}'$ and $\mathbf{y} \nleq \mathbf{x}$ then there are c.e. degrees $\mathbf{a}_i$ and Δ_2^0 Turing degrees $\mathbf{c}_i, \mathbf{b}_i$, $\mathbf{g}_i$ for $i = 1, 2$ such that:*

1. *$\mathbf{b}_i$ and $\mathbf{c}_i$ are elements of $\mathcal{Z}$.*
2. *$\mathbf{g}_i$ is the least element below $\mathbf{a}_i$ which joins $\mathbf{b}_i$ above $\mathbf{c}_i$.*
3. *$\mathbf{x} \leq \mathbf{g}_1 \vee \mathbf{g}_2$.*
4. *$\mathbf{y} \nleq \mathbf{g}_1 \vee \mathbf{g}_2$.*

We state and prove our main result.

Theorem 3.3. *There are parameters $\vec{\mathbf{p}}$ in $\mathcal{D}_T(\leq \mathbf{0}')$ which define an indexing of the Δ_2^0 degrees.*

Proof. Let $\{Z_i\}_{i<\omega}$ be the uniformly low sequence of sets, representing the uniformly low set of degrees $\mathcal{Z}$ obtained in Theorem 3.2. By Theorem 2.1 and Theorem 2.2 we can fix finitely many Δ^0_2 parameters $\vec{\mathbf{p}}$ which code a model $\mathcal{M}$, an indexing $\varphi_{\mathcal{Z}}$ of $\mathcal{Z}$ via $\{Z_i\}_{i<\omega}$ and an indexing $\varphi_{\mathcal{R}}$ of the c.e. degrees:

$$\varphi_{\mathcal{Z}}(i^{\mathcal{M}}) = d_T(Z_i) \quad \text{and} \quad \varphi_{\mathcal{R}}(i^{\mathcal{M}}) = d_T(W_i).$$

We will show that the function $\varphi : \mathbb{N}^{\mathcal{M}} \to \mathcal{D}_T(\leq \mathbf{0}')$, given by $\varphi(i^{\mathcal{M}}) = d_T(X_i)$ is definable in $\mathcal{D}_T(\leq \mathbf{0}')$ from the parameters $\vec{\mathbf{p}}$.

Fix a Δ^0_2 degree $\mathbf{i}$. The property $\mathbf{i} \in \mathbb{N}^{\mathcal{M}}$ is definable from the parameters $\vec{\mathbf{p}}$, so we can assume that $\mathbf{i} = i^{\mathcal{M}}$ for a fixed natural number i. We have different cases depending on the nature of the number i. We have an effective way to translate an arithmetic statement about i, say $\psi(i)$, into a degree theoretic statement $\hat{\psi}(\mathbf{i}, \vec{\mathbf{p}})$ so that:

$$\mathbb{N} \models \psi(i) \text{ if and only if } \mathcal{D}_T(\leq \mathbf{0}') \models \hat{\psi}(\mathbf{i}, \vec{\mathbf{p}}).$$

We will say that $\psi(i)$ *is true in* $\mathcal{M}$ to mean that $\psi(\mathbf{i}, \vec{\mathbf{p}})$ is true in $\mathcal{D}_T(\leq \mathbf{0}')$.

Case 1: Suppose that in $\mathcal{M}$ the following statement is true:

"$\{i\}^{\emptyset'}$ is not a $\{0,1\}$-valued total function".

In this case $\varphi(\mathbf{i}) = \mathbf{0}$.

Case 2: Suppose that in $\mathcal{M}$ the following statement is true:

"$\{i\}^{\emptyset'}$ is the characteristic function of a low set."

In this case we use Theorem 3.2. Note that as $\{Z_i\}_{i<\omega}$ is a sequence uniformly computable from $\mathbf{0}'$, we can fix a total computable function g, such that $Z_e = \{g(e)\}^{\emptyset'}$ for every e. We can define $\varphi(\mathbf{i})$ as the largest degree $\mathbf{x}$ with the following property: For every list of degrees $\mathbf{a}_1, \mathbf{a}_2$, $\mathbf{b}_1, \mathbf{b}_2$, $\mathbf{c}_1, \mathbf{c}_2$ $\mathbf{g}_1$ and $\mathbf{g}_2$ if:

- $\mathbf{g}_1$ is the least element below $\mathbf{a}_1$ which joins $\mathbf{b}_1$ above $\mathbf{c}_1$;
- $\mathbf{g}_2$ is the least element below $\mathbf{a}_2$ which joins $\mathbf{b}_2$ above $\mathbf{c}_2$;
- There are elements e_{a_1}, e_{b_1} and e_{c_1} such that $\varphi_{\mathcal{R}}(e^{\mathcal{M}}_{a_1}) = \mathbf{a}_1$, $\varphi_{\mathcal{Z}}(e^{\mathcal{M}}_{b_1}) = \mathbf{b}_1$ and $\varphi_{\mathcal{Z}}(e^{\mathcal{M}}_{c_1}) = \mathbf{c}_1$;
- There are elements e_{a_2}, e_{b_2} and e_{c_2} such that $\varphi_{\mathcal{R}}(e^{\mathcal{M}}_{a_2}) = \mathbf{a}_2$, $\varphi_{\mathcal{Z}}(e^{\mathcal{M}}_{b_2}) = \mathbf{b}_2$ and $\varphi_{\mathcal{Z}}(e^{\mathcal{M}}_{c_2}) = \mathbf{c}_2$;
- In $\mathcal{M}$ the following statement is true:

  "$d_T(\{i\}^{\emptyset'})$ is bounded by the join of the least Turing degree below $d_T(W_{e_{a_1}})$ which joins $d_T(\{g(e_{b_1})\}^{\emptyset'})$ above $d_T(\{g(e_{c_1})\}^{\emptyset'})$ and the least Turing degree below $d_T(W_{e_{a_2}})$ which joins $d_T(\{g(e_{b_2})\}^{\emptyset'})$ above $d_T(\{g(e_{c_2})\}^{\emptyset'})$".

then $\mathbf{x} \leq \mathbf{g}_1 \vee \mathbf{g}_2$.

Case 3: Suppose that in $\mathcal{M}$ the following statement is true:

"$\{i\}^{\emptyset'}$ is the characteristic function of a non low set."

In this case we apply Theorem 3.1 and the previous case. Let e_1, e_2, e_3, e_4 be natural numbers such that in $\mathcal{M}$ the following statement is true:

"For each $j = 1, 2, 3, 4$ the function $\{e_j\}^{\emptyset'}$ is the characteristic function of a low set and $d_T(\{i\}^{\emptyset'}) = (d_T(\{e_1\}^{\emptyset'}) \vee d_T(\{e_2\}^{\emptyset'})) \wedge (d_T(\{e_3\}^{\emptyset'}) \vee d_T(\{e_4\}^{\emptyset'}))$."

Under the previous case we have already shown that $\varphi(e_j^{\mathcal{M}})$ is definable with parameters $\vec{\mathbf{p}}$. We define $\varphi(\mathbf{i})$ as $(\varphi(e_1^{\mathcal{M}}) \vee \varphi(e_2^{\mathcal{M}})) \wedge (\varphi(e_3^{\mathcal{M}}) \vee \varphi(e_4^{\mathcal{M}}))$. □

4. Applications

Biinterpretability with parameters has many consequences for $\mathcal{D}_T(\leq \mathbf{0}')$ and its automorphism group. In this section we state and prove them.

Definition 3. *An* automorphism base *of a structure $\mathcal{A}$ with domain A is a set $B \subseteq A$, such that for every pair of automorphisms π_1 and π_2 of $\mathcal{A}$ if π_1 and π_2 agree on all elements in B then $\pi_1 = \pi_2$.*

The relationship between an indexing of the Δ^0_2 degrees and automorphism bases is given by the following.

Theorem 4.1. *If $\vec{\mathbf{p}}$ are parameters that define an indexing of the Δ^0_2 degrees then $\vec{\mathbf{p}}$ is an automorphism base for $\mathcal{D}_T(\leq \mathbf{0}')$.*

Proof. Consider an automorphism π of $\mathcal{D}_T(\leq \mathbf{0}')$. Let $\vec{\mathbf{p}}$ be the Δ^0_2 parameters which code a standard model of arithmetic $\mathcal{M}$ and an indexing φ of the Δ^0_2 degrees. Then $\pi(\vec{\mathbf{p}})$ are Δ^0_2 degrees which also code a model of arithmetic $\mathcal{M}_\pi$ and a function: $\varphi_\pi : \mathbb{N}_{\mathcal{M}_\pi} \to \mathcal{D}_T(\leq \mathbf{0}')$ which is onto. As every element in the domain of the model of arithmetic is definable from the parameters that code it, it follows that for every natural number e we have that $\pi(e^{\mathcal{M}}) = e^{\mathcal{M}_\pi}$. This allows us to conclude that the image of $\vec{\mathbf{p}}$ determines the image of every other Δ^0_2 degree, as for every natural number e we have that $\pi(d_T(\{e\}^{\emptyset'})) = \pi(\varphi(e^{\mathcal{M}})) = \varphi_\pi(\pi(e^{\mathcal{M}})) = \varphi_\pi(e^{\mathcal{M}_\pi})$. In particular if two automorphism agree on the elements of $\vec{\mathbf{p}}$ then they are identical. □

Definition 4. *Let π be a function on the Δ^0_2 degrees. A presentation of π is a function Π on natural numbers, such that for every e we have that $d_T(\Pi(\{e\}^{\emptyset'})) = \pi(d_T(\{e\}^{\emptyset'}))$.*

Corollary 4.2. *$\mathcal{D}_T(\leq \mathbf{0}')$ has a finite automorphism base. The automorphism group of $\mathcal{D}_T(\leq \mathbf{0}')$ is countable and every automorphism of $\mathcal{D}_T(\leq \mathbf{0}')$ has an arithmetically definable presentation.*

Proof. By Theorem 3.3 there are finitely many parameters $\vec{\mathbf{p}}$ which define an indexing of the Δ^0_2 degrees. By Theorem 4.1 they are an automorphism base for $\mathcal{D}_T(\leq \mathbf{0}')$. There are only countably many choices for the image of $\vec{\mathbf{p}}$ and hence only countably many possible automorphisms.

Given any automorphism π of $\mathcal{D}_T(\leq \mathbf{0}')$, the image $\pi(\vec{\mathbf{p}})$ is a finite sequence of Δ^0_2 degrees. Fix Δ^0_2 sets that represent the degrees in $\vec{\mathbf{p}}$ and in $\pi(\vec{\mathbf{p}})$ and corresponding arithmetic definitions for these sets. The least index of a fixed fixed Δ^0_2 set is an arithmetically definable singleton. The functions φ and φ_π described above corresponds to an arithmetically definable function on indices, because the relation "$\{e_1\}^{\emptyset'} \leq_T \{e_2\}^{\emptyset'}$" is arithmetically definable. The procedure, described in the proof of Theorem 4.1, for determining $\pi(d_T(\{e\}^{\emptyset'}))$ from the image of $\vec{\mathbf{p}}$ is therefore arithmetically definable and gives an arithmetic presentation of π. □

Definition 5. *We say that a structure is* atomic *if the complete type of every tuple is axiomatized by a single formula.*

Theorem 4.3. *$\mathcal{D}_T(\leq \mathbf{0}')$ is atomic.*

Proof. We show that for every tuple $\mathbf{x}$ there is a formula $\chi_{\vec{\mathbf{x}}}$ which determines the type of $\vec{\mathbf{x}}$ in $\mathcal{D}_T(\leq \mathbf{0}')$. Let $\vec{\mathbf{p}}$ be parameters that code a model of arithmetic $\mathcal{M}$ and an indexing φ of the Δ^0_2 degrees. The indexing φ is far from injective, but we can use it to define an injection $\theta : \mathcal{N}^{\mathcal{M}} \to \mathcal{D}_T(\leq \mathbf{0}')$. In arithmetic we can define a function $f : \mathbb{N} \to \mathbb{N}$ where $f(n)$ is the least index of a Δ^0_2 set which is not Turing equivalent to any of the sets with indices $f(m)$ for $m < n$. Let $\theta(e^{\mathcal{M}}) = \varphi(f(e)^{\mathcal{M}})$. Now $\vec{\mathbf{x}} = (\mathbf{x}_1, \dots \mathbf{x}_k)$ is the tuple $(\theta(e_1^{\mathcal{M}}), \dots \theta(e_k^{\mathcal{M}}))$ for a unique tuple $(e_1, \dots e_k)$.

We define the formula $\chi_{\vec{\mathbf{x}}}$ to express exactly this relationship: there are parameters $\vec{\mathbf{p}}$ which code a standard model of arithmetic $\mathcal{M}$, such that the relation θ coded by $\vec{\mathbf{p}}$ using the definition that can be derived by combining the proof of Theorem 3.3 and the description of the function f above is a bijection $\theta : \mathbb{N}^{\mathcal{M}} \to \Delta^0_2$, such that:

- For every i and j the model $\mathcal{M}$ satisfies the translation of the arithmetic formula $\{f(i)\}^{\emptyset'} \leq_T \{f(j)\}^{\emptyset'}$ if and only if $\theta(i^{\mathcal{M}}) \leq \theta(j^{\mathcal{M}})$.
- For every $j \leq k$, $\theta(e_j^{\mathcal{M}}) = \mathbf{x}_j$.

To guarantee that $\mathcal{M}$ is a standard model of arithmetic we ask that in addition for any other set of parameters $\vec{\mathbf{q}}$ which also code a model of arithmetic $\mathcal{N}$ and a bijection $\psi : \mathbb{N}^{\mathcal{N}} \to \Delta^0_2$ with the same property, we have that the following is true. Let $\lambda : \mathbb{N}^{\mathcal{M}} \to \mathbb{N}^{\mathcal{N}}$ be the bijection defined by $\lambda(e^{\mathcal{M}}) = \psi^{-1}(\theta(e^{\mathcal{M}}))$. We require that the image of the interval $[0^{\mathcal{M}}, n^{\mathcal{M}}]$ into $\mathcal{N}$ under λ is bounded in $\mathcal{N}$. In other words we ask that for every element $n^{\mathcal{M}} \in \mathbb{N}^{\mathcal{M}}$ there is an element $\mathbf{b} \in \mathbb{N}^{\mathcal{N}}$ such that for every $m \leq n$, the element $\lambda(m^{\mathcal{M}}) <_{\mathcal{N}} \mathbf{b}$.

Now if $\vec{\mathbf{x}}$ and $\vec{\mathbf{y}}$ satisfy the same formula $\chi_{\vec{\mathbf{x}}}$ then $\vec{\mathbf{x}}$ and $\vec{\mathbf{y}}$ are automorphic. Indeed, let $\mathcal{M}_x$, θ_x be the model of arithmetic and bijection witnessing $\chi_{\vec{\mathbf{x}}}(\vec{\mathbf{x}})$ and $\mathcal{M}_y$, θ_y be the model of arithmetic and bijection witnessing $\chi_{\vec{\mathbf{x}}}(\vec{\mathbf{y}})$. The automorphism π works as follows: on a degree $\mathbf{z}$, it finds the natural number in the first model $e^{\mathcal{M}_x}$ such that $\theta_x(e^{\mathcal{M}_x}) = \mathbf{z}$ and then maps $\mathbf{z}$ to $\theta_y(e^{\mathcal{M}_y})$. As θ_x and θ_y are bijections, it follows that π is a bijection and $\pi(\vec{\mathbf{x}}) = \vec{\mathbf{y}}$. Furthermore, for all pairs of degrees $\mathbf{z}_1 = \theta_x^{-1}(e_1^{\mathcal{M}})$ and $\mathbf{z}_2 = \theta_x^{-1}(e_2^{\mathcal{M}})$ we have that $\mathbf{z}_1 \leq \mathbf{z}_2$ if and only if $\{f(e_1)\}^{\emptyset'} \leq_T \{f(e_2)\}^{\emptyset'}$ if and only if $\pi(\mathbf{z}_1) \leq \pi(\mathbf{z}_2)$, so π is an automorphism. □

We next consider consequences of biinterpretability with parameters to first order definability in $\mathcal{D}_T(\leq \mathbf{0}')$.

Definition 6. *Let $R \subseteq \mathbb{N}^k$.*

1. *R is* degree invariant *if whenever $(e_1, \dots e_k) \in R$ and $\{e_1\}^{\emptyset'} \equiv_T \{i_1\}^{\emptyset'}$, $\dots \{e_k\}^{\emptyset'} \equiv_T \{i_k\}^{\emptyset'}$ then $(i_1, \dots i_k) \in R$.*
2. *If R is degree invariant then R* induces a relation $\mathcal{R}$ *on $\mathcal{D}_T(\leq \mathbf{0}')$ defined by $(\mathbf{x}_1, \dots \mathbf{x}_k) \in \mathcal{R}$ if and only if $\mathbf{x}_1 = d_T(\{e_1\}^{\emptyset'})$, $\dots \mathbf{x}_k = d_T(\{e_k\}^{\emptyset'})$ and $(e_1, \dots e_k) \in R$.*

Theorem 4.4.

1. *Every relation on $\mathcal{D}_T(\leq \mathbf{0}')$ that is induced by an arithmetically definable degree invariant relation is definable in $\mathcal{D}_T(\leq \mathbf{0}')$ with finitely many Δ^0_2 parameters.*
2. *Every relation on $\mathcal{D}_T(\leq \mathbf{0}')$ that is induced by an arithmetically definable degree invariant relation and which is in addition invariant under automorphisms is definable in $\mathcal{D}_T(\leq \mathbf{0}')$ without parameters.*

Proof. Let $R \subseteq \mathbb{N}^k$ be an arithmetically definable degree invariant relation and $\mathcal{R}$ be the induced relation on $\mathcal{D}_T(\leq \mathbf{0}')$. Let $\vec{\mathbf{p}}$ be the parameters which code a model of arithmetic $\mathcal{M}$ and an indexing of the Δ_2^0 degrees φ as in Theorem 3.3. Let $\chi(X_1, \dots X_k)$ be the formula in arithmetic which defines R. Then $\mathcal{R}$ is definable in $\mathcal{D}_T(\leq \mathbf{0}')$ with parameters $\vec{\mathbf{p}}$ by the following formula: $(\mathbf{x}_1, \dots \mathbf{x}_k) \in \mathcal{R}$ if and only if there are elements $e_1^{\mathcal{M}}, \dots e_k^{\mathcal{M}}$ in the domain of the model $\mathcal{M}$ such that $\mathbf{x}_i = \varphi(e_i^{\mathcal{M}})$ for $i \leq k$ and such that $\chi(e_1, \dots e_k)$ is true in $\mathcal{M}$.

If $\mathcal{R}$ is in addition invariant under automorphisms then we use the method from the proof of Theorem 4.3. Recall that f is a function with the property that $f(n)$ is the least index of a Δ_2^0 set which is not Turing equivalent to any of the sets with indices $f(m)$ for $m < n$. We define $(\mathbf{x}_1, \dots \mathbf{x}_k) \in \mathcal{R}$ if and only if there are parameters $\vec{\mathbf{p}}$ which code a standard model of arithmetic $\mathcal{M}$ and a bijection $\theta : \mathbb{N}^{\mathcal{M}} \to \Delta_2^0$, such that:

- For every i and j the model $\mathcal{M}$ satisfies the translation of the arithmetic formula $\{f(i)\}^{\emptyset'} \leq_T \{f(j)\}^{\emptyset'}$ if and only if $\theta(i^{\mathcal{M}}) \leq \theta(j^{\mathcal{M}})$.
- There are elements $e_1^{\mathcal{M}}, \dots e_k^{\mathcal{M}}$ in the domain of the model $\mathcal{M}$ such that $\mathbf{x}_i = \theta(e_i^{\mathcal{M}})$ for $i \leq k$ and such that $\chi(f(e_1), \dots f(e_k))$ is true in $\mathcal{M}$.

Let $\vec{\mathbf{p}}$ be the parameters that code a model of arithmetic and a bijective indexing of the Δ_2^0 degrees θ. If $(\mathbf{x}_1, \dots \mathbf{x}_k) \in \mathcal{R}$ then the parameters $\vec{\mathbf{p}}$ witness that $(\mathbf{x}_1, \dots \mathbf{x}_k)$ satisfy this formula. If on the other hand $(\mathbf{x}_1, \dots \mathbf{x}_k)$ satisfy the formula above as witnessed by parameters $\vec{\mathbf{q}}$ then there is a tuple $(\mathbf{y}_1 \dots \mathbf{y}_k)$ that satisfies the formula as witnessed by $\vec{\mathbf{p}}$. As $\vec{\mathbf{p}}$ defines an indexing of the Δ_2^0 degrees, we have that $(\mathbf{y}_1 \dots \mathbf{y}_k) \in \mathcal{R}$. Furthermore $(\mathbf{y}_1 \dots \mathbf{y}_k)$ is automorphic to $(\mathbf{x}_1, \dots \mathbf{x}_k)$. As $\mathcal{R}$ is invariant under automorphisms, it follows that $(\mathbf{x}_1, \dots \mathbf{x}_k) \in \mathcal{R}$. □

Corollary 4.5. *$\mathcal{D}_T(\leq \mathbf{0}')$ is rigid if and only if it is biinterpretable with first order arithmetic.*

Proof. If $\mathcal{D}_T(\leq \mathbf{0}')$ has no nontrivial automorphisms then every relation on $\mathcal{D}_T(\leq \mathbf{0}')$ is invariant under automorphisms. So by Theorem 4.4 every relation $\mathcal{R}$, induced by a degree invariant relation R definable in first order arithmetic, is definable in $\mathcal{D}_T(\leq \mathbf{0}')$. In particular, consider the parameters $\vec{\mathbf{p}} = (\mathbf{p}_1 \dots \mathbf{p}_k)$ which define an indexing φ of the Δ_2^0 degrees. The relation $R = \{(e_1, \dots e_k) \mid (\forall i \leq k)\{e_i\}^{\emptyset'} \in \mathbf{p}_i\}$ is arithmetically definable. Here we use once again that $\mathbf{p}_i$ is a Δ_2^0 degree, hence we can give an arithmetic definition for some index of a member of this degree. The relation R induces $\mathcal{R} = \{\vec{\mathbf{p}}\}$. It follows that the indexing φ is definable without parameters.

If there is a definable indexing φ of the Δ_2^0 degrees then by Theorem 4.1 the empty set is an automorphism base of $\mathcal{D}_T(\leq \mathbf{0}')$. In other words there can be only one automorphism, the identity. It follows that $\mathcal{D}_T(\leq \mathbf{0}')$ is rigid. □

The applications listed in this section give a clearer path towards understanding the structure of the Δ_2^0 Turing degrees, revealing deep connections between definability, rigidity and biinterpretability. By analyzing the complexity of the formula χ for tuples of length 1 in Theorem 4.3 we obtain a number b such that:

(1) There is a sequence of formulas with one free variable and of fixed quantifier complexity b: $\varphi_1(\mathbf{x}), \varphi_2(\mathbf{x}), \dots$ such that:
  - Every Δ_2^0 degree $\mathbf{d}$ satisfies at least one such formula.

- If any one of these formulas is satisfied by two different Δ^0_2 degrees then $\mathcal{D}_T(\leq \mathbf{0}')$ has a nontrivial automorphism.

(2) If $\mathbf{d}$ is a fixed point of an automorphism of the structure $\mathcal{D}_T(\leq \mathbf{0}')$ then $\mathbf{d}$ is first order definable in $\mathcal{D}_T(\leq \mathbf{0}')$ by a formula with complexity b.

(3) The following two statements are equivalent:
- No intermediate degree is definable in $\mathcal{D}_T(\leq \mathbf{0}')$.
- No intermediate degree is definable in $\mathcal{D}_T(\leq \mathbf{0}')$ by a formula with quantifier complexity b.

The equivalent statements listed in (3) above imply rigidity for $\mathcal{D}_T(\leq \mathbf{0}')$ in a very strong way. They allow us, on the other hand, to formulate a clear technical route to settling the automorphism problem for the structure $\mathcal{D}_T(\leq \mathbf{0}')$:

Question 1. *Let $b > 0$ be a natural number. Given a formula $\varphi(\mathbf{x})$ of quantifier complexity b and an intermediate Turing degree $\mathbf{d}$, such that $\varphi(\mathbf{d})$ is true in $\mathcal{D}_T(\leq \mathbf{0}')$, does there exist an intermediate degree $\mathbf{d}^* \neq \mathbf{d}$ such that $\varphi(\mathbf{d}^*)$ is also true of $\mathcal{D}_T(\leq \mathbf{0}')$.*

This question has an easy answer when $b = 1$. Any existential formula true of an intermediate Δ^0_2 degree $\mathbf{d}$ can be realized by a Δ^0_2 degree $\mathbf{g}$ that is 1-generic relative to $\mathbf{d}$. A negative answer to this question at any level b would provide us with a definable intermediate degree. The most interesting b is the one from the discussion above: a positive answer for all formulas at that level of complexity would show that every intermediate point has a nontrivial orbit; in the other extreme, there are finitely many formulas of that level, such that a negative answer for all of them would imply rigidity for $\mathcal{D}_T(\leq \mathbf{0}')$ and, by Slaman and Soskova [8], rigidity for the global structure of the enumeration degrees.

5. Low degrees determine all Δ^0_2 degrees

This section is devoted to the proof of Theorem 3.1. We will prove that every Δ^0_2 degree $\mathbf{y}$ can be represented as $(\mathbf{x}_1 \vee \mathbf{x}_2) \wedge (\mathbf{x}_3 \vee \mathbf{x}_4)$, where $\mathbf{x}_1, \dots \mathbf{x}_4$ are low degrees. If $\mathbf{y} = \mathbf{0}'$ then this theorem is an easy consequence of Sacks Splitting Theorem [3]: there are low degrees $\mathbf{a}$ and $\mathbf{b}$ such that $\mathbf{a} \vee \mathbf{b} = \mathbf{0}'$, so we can set $\mathbf{x}_1 = \mathbf{x}_3 = \mathbf{a}$ and $\mathbf{x}_2 = \mathbf{x}_4 = \mathbf{b}$. For incomplete degrees $\mathbf{y}$ we reason as follows.

We start with a definition of a functional C which maps a pair of sets Y and G to a set $C(Y, G)$ as follows. Let $G = G_{even} \oplus G_{odd}$:

$$C(Y,G)(n) = \begin{cases} G_{even}(n-m) & \text{if } G_{odd}(n) = 0 \text{ and } G_{odd} \upharpoonright n \text{ has } m \text{ elements;} \\ Y(m) & \text{if } G_{odd}(n) = 1 \text{ and } n \text{ is the } m\text{-th element of } G_{odd}. \end{cases}$$

Note that as long as G_{odd} is infinite, $G \oplus Y \equiv_T G_{odd} \oplus C(Y, G)$. We furthermore show the following property:

Lemma 5.1. *If G is 1-generic relative to Y then so is $C(Y, G)$.*

Proof. Suppose that W is c.e. in Y and W is dense in $C(Y, G)$, i.e every initial segment of $C(Y, G)$ has an extension in W. We need to show that $C(Y, G)$ meets W. We extend the definition of C so that it can have a finite binary string as a second parameter.

$$C(Y,\tau)(n) = \begin{cases} \tau_{even}(n-m) & \text{if } \tau_{odd}(n) = 0 \text{ and } \tau_{odd} \upharpoonright n \text{ has } m \text{ elements;} \\ Y(m) & \text{if } \tau_{odd}(n) = 1 \text{ and } n \text{ is the } m\text{-th element of } \tau_{odd}; \\ \uparrow & \text{if } \tau_{odd}(n) \uparrow. \end{cases}$$

Consider the c.e. in Y set $W^* = \{\sigma \oplus \tau \mid C(Y, \sigma \oplus \tau) \in W\}$. This set is dense in G. Indeed, fix $\sigma \oplus \tau \preceq G$. Then $C(Y, \sigma \oplus \tau)$ is an initial segment of $C(Y, G)$, hence there is an extension of $C(Y, \sigma \oplus \tau)$, say $C(Y, \sigma \oplus \tau)\hat{}\gamma$ in W. We can represent $C(Y, \sigma \oplus \tau)\hat{}\gamma$ as $C(Y, (\sigma\hat{}\gamma) \oplus (\tau\hat{}00\dots0))$, where the number of zeros added at the end of τ is the length of γ. Thus $(\sigma\hat{}\gamma) \oplus (\tau\hat{}00\dots0)$ is an extension of $\sigma \oplus \tau$ in W^*. As G is 1-generic relative to Y, it follows that some initial segment $\sigma \oplus \tau$ of G is in W^*, but then $C(Y, \sigma \oplus \tau)$ is an initial segment of $C(Y, G)$ in W. □

Let $\mathbf{y} < \mathbf{0}'$ be given and fix a set $Y \in \mathbf{y}$. We will use the well known fact that every non-computable c.e. set computes a 1-generic set, hence relativizing if Z is a set that is c.e. in Y and $Y <_T Z$ then Z computes a 1-generic relative to Y. See Downey and Hirschfeldt [1] for a proof of this fact. As $\emptyset'$ is c.e. relative to Y and $Y <_T \emptyset'$ there is a set $G \leq_T \emptyset'$ which is 1-generic relative to Y. We split G into odd and even bits: $G = G_0 \oplus G_1$. Then G_0 and G_1 are mutually generic relative to Y and so $G_0 \oplus Y$ and $G_1 \oplus Y$ form a minimal pair above Y. We set X_1 to be the odd bits in G_0 and $X_2 = C(Y, G_0)$. Similarly X_3 is the odd bits in G_1 and $X_4 = C(Y, G_1)$. The sets X_i, $i = 1\dots4$ are 1-generic and hence low. Now we have the required property:

$$\mathbf{y} = (d_T(X_1) \vee d_T(X_2)) \wedge (d_T(X_3) \vee d_T(X_4)).$$

6. A uniformly low set of Turing degrees

This section is devoted to the proof of Theorem 3.2. We wish to prove that there is a uniformly low set $\mathcal{Z}$ of Turing degrees, bounded by a low $\mathbf{z} < \mathbf{0}'$, such that if $\mathbf{x}, \mathbf{y}$ are Δ^0_2 degrees, $\mathbf{x}$ is low and $\mathbf{y} \not\leq \mathbf{x}$, then there are $\mathbf{g}_i < \mathbf{0}'$, c.e. degrees $\mathbf{a}_i$ and Δ^0_2 Turing degrees $\mathbf{c}_i, \mathbf{b}_i$ for $i = 1, 2$ such that:

(1) $\mathbf{b}_i$ and $\mathbf{c}_i$ are elements of $\mathcal{Z}$.
(2) $\mathbf{g}_i$ is the least element below $\mathbf{a}_i$ which joins $\mathbf{b}_i$ above $\mathbf{c}_i$.
(3) $\mathbf{x} \leq \mathbf{g}_1 \vee \mathbf{g}_2$.
(4) $\mathbf{y} \not\leq \mathbf{g}_1 \vee \mathbf{g}_2$.

Note, that if we leave out the requirement that $\mathbf{b}_i$ and $\mathbf{c}_i$ are elements of a uniformly low set $\mathcal{Z}$, we could solve the problem trivially by setting $\mathbf{a}_i = \mathbf{0}'$, $\mathbf{b}_i = \mathbf{0}$ and $\mathbf{c}_i = \mathbf{x}$. The design of our construction is therefore very much influenced by the necessity to realize the first requirement.

The first step in our proof is to design a construction which takes as parameters approximations to sets X and Y and produces approximations to sets G_i, A_i, B_i, C_i. If X is low, as witnessed by the given approximation, and Y is Δ^0_2 and not computable from X then the sets G_i, A_i, B_i, C_i will have the necessary structural properties. Once we have such a construction we will run it simultaneously relative to all possible parameters and interweave that with lowness requirements to obtain the uniformly low set $\mathcal{Z}$.

We start with the description of the construction. Let $\{X[s]\}_{s<\omega}$ and $\{Y[s]\}_{s<\omega}$ be uniformly computable sequences of finite sets.

Definition 7. *A Δ^0_2 approximation $\{X[s]\}_{s<\omega}$ to a set X is called a low approximation, if it has the following property: for every Turing functional Ξ and natural number x if there are infinitely many stages s such that $\Xi^X(x)[s]\downarrow$ then $\Xi^X(x)\downarrow$.*

We construct c.e sets A_i, Δ^0_2 sets B_i and C_i and uniformly computable sequences $\{G_i[s]\}_{s<\omega}$ so that if $\{X[s]\}_{s<\omega}$ is a low Δ^0_2 approximation to a set X and $\{Y[s]\}_{s<\omega}$

is a Δ^0_2 approximation to a set $Y \not\leq_T X$ then $\{G_i[s]\}_{s<\omega}$ are Δ^0_2 approximations to sets G_i, where $i = 1, 2$, such that the following requirements are satisfied:

(1) Λ_i: There is a Turing functional Λ_i such that $G_i = \Lambda_i^{A_i}$.
(2) Γ_i: There is a Turing functional Γ_i such that $C_i = \Gamma_i^{B_i,G_i}$.
(3) Ω: There is a Turing functional Ω such that $X = \Omega^{G_1,G_2}$.
(4) Let $(\Theta_e, \Phi_e)_{e<\omega}$ be a listing of all pairs of Turing functionals. For every e we have the requirement:
$\mathcal{R}^i_e$: If $\Theta_e^{\Phi_e^{A_i},B_i} = C_i$ then there is a functional Δ_e such that $G_i = \Delta_e^{\Phi_e^{A_i}}$.
(5) Let $(\Psi_e)_{e<\omega}$ be a listing of all Turing functionals. For every e we have the requirement:
$\mathcal{Q}_e : \Psi_e^{G_1,G_2} \neq Y$.

6.1. **Description of strategies.** Suppose that $\{X[s]\}_{s<\omega}$ is a low Δ^0_2 approximation to a set X and $\{Y[s]\}_{s<\omega}$ is a Δ^0_2 approximation to a set $Y \not\leq_T X$. The construction is in stages. We start stage s by visiting the global strategies: Λ_0, Λ_1, Γ_0, Γ_1 and Ω.

The Λ-strategies. The Λ_i-strategy is a simple marker strategy. During the construction we dynamically assign markers $\lambda_i(n)$ to every natural number n. At the beginning of a stage we check Λ_i for errors. If $\Lambda_i^{A_i}(n)\uparrow$ then we define it to be equal to $G_i(n)$ with use $A_i \upharpoonright \lambda_i(n)$, where $\lambda_i(n)$ is a new fresh number. A fresh number is defined as follows: let N be the largest number mentioned so far in the construction; a fresh number is larger than 2N. This way we not only use a new number every time, but also leave enough space between any two numbers that are used. Here we mean $A \upharpoonright x$ to be the initial segment of A of length $x + 1$. During the construction, whenever we change the value of $G_i(n)$, we will automatically also enumerate the marker $\lambda_i(n)$ in A_i, so that whenever $\Lambda_i^{A_i}(n)$ is defined, it is consistent with $G_i(n)$. This will be the only reason that A_i changes. So, as long as $G_i(m)$ changes only finitely many times for every $m \leq n$ we will have that $\Lambda^{A_i}(n)$ is eventually defined.

The Γ-strategies. The strategy for the construction of Γ_i is slightly more complicated for the following reason. The Δ^0_2 sets B_i and G_i can change many times, not necessarily only in response to changes in the set C_i. We cannot raise the value of the use of an element up every time this happens, or else we risk to make Γ_i partial. Instead to every element c we will attach two markers $g_i(c)$ and $\gamma_i(c)$. The marker $g_i(c)$ will serve as the G_i-use of c. It is usually set to 0, however an $\mathcal{R}^i_e$-strategy can preemptively define its value to be a larger number once during the construction, before any Γ_i computations have been defined for c. The marker $\gamma_i(c)$ is the B_i-use of c and it is raised only if we observe a change in $G_i \upharpoonright g_i(c)$ or a change in $C_i \upharpoonright c$. The Γ_i-strategy only defines new computations, it never changes the approximation to any set. At stage s it ensures that $\Gamma_i^{B_i,G_i}(c)$ is defined for all $c < s$. The strategies for $\mathcal{R}^i$ requirements will ensure that Γ_i is always correct.

The Ω-strategy. The strategy to define Ω will once again be able to move the use up every time we see that some computation is undefined. It will follow from the construction that G_1 and G_2 almost always change only in response to X changes. We will maintain that any Ω computation ever defined will use an initial segment of the oracle ending in 0. Thus enumerating the last bit of the use in the oracle set will invalidate the computation. At stage s we ensure that Ω^{G_1,G_2} is correctly defined for every $x < s$, by changing the approximation to G_1 or G_2 if necessary to invalidate wrong computations. Which set we change will depend on the desire to preserve the work of the highest possible number of $\mathcal{R}$-strategies. If $\Omega^{G_1,G_2}(n)\uparrow$ we define a new computation for n with a fresh Ω-use (also referred to as an Ω-marker) $\omega(n)$.

Now we turn to the two more difficult requirements $\mathcal{R}_e^i$ and $\mathcal{Q}_e$. We will handle them using a tree of strategies. We will denote $\mathcal{R}_e^i$ strategies by α (or α^i), and $\mathcal{Q}_e$ strategies by β. We order the requirements by priority linearly as follows:

$$\mathcal{R}_0^1 < \cdots < \mathcal{R}_e^1 < \mathcal{R}_e^2 < \mathcal{Q}_e < \mathcal{R}_{e+1}^1 \ldots$$

Strategies of level n in the tree are assigned the n-th requirement in the priority ordering. The branching in the tree is determined by the outcomes of the strategies. The lexicographical ordering of nodes in the tree induces a priority ordering of the strategies.

The $\mathcal{R}$-strategies. Suppose that α is working towards satisfying the requirement $\mathcal{R}_e^1$. We will drop the subscript e in the discussion below. The strategy α tries to establish and preserve a difference between $\Theta^{\Phi^{A_1},B_1}$ and C_1. At the same time it builds an operator Δ, so that if no diagonalization is possible then $\Delta^{\Phi^{A_1}} = G_1$.

The strategy will have three possible outcomes: $f <_L i <_L w$. The outcome f will signify that the strategy was successful in diagonalizing $\Theta^{\Phi^{A_1},B_1}$ against C_1. In order to preserve this diagonalization at further stages, the strategy will need to preserve some initial segment of A_1 and B_1. The preservation of B_1 can be achieved through initialization of lower priority strategies and the general rule that every time a strategy restarts its work, it deals with fresh numbers. The preservation of an initial segment of A_1 is more difficult, as changes in X, that are out of our control, result in changes in G_1 (done by the Ω-strategy or by the $\mathcal{Q}$-strategies), and therefore changes in A_1. In order to preserve A_1, the strategy will have to set things up as follows.

When α is first visited after initialization it only records the stage in a parameter s_0 and set a request to the Ω-strategy to preserve $G_1 \upharpoonright r_\alpha$, where $r_\alpha = \omega(s_0)$. Every time this restraint is violated, the strategy α will be initialized and will start its work from the beginning. Assuming that X is Δ_2^0, and that higher priority $\mathcal{R}$-strategies are not initialized infinitely often and stop raising their restraints, eventually Ω will start respecting α's restraints and α will not be initialized for this reason any longer.

This setup allows α to assume that the use in the Ω-computation of s_0, denoted by $\omega(s_0)$, is fixed and $G_i \upharpoonright \omega(s_0)$, and hence $A_1 \upharpoonright \omega(s_0)$, do not change at further stages. The strategy α will be designed so that it does not change the sets B_1 or C_1 on numbers less than s_0. This will be important to ensure the uniform lowness of all parameters B_i and C_i. The strategy also selects a threshold d_α as the least number which is greater than $\omega(s_0)$ and not in G_2: $\omega(s_0)+1$. Note that by the way that we select fresh numbers, d_α is not an ω-marker and is smaller than the ω-marker of s_0+1. Now α can preserve an initial segment of G_1 and of A_1 by enumerating the threshold d_α in G_2. This threshold will be essential in the proof of the correctness of the Γ_1-strategy.

Next α inspects the greatest common initial segment of $\Theta^{\Phi^{A_1},B_1}$ and C_1. Its length is called the *length of agreement*. It compares this length with the greatest one observed at a previous *true* stage, i.e. stage at which the strategy is visited and activated. If the length of agreement is bounded then $\Theta^{\Phi^{A_1},B_1} \neq C_1$. So, while the current length of agreement is smaller than any of the previous ones the strategy has outcome w.

If the length of agreement has grown, we say that the stage is *expansionary.* At expansionary stages the strategy first searches for a possibility for diagonalization, and if none is found progresses on the construction of Δ. As these actions will be related we start by explaining the method by which α constructs the operator Δ. The strategy is equipped with an infinite computable set $\mathcal{C}$ of natural numbers, called *the list of chits*[1]. The chits for different strategies are disjoint. The chits are used to define computations in Δ. Whenever we want to define a new value for $\Delta^{\Phi^{A_1}}(n)$ we first assign a new chit $c(n) \notin C_1$ to the number n, wait until we see that $\Theta^{\Phi^{A_1},B_1}(c(n))$ is defined and then use the same initial segment of Φ^{A_1} to define $\Delta^{\Phi^{A_1}}(n)$. So for every computation in Δ, we have that if $\Delta^{\tau_\Phi}(n)\downarrow$ then there is some chit $c(n)$ and some finite binary string τ_B such that $\Theta^{\tau_\Phi,\tau_B}(c(n))\downarrow= 0$. We will say that the computation of $\Delta(n)$ is defined via the triple $(c(n), \tau_\Phi, \tau_B)$. If $\Delta^{\Phi^{A_1}}(n)$ is defined and is not equal to $G_1(n)$, then we can diagonalize by preserving G_1 and A_1, ensuring $\tau_B \subseteq B_1$ and enumerating $c(n)$ into C_1.

In order to use this plan for diagonalization, we need to ensure that it is not in conflict with the strategy Γ_1, the strategy that is trying to prove that C_1 is computable from $G_1 \oplus B_1$ via Γ_1. For this reason the construction will be designed so that every computation for $c(n)$ in Γ_1 which has B_1-use smaller than $|\tau_B|$ assumes the same value of $G_1(n)$, namely $\Delta^{\tau_\Phi}(n)$. This is where the marker $g_1(c(n))$ will be used. When a new chit is assigned to n, it is selected to be a number that Γ_1 has not yet been defined on and its marker $g_1(c(n))$ is set to be larger than n. If $G_1(n)$ changes before this chit is used to define a computation in Δ, we will cancel it and pick a new chit for n. Thus when we diagonalize via a chit $c(n)$, the computations in Γ_1, defined before $c(n)$ is used to define Δ, are invalid on account of G_1. Any remaining valid computations in Γ_1 for $c(n)$ can be invalidated with a change in B_1 above the initial segment that needs to be preserved. This second change will be at a number that we denote as $b_{c(n)}$.

A final point to take into consideration is the impact of higher priority strategies. If a higher priority $\mathcal{R}_1$ strategy α' acts to diagonalize and changes the approximation to B_1, this might not take into account the fact that a lower priority strategy α is counting on some $b_c \in B_1$ for a chit c that it used to diagonalize. To remedy this we will maintain that $B_1(b_c) = C_1(c)$, i.e. α' will be responsible to change the value at $C_1(c)$ if necessary to keep the Γ-computations correct.

The actions of α at an expansionary stage are therefore as follows: First it examines the definition of Δ_α, searching for a number n with $\Delta^{\Phi^{A_1}}(n) \neq G_1(n)$. If it finds such a number then it uses the chit $c(n)$ to diagonalize and sends a request to the Ω-strategy r_α, asking that $G_1 \upharpoonright r_\alpha$ is preserved at further stages. The strategy has outcome f, until (if ever) it is injured by a higher priority strategy. If no such number is found, the strategy considers all numbers less than the current stage, defines chits for them and if possible new values for the Δ. In this case it has outcome i.

The $\mathcal{Q}$-strategies. Let β be a $\mathcal{Q}$-strategy working on the requirement $\mathcal{Q}_e$. We drop the index e in the discussion below. The strategy β will have one outcome f.

[1]The word "chit" is borrowed from [2]. Another commonly used term for similar concepts is "agitator".

The strategy will cancel all lower priority strategies if it acts. In order to diagonalize the strategy will try to build a Turing functional Ξ such that $\Xi^X = Y$. As $Y \not\leq_T X$ we know that this attempt will fail.

Just like the $\mathcal{R}$-strategies, β records the first stage it is visited after initialization in a parameter s_0. It then waits until a stage s_1, such that Ω has correctly defined its functional on numbers $x < s_0$. This will allow the Ω-strategy sufficient room to ensure its functional is total. At stage s_1 the strategy starts its real work. It monitors the length of agreement between Ψ^{G_1,G_2} and Y. At expansionary stages it translates the computation of $\Psi^{G_1,G_2}(n)$ into a computation $\Xi^X(n)$ by using an initial segment of X that is long enough, as to ensure that if X does not change on this initial segment, then neither will G_1 or G_2. The idea is to argue that if the length of agreement between Ψ^{G_1,G_2} and Y is unbounded then, so is the length of agreement between Ξ^X and Y. But since X is low as witnessed by the given approximation, this would mean that $\Xi^X = Y$, contradicting $Y \not\leq_T X$. Thus the length of agreement will eventually remain bounded and the impact of this strategy on the whole construction would be finitary.

As X is Δ_2^0 we are faced with the following difficulty: a computation $\Xi^X(n)$ can be invalidated via a change in X. This results in a change in $G_1 \oplus G_2$ and possibly a new computation in $\Psi^{G_1,G_2}(n)$. Later on X can revert to its old state, breaking the connection between computations in Ψ and Ξ. To restore this connection, we must change G_1 and G_2 back to their old state as well.

The strategy keeps a list $\mathcal{P}$ of promises. A promise has the form $\langle \tau_X, \tau_G \rangle$ and means that we promise: if $\tau_X \preceq X$ then $\tau_G \preceq G_1 \oplus G_2$. During any visit (after stage s_1) the first action of the strategy is to ensure that all promises are kept by changing the approximation to $G_1 \oplus G_2$ according to the promise list and the current approximation to the set X.

Next β examines the length of agreement between Ψ^{G_1,G_2} and Y. If the stage is not expansionary, it does nothing further. If the stage is expansionary, the strategy defines Ξ on all elements x below the length of agreement, adding new entries to the list of promises.

If X and Y are not as expected. We take a moment to consider happens with the construction if $\{X[s]\}_{s<\omega}$ is not a low Δ_2^0 approximation to a set X, or if $\{Y[s]\}_{s<\omega}$ is not a Δ_2^0 approximation to a set $Y \not\leq_T X$. The danger that this situation poses to our construction is that some $\mathcal{Q}$-strategy may act infinitely often, or that the approximation to $\{X[s]\}_{s<\omega}$ changes infinitely often at a particular number n. The effect of this will be that all but finitely many $\mathcal{R}$-strategies are initialized infinitely often. This means that their parameter s_0 will grow unboundedly and such strategies will only be allowed modify the sets B_i and C_i on larger and larger numbers. This scenario will not interfere with the requirement that the degrees of B_i and C_i belong to a uniformly low set.

6.2. **Construction.** We combine the ideas described above in a formal construction. We have global strategies : Λ_0, Λ_1, Γ_0, Γ_1 and Ω and a tree $T \preceq \{f, i, w\}^{<\omega}$ of $\mathcal{R}$ and $\mathcal{Q}$ strategies. Every strategy α in the tree T is given higher priority than all its successors and all nodes to the right of it in T.

To every strategy we attach parameters as follows:

(1) Λ_i and Ω have one parameter each: the functional that they are building. They also assign markers $\lambda_i(n)$ and $\omega(n)$ for all natural numbers n.

(2) Γ_i, in addition to its functional, assigns two types of markers to all numbers n: $g_i(n)$ tells us how much of G_i to use in a computation for n and $\gamma_i(n)$ tells us how much of B_i to use in a computation for n.
(3) An $\mathcal{R}^i_e$-strategy α has the following parameters: $s_0(\alpha)$, the stage when α started work after being initialized; r_α the restraint that it requests of the Ω-strategy; a threshold d_α, used to ensure that G_i and A_i can be restrained; a functional Δ_α; numbers l^-_α and l_α recording the development of length of agreement between $\Theta_e^{\Phi_e^{A_1},B_1}$ and C_i; a list of chits $\mathcal{C}_\alpha$; and a request to the Ω-strategy r_α, asking for no more changes in G_1 on numbers $y < r_\alpha$. In addition the strategy will dynamically assign chits $c(n)$ to natural numbers n and record the stage at which this happened in $s_{c(n)}$. If $c(n)$ is ever enumerated in C_i, we attach to it one of its Γ_i-markers $b_{c(n)}$. When α is *initialized* all parameters become undefined.
(4) A $\mathcal{Q}_e$ strategy β has the following parameters: $s_0(\beta)$, the stage when β started work after being initialized; $s_1(\beta)$ the stage after the last time β is restarted; a list of promises $\mathcal{P}_\beta$; a functional Ξ_β; numbers l^-_β and l_β, recording the development of the length of agreement between $\Psi_e^{G_1,G_2}$ and Y. When β is *initialized* all parameters become undefined. When β is *restarted* all parameters except $s_0(\beta)$ become undefined.

Construction:
At stage 0 we initialize all strategies and set $A_i = B_i = C_i = G_i = \emptyset$. At stage $s \geq 0$ all parameters inherit their values from the previous stage unless they are explicitly modified during stage s. We start stage s by visiting the global strategies:

Step I: Ω*:* Scan all $x < s$.

(1) If $\Omega^{G_1,G_2}(x)\uparrow$ then let $\omega(x)$ be a fresh number and set $\Omega^{G_1,G_2}(x) = X(x)$ with use $(G_1 \restriction \omega(x)) \oplus (G_2 \restriction \omega(x))$ and move to $x+1$. Otherwise go to 2.
(2) If $\Omega^{G_1,G_2}(x)\downarrow\neq X(x)$ then α be the highest priority strategy such that $r_\alpha > \omega(x)$. If α is an $\mathcal{R}^i_e$ strategy then enumerate $\omega(x)$ in G_{1-i} and enumerate $\lambda_{1-i}(\omega(x))$ in A_{1-i}. Initialize all α-strategies whose requests are not kept. Go to step 1.

Step II: Λ_i*:* Scan all elements $n < s$. If $\Lambda_i^{A_i}(n)\uparrow$ then let $\lambda_i(n)$ be a fresh number and define $\Lambda_i^{A_i}(n) = G_i(n)$ with use $A_i \restriction \lambda_i(n)$.

Step III: Γ_i*:* Scan all $c < s$. If $\Gamma_i^{B_i,G_i}(c)\uparrow$ then first check if any of the following is true: $\gamma_i(c)\uparrow$; there is a number $d \leq c$ such $C_1(d)[s-1] \neq C_1(d)[s]$; there is a number $n < g_i(c)$, such that $G_i(n)[s] \neq G_i(n)[s]$. If so, then let $\gamma_i(c)$ be a fresh number. Otherwise do not change the value of $\gamma_i(c)$. Set $\Gamma_i^{B_i,G_i}(c) = C_i(c)$ with use $(B_i \restriction \gamma_i(c), G_i \restriction g_i(c))$. If $g_i(s)\uparrow$, set $g_i(s) = 0$.

Step IV: Construction of δ*:* We construct a finite path $\delta[s]$ in T. The path $\delta[s]$ is defined inductively. We set $\delta[s] \restriction 0 = \emptyset$. Once we have constructed $\delta[s] \restriction k$, we check if $k = s$ or if the strategy $\delta[s] \restriction k$ ends the current stage. If so, then $\delta[s] = \delta[s] \restriction k$. Otherwise the strategy produces an outcome o, and $\delta[s] \restriction k+1 = (\delta[s] \restriction k)\hat{\ }o$. We have two cases depending on the type of the strategy $\delta[s] \restriction k$:

Case $\mathcal{R}^i_e$*:* Suppose that $\delta[s] \restriction k$ is an $\mathcal{R}^1_e$-strategy α. Let s^- be the previous stage at which α was visited and o^- be the outcome that α had at stage s^-. (If α has never been visited, or if α was just initialized then $s^- = 0$ and $o^- = w$). Pick the first case which applies to α:

(1) If $s_0(\alpha)$ is not defined then set $s_0(\alpha) = s$. End this stage.
(2) If r_α is not defined then set $r_\alpha = \omega(s_0(\alpha))$ and $d_\alpha = \omega(s_0(\alpha)) + 1$. Set $\mathcal{C}_\alpha = \{\langle\hat{\alpha}, n\rangle \mid n > s\}$ where $\hat{\alpha}$ is the code of α in some fixed computable coding of all nodes on the tree of strategies. End this stage.
(3) If $o^- = f$ then let the outcome be f.
(4) Let l^-_α be the greatest element of the set $\{0\} \cup \{l_\alpha[t] \mid s_1(\alpha) \leq t < s$ and t is an α true stage$\}$ and l_α be the maximal common initial segment of $\Theta_e^{\Phi_e^{A_1}, B_1}[s]$ and $C_1[s]$. We assume that if $\Theta^{\Phi^{A_1}, B_1}(n)[s] \downarrow$ then $n < s$, $\theta(n) < s$ and for all $m < n$ $\Theta^{\Phi^{A_1}, B_1}(m)[s] \downarrow$. If $l^-_\alpha \geq l_\alpha$ then let the outcome be w.
(5) If there is an element n, such that $\Delta_\alpha^{\Phi_e^{A_1}}(n) \downarrow \neq G_1(n)$ then pick the least such n. Let $\Delta_\alpha^{\Phi_e^{A_1}}(n) \downarrow$ via the triple (c, τ_Φ, τ_B). Enumerate c in C_1 and make $\tau_B \preceq B_1$, by changing the approximation to B_1. Let b_c be the least marker $\gamma_1(c)$ that is larger than $|\tau_B|$. Enumerate b_c in B_1. Note, that in addition to b_c we will only change B_1 on numbers that lower priority strategies have used for their own chits. For every number n such that $s_0(\alpha) < n < |\tau_B|$ if $n = b_{c'}$ for some chit $c' > s_0(\alpha)$ (belonging to a different strategy), then set $C_1(c') = B_1(b_{c'})$. (This will ensure that Γ^{G_1, B_1} and C_1 agree on c'). Enumerate d_α in G_2 and $\lambda_2(d_\alpha)$ in A_2. Initialize all lower priority strategies and let the outcome be f.
(6) Scan all $n \leq s$ and perform the following actions for each n:

If $c(n) \uparrow$ or if $G_1(n)[s] \neq G_1(n)[t]$ at some stage $t \geq s_{c(n)}$ then let $c(n)$ be a fresh number in $\mathcal{C}_\alpha$ (one which Γ_1 has not yet interacted with) and set $g_1(c(n)) = n + 1$ and $s_{c(n)} = s$. If $\Delta_\alpha^{\Phi_e}(n) \uparrow$ and $c(n) < l_\alpha$ then let $\tau_\Phi = \Phi_e^{A_1} \upharpoonright \theta(c(n))$ and $\tau_B = B_1 \upharpoonright \theta(c(n))$. Define $\Delta_\alpha^{\Phi_e}(n) = G_1(n)$ with use τ_Φ and say that this computation is defined via the triple $(c(n), \tau_\Phi, \tau_B)$.

Once all elements are scanned α ends with outcome i.

If $\delta[s] \upharpoonright k$ is an $\mathcal{R}^2_e$-strategy, the instructions are the same as above with G_1, A_1, B_1, C_1 swapped with G_2, A_2, B_2, C_2.

Case $\mathcal{Q}_e$: Suppose that $\delta[s] \upharpoonright k$ is a $\mathcal{Q}_e$-strategy β. Pick the first case which applies to β:

(1) If $s_0(\beta)$ is not defined then set $s_0(\beta) = s$. End this stage.
(2) If $s_1(\beta)$ is not defined then set $s_1(\beta) = s$. End this stage.
(3) If $X \upharpoonright s_0(\beta)$ changed at a stage t such that $s_0(\beta) < t \leq s$ then restart β. End this stage.
(4) If there is a promise $\langle\tau_X, \tau_G\rangle \in \mathcal{P}_\beta$ which is not currently kept: $\tau_X \preceq X$ and $\tau_G \not\preceq (G_1 \oplus G_2)$, then change the value of $(G_1 \oplus G_2)$ to make $\tau_G \preceq G_1 \oplus G_2$ (enumerate the corresponding Λ-uses in A_1 or A_2). End this stage.
(5) Let l^-_β be $\max(\{0\} \cup \{l_\beta[t] \mid s_1(\beta) \leq t < s$ and t is β-true$\})$ and l_β be the maximal common initial segment of $\Psi_\beta^{G_1, G_2}[s]$ and $Y[s]$. Here we similarly assume that if $\Psi_\beta^{G_1, G_2}(x)[s] \downarrow$ then $x < s$, $\psi(x) < s$ and for all $y < x$ $\Psi_\beta^{G_1, G_2}(y)[s] \downarrow$. If $l_\beta > l^-_\beta$ then scan all elements $x < l_\beta$ and perform the following actions for each:

If $\Xi^X(x)\uparrow$ then we define it as $Y(x)$ with use $X \upharpoonright \xi(x)$, where $\xi(x) > \psi(x)$ is a fresh number. Enumerate in $\mathcal{P}_\beta$ the promise $\langle X \upharpoonright \xi(x), (G_1 \oplus G_2) \upharpoonright \psi(x)\rangle$.

End this stage.

(6) If none of the above cases hold, β has outcome f.

We end a stage s by canceling all strategies of lower priority than $\delta[s]$ and proceed to stage $s+1$.

End of construction

6.3. **Uniformly low parameters.** We have described the construction given a particular pair of approximations $\{X[s]\}_{s<\omega}$ and $\{Y[s]\}_{s<\omega}$. Note that the construction can be viewed as a function $\mathcal{K}$ which takes as input the stage s and outputs $A_i[s+1], B_i[s+1], C_i[s+1], G_i[s+1]$ and the stage-$s+1$-values of all parameters for all strategies. Even if the given sequences of sets $\{X[s]\}_{s<\omega}$ and $\{Y[s]\}_{s<\omega}$ do not have the expected properties, the function $\mathcal{K}$ is well defined. We combine all of these constructions together with the additional requirement that the obtained sequence consisting of all possible constructed sets B_i and C_i is uniformly low.

For every $j = \langle j_x, j_y\rangle$ let $\{X_j[s]\} = \{j_x\}^{\emptyset'}[s]$ and $\{Y_j[s]\} = \{j_y\}^{\emptyset'}[s]$. We will denote the construction relative to $\{X_j[s]\} = \{j_x\}^{\emptyset'}[s]$ and $\{Y_j[s]\} = \{j_y\}^{\emptyset'}[s]$ as $\mathcal{K}_j$. We would like to ensure that in addition $Z = \bigoplus_{j<\omega} B_{j,1} \oplus B_{j,2} \oplus C_{j,1} \oplus C_{j,2}$ is low.

Let $\{\Upsilon_i\}$ be a listing of all Turing functionals. For every $e = \langle i, x\rangle$ we will have the requirement $\mathcal{L}_e$: if there are infinitely many stages s such that $\Upsilon_i^Z(x)[s]\downarrow$ then $\Upsilon_i^Z(x)\downarrow$.

The requirement $\mathcal{L}_e$ is in conflict with $\mathcal{R}$-strategies from the construction $\mathcal{K}_j$, as these strategies modify the sets that the requirement wishes to preserve. $\mathcal{L}_e$ will be given higher priority than any strategy α in $\mathcal{K}_j$ where $j \geq e$. Whenever $\mathcal{L}_e$ injures these strategies, it will initialize them. In addition at stage s, it will have higher priority than $\mathcal{R}$-strategies α in $\mathcal{K}_j$ with $j < e$, such that the parameter $s_0(\alpha)$ is currently larger than e. Whenever $\mathcal{L}_e$ injures these strategies, it will only restart them. This means that they will keep the value of the parameter s_0, and hence their priority with respect to other $\mathcal{L}$-requirements, but restart their work from the beginning and so preserve whatever computation $\mathcal{L}_e$ would like them to.

If X_j and Y_j are not as expected then $\mathcal{R}$-strategies in $\mathcal{K}_j$ can be initialized infinitely often, giving more and more $\mathcal{L}$-strategies higher priority. If X_j and Y_j are as expected then there will be an infinite branch in the tree T_j, *the true path*, such that $\mathcal{R}$-strategies along it can be initialized only finitely many times and restarted only finitely many times. We will show that they will ultimately succeed.

Construction:

We will say that $\mathcal{L}_e$ requires attention at stage s if $\mathcal{L}_e$ is not satisfied at stage s and $\Upsilon_i^Z[s](x)\downarrow$. Initially all requirements $\mathcal{L}_e$ are not satisfied.

At stage s check if there is a requirement $\mathcal{L}_e$ with $e < s$ which requires attention at stage s and if there is, pick the least one. For all $j < e$ restart all $\mathcal{R}$-strategies α in the tree T_j which currently have $s_0(\alpha) \geq e$. When a strategy α is *restarted* all parameters except $s_0(\alpha)$ become undefined. For all $j \geq e$ initialize all strategies in T_j. Announce that $\mathcal{L}_e$ is satisfied.

Run the constructions $\mathcal{K}_j$ for all $j < s$. Let $Z[s+1] = \bigoplus_j B_{j,1} \oplus B_{j,2} \oplus C_{j,1} \oplus C_{j,2}[s+1]$. If $\mathcal{L}_e$ is a strategy that was satisfied, $e = \langle i, x\rangle$ and $\Upsilon_i^Z(x)\uparrow$ or the computation $\Upsilon_i^Z(x)$ has changed since the previous stage then announce that $\mathcal{L}_e$ is not satisfied.

End of Construction

6.4. **Verification.**

Lemma 6.1. *If an $\mathcal{R}$-strategy α modifies the approximation to B_i or C_i it is on a number larger than the stage at which α was last restarted.*

Proof. After a strategy is restarted and before it starts its work, it goes through Case (2) in the construction, redefining its list of chits to all be larger than the current stage and initializing all lower priority strategies. When in Case (5) it modifies the construction of B_i it is to make some binary string τ_B that was an initial segment of B_i after the restart, once again an initial segment of B_i. Higher priority strategies cannot have caused τ_B to not be an initial segment of B, as that would initialize this strategy. So a difference can only be caused by be lower priority strategies, that are enumerating markers b'_c for their own chits c'. These are defined after the stage of the restart and hence are larger than the stage of the restart. □

Lemma 6.2. *For every e, $\mathcal{L}_e$ is satisfied and requires attention finitely often.*

Proof. Fix $e = \langle i, x\rangle$. Assume that the statement is true for $e' < e$. For every $j < e$ there are finitely many pairs of $(s_0(\alpha_j), \alpha_j)$ such that α_j is an $\mathcal{R}$-strategy in $\mathcal{K}_j$ with $s_0(\alpha_j)[s] < e$ at some stage s. For each such pair the strategy α_j can change the approximation to $B_{j,i}$ and $C_{j,i}$ only once on numbers greater than $s_0(\alpha)$ when it performs an attack under Case (6). In order for it to perform a new attack it must be restarted by a higher priority $\mathcal{L}$-strategy or initialized. By the induction hypothesis there will be a stage after which these strategies are not restarted as strategies $\mathcal{L}_{e'}$ of higher priority do not require attention. If a strategy α is initialized then it will have a different larger value of the parameter $s_0(\alpha)$ and so this too can happen only finitely often before this value becomes larger than e. Suppose that s is a stage after all of these finitely many strategies have stopped changing the approximation to $B_{j,i}$ and $C_{j,i}$ and after all $\mathcal{L}_{e'}$ where $e' < e$ have stopped requiring attention. If at all $t \geq s$ we have $\Upsilon_i^Z[t]\uparrow$ then $\mathcal{L}_e$ is satisfied and does not require attention after stage s.

If at stage $t \geq s$ we see that $\Upsilon_i^Z[t]\downarrow$ then we restart all strategies with $s_0(\alpha) \geq e$ in T_j, where $j < e$ and all strategies in T_j for $j \geq e$. These strategies are activated again after stage t and by Lemma 6.1 can only change the approximation to Z on numbers larger than t. Thus the computation $\Upsilon_i^Z(x)[t]$ is preserved at all further stages and $\mathcal{L}_e$ does not require attention at any further stage. □

Let $\mathcal{K}$ be the construction relative to a low Δ_2^0 approximation to a set X and a Δ_2^0 approximation to a set $Y \not\leq_T X$. We prove that the construction $\mathcal{K}$ satisfies all requirements in a series of lemmas. We start with a technical lemma that deals with $\mathcal{Q}$-strategies and their promises. Once we have established this lemma, we can show that the requirements Λ_i are satisfied and assuming that there is a true path that $\mathcal{Q}_e$ requirements are satisfied. Next we establish a technical fact about

$\mathcal{R}$-strategies, which in turn will allow us to prove that $\mathcal{R}$-strategies on the true path satisfy their requirements. We combine these statements to establish the true path and to show that Ω is successful. Finally we show that the Γ_i strategies succeed as well.

Lemma 6.3. *Suppose that β is a $\mathcal{Q}_e$-strategy, visited at stage s.*

(1) *The promises that β makes are consistent: i.e. if $\langle \tau_X, \tau_G\rangle, \langle \tau'_X, \tau'_G\rangle \in \mathcal{P}_\beta[s]$ and $\tau_X \preceq \tau'_X$ then τ_G and τ'_G are compatible.*
(2) *For every promise $\langle \tau_X, \tau_G\rangle \in \mathcal{P}_\beta[s]$ we have that $X[s] \upharpoonright s_0(\beta) \preceq \tau_X$ and $(G_1 \oplus G_2)[s] \upharpoonright \omega(s_0(\beta))$ is compatible with τ_G.*
(3) *If β at stage s changes the value of $G_1 \oplus G_2$ on a number o then o is the use in some Ω-computation and not a threshold.*
(4) *Suppose that $\langle \tau_X, \tau_G\rangle \in \mathcal{P}_\beta[s'] \cap \mathcal{P}_\beta[s]$, where $s' < s$. If β keeps this promise stage s' and $(X \upharpoonright |\tau_X|)[s'] = (X \upharpoonright |\tau_X|)[t]$ for all $t \in [s', s]$ then for all $t \in [s', s]$ we have $\tau_G \preceq (G_1 \oplus G_2)[t]$.*

Proof. (1): We always first ensure that all previous promises are kept before we make new promises. If $\langle \tau_X, \tau_G\rangle, \langle \tau'_X, \tau'_G\rangle \in \mathcal{P}_\beta$ and $\tau_X \preceq \tau'_X$ then $\langle \tau_X, \tau_G\rangle$ was made at stage s^0 before the promise $\langle \tau'_X, \tau'_G\rangle$ was made at stage s'. As τ_X and τ'_X are compatible, it follows that $\tau_G \preceq (G_1 \oplus G_2)[s_2]$: either $s^0 = s'$ or $s^0 < s'$ and Case (4) does not apply to β at stage s' i.e. all promises are kept. As τ'_G is also selected as an initial segment of $(G_1 \oplus G_2)[s']$, it follows that τ_G and τ'_G are compatible.

(2): Suppose $\langle \tau_X, \tau_G\rangle \in \mathcal{P}_\beta[s]$ is a promise made at stage s'. Then at stages $t \in [s', s]$ the strategy β was not initialized or restarted or else $\mathcal{P}_\beta[s']$ would be cancelled. Hence $X[s'] \upharpoonright s_0(\beta) = X[s] \upharpoonright s_0(\beta)$ and $(G_1 \oplus G_2)[s'] \upharpoonright \omega(s_0(\beta)) = (G_1 \oplus G_2)[s] \upharpoonright \omega(s_0(\beta))$. As τ_X is selected as an initial segment of $X[s']$ of length larger than $s_0(\beta)$ and τ_G is selected as an initial segment of $(G_1 \oplus G_2)[s']$, the statement follows.

(3): Suppose that at stage s the strategy β changes the value of $G_1 \oplus G_2$ at a number o. Then o is part of a promise made by β at some previous stage $s' < s$ and β is not initialized at stages in the interval $[s', s]$. We have that $o < s'$ and there is a least intermediate stage t: $s' < t \leq s$ at which the value of $G_1 \oplus G_2$ is changed at o by a strategy γ. Strategies that change G_1 or G_2 are Ω, $\mathcal{R}$-strategies or $\mathcal{Q}$-strategies. As β is not initialized at stage t, γ cannot be a higher priority $\mathcal{Q}$- or $\mathcal{R}$-strategy. All lower priority strategies are initialized at stage s' and hence do not change G_1 or G_2 on elements less than s' at further stages. This follows by construction for $\mathcal{R}$-strategies and by the previous statement in this Lemma for $\mathcal{Q}$-strategies. This leaves Ω as the only possibility and so o is the use in some Ω-computation.

(4): Finally let $\langle \tau_X, \tau_G\rangle \in \mathcal{P}_\beta[s'] \cap \mathcal{P}_\beta[s]$, where $s' < s$ and suppose that this promise was made at stage $s^0 \leq s'$. At stage s^0 we know that Ω^{τ_G} is an initial segment of $X[s^0]$ and by our choice of uses for Ξ-computations (as fresh numbers) it follows that $\Omega^{\tau_G} \preceq \tau_X$. At stage s' we keep this promise, hence $\Omega^{\tau_G} \preceq \tau_X \preceq X[s']$. By the previous part of this Lemma it follows that only the Ω-strategy can initiate a series of changes in $G_1 \oplus G_2$ at a promised number at a stage $t \in (s', s]$. It will do so only if $\Omega^{\tau_G} \not\preceq X[t]$. But since by assumption $(X \upharpoonright |\tau_X|)[t] = (X \upharpoonright |\tau_X|)[s'] = \tau_X$ it follows that τ_G remains an initial segment of $(G_1 \oplus G_2)$ at all stages in the interval $[s', s]$. □

Corollary 6.4. *G_i is Δ^0_2 and $\Lambda_i^{A_i} = G_i$.*

Proof. Fix a number n in G_i. The value of $G_i(n)$ changes at most once if n is a threshold for an $\mathcal{R}$ strategy. If n is an ω marker $o(x)$ then by Lemma 6.3 the number of times that $G_i(n)$ can change is bounded by the number of times $X \restriction x$ can change. This gives us that G_i is Δ^0_2. Assume inductively that $\Lambda_i^{A_i}(m) \downarrow= G_i(m)$ for $m < n$. Every time $G_i(n)$ changes the previous Λ_i-computation is invalidated forever as the use $\lambda_i(n)$ is enumerated in A_i. Let s be a stage such that $G_i \restriction n$ does not change at further stages. Then $\lambda_i(n)$ does not change at further stages and so $\Lambda_i(n) \downarrow= G_i(n)$. □

Lemma 6.5. *Let β be a $\mathcal{Q}_e$-strategy which is not initialized after stage s_0 and visited infinitely often. Then $\mathcal{Q}_e$ is satisfied and there is a stage s_β after which β does not end stages at which it is visited.*

Proof. After stage s_0 the strategy β has a fixed parameter $s_0(\beta)$. As X is Δ^0_2 there will be a least stage $s_1 \geq s_0$ after which $X \restriction s_0(\beta)$ does not change. At the next β-true stage after s_1 the parameter $s_1(\beta)$ attains its final value. After stage $s_1(\beta)$ Cases (1), (2) and (3) do not apply for β.

Suppose that there are infinitely many β-true expansionary stages, i.e. stages at which Case (5) applies to β. We show that $\Xi^X = Y$, contradicting our choice of X and Y. Fix a natural number x. Let $s_x > s_1(\beta)$ be the β-true stage when $\Xi^X(x)$ is first defined. We will show that at every expansionary stage $t \geq s_x$ we have that $\Xi^X(x)[t] \downarrow= Y(x)[t]$. At stage s_x this is true by construction. Suppose that it is true at all β-true expansionary stage $s_x < t < s$ and consider s. If at stage s, $\Xi^X(x) \uparrow$ then we define it to be equal to $Y(x)[s]$. Suppose that $\Xi^X(x)[s] \downarrow$. Then this computation was defined at a previous expansionary stage t, such that $s_x \leq t < s$ using the computation $\Psi_e^{G_1,G_2}(x)[t]$. At stage t the promise $\langle X \restriction \xi(x)[t], (G_1 \oplus G_2) \restriction \psi_e(x)[t] \rangle$ was made. At stage s the construction ensures that all promises are kept. As $(X \restriction \xi(x))[t] \preceq X[s]$, it follows that $(G_1 \oplus G_2) \restriction \psi_e(x)[t] \preceq G_1 \oplus G_2[s]$. Hence:

$$\Xi^X(x)[s] = \Xi^X(x)[t] = \Psi_e^{G_1,G_2}(x)[t] = \Psi_e^{G_1,G_2}(x)[s] = Y(x)[s].$$

By our choice of low approximation to X and the fact that Y is Δ^0_2 it follows that $\Xi^X(x) \downarrow= Y(x)$.

We have shown that there can be only finitely many expansionary stages. Let s_e be the last β-true expansionary stage. Then for every stage $t > s_e$ the set of promises does not change: $\mathcal{P}_\beta[t] = \mathcal{P}_\beta[s_e]$ and the value of l_β^- does not change: $l_\beta^-[t] = l_\beta[s_e]$. Denote these final values by $\mathcal{P}_\beta$ and l_β^- respectively. Let n be the length of the largest τ_X in a promise $\langle \tau_X, \tau_G \rangle \in \mathcal{P}_\beta$. Let s_X be the least stage after s_e such that $X \restriction n$ does not change after stage s_X. There are finitely many promises $\langle \tau_X, \tau_G \rangle \in \mathcal{P}_\beta$ with $\tau_X \preceq X$. By part (1) of Lemma 6.3 they are all compatible. Suppose that at stage $s \geq s_X$ the strategy β keeps a promise $\langle \tau_X, \tau_G \rangle$ by passing through Case (4). By part (4) of Lemma 6.3 it follows this promise is kept at all further stages. Thus after finitely many passes through Case (4) all of these promises are kept forever. Let $s_\beta \geq s_e$ be the least stage such that Case (4) does not apply to β at any further stage. Then at stages $t > s_\beta$, the strategy β always ends in Case (6), it does not end stages prematurely. Furthermore, at all stages $t > s_\beta$ we have that the length of agreement between $\Psi_e^{G_1,G_2}[t]$ and $Y[t]$ is bounded by l_β^-. It follows that $\Psi^{G_1,G_2} \neq Y$; the requirement $\mathcal{Q}_e$ is satisfied. □

Lemma 6.6. *Let α be an $\mathcal{R}_e^i$-strategy visited at stage s and suppose that α diagonalizes via Case (6) at stage s. Then $G_i(n)[t] = G_i(n)[s]$ for all $n \in [d_\alpha, s]$ and all $t \geq s$.*

Proof. Without loss of generality we assume that $i = 1$. The strategy α selects its threshold d_α at stage $s_1 < s$ as $(\omega(s_0(\alpha)) + 1)[s_1]$. The strategy is not initialized until at stage s it enumerates d_α in G_2. First we show that d_α remains in G_2 at all further stages. By construction when an $\mathcal{R}$-strategy selects the value of its parameter s_0 it ends the stage. Thus α is the unique strategy whose threshold is selected relative to the use of s_0. Note that d_α is not the use in any Ω-computation. Thus by Proposition 6.3 no $\mathcal{Q}$-strategy can extract d_α from G_2 at any further stage.

Let $n \in [d_\alpha, s]$ be a number and towards a contradiction assume that $G_1(n)[t] \neq G_1(n)[s]$, where $t > s$. The possible roles of n are - a threshold for an $\mathcal{R}$-strategy α', an ω-use and a member of a promise in a $\mathcal{Q}$-strategy β. We rule these out in turn: If n is a threshold for α' then it is selected at a stage in the interval $[s_1, s]$. As α is not initialized until stage s, α' cannot be of higher priority. If α' is of lower priority than α then it is initialized at stage s, its threshold is canceled and hence α' cannot change the value of G_1 at d_α. If n is the ω-use of some computation in Ω then this computation is defined in the interval $[s_1, s]$ and hence must involve the fact that $G_2(d_\alpha) = 0$ at these stages. After stage s this computation is invalid as $d_\alpha \in G_2$ and hence Ω will never need to change the value of $G_1(n)$. Finally suppose that β changes the approximation to $G_1(n)$ after stage s, due to a promise $\langle \tau_X, \tau_G \rangle$ with $\tau_G(2n) \neq G_1(n)[s]$. Assume that β is the first $\mathcal{Q}$-strategy to do so. Consider the stage t_0 when β made this promise. It must be that $t_0 > s_0(\alpha)[s_1]$ as $n \geq d_\alpha$. At this stage β observed that $G_1(n)$ has a different value, i.e. $G_1(n)[t_0] \neq G_1(n)[s]$ and this is not due to β's action. As we have already ruled out any other reason for such a change at stages greater than s, it follows that $t_0 \leq s$. Whenever β makes a new promise, it ends the stage thereby initializing all lower priority strategies, thus β cannot be of higher priority than α. But then β is initialized at stage s and its list of promises is emptied. □

Lemma 6.7. *Let α be an $\mathcal{R}_e^i$-strategy which is not initialized after stage s_0 and visited infinitely often. Then $\mathcal{R}_e^i$ is satisfied and there is a stage s_α after which α does not end stages at which it is visited.*

Proof. Again for concreteness we will assume that $i = 1$. After stage s_0 the parameter $s_0(\alpha)$ does not change and higher priority $\mathcal{R}$- and $\mathcal{Q}$-strategies do not make any further changes to any of the global parameters: A_i, C_i, B_i, and G_i. There will be a stage s_1 such that strategies $\mathcal{L}_i$ where $i < s_0(\alpha)$ do not require attention and α will not be restarted any longer. The parameters r_α, C_α, and $d_\alpha = \omega(s_0(\alpha))[s_1] + 1$ attain their final values and the Ω-strategy preserves α's restraint r_α, so $G_1 \upharpoonright d_\alpha - 1$ and hence $A_1 \upharpoonright d_\alpha$ do not change at any further stage. Similarly $B_i \upharpoonright s_0(\alpha)$ and $C_i \upharpoonright s_0(\alpha)$ do not change at any further stage. After stage s_1 Cases (1) and (2) do not apply for α.

Suppose that Case (3) applies for α at a least stage s_2. Then at all further stages $t \geq s_2$ α ends with outcome f and does not initialize lower priority strategies. Consider the α-true stage before s_2, call it s. At stage s Case (5) applies to α. There is an element n such that $\Delta_\alpha^{\Phi_e^{A_1}}(n) \downarrow$ via the triple (c, τ_Φ, τ_B). This computation was observed at a previous stage s', such that $s_0(\alpha) < s' < s$ and so $B_1[s] \upharpoonright s_0(\alpha) \preceq \tau_B$. At stage s α changes the value of B_1 on numbers larger than $s_0(\alpha)$ to ensure that

$\tau_B \preceq B_1[s]$. Thus $\Theta_e^{\Phi_e^{A_1},B_1}(c) \downarrow= 0$. By Lemma 6.6 $G_1 \upharpoonright s$ and hence $A_1 \upharpoonright s$ is preserved at all further stages. At stage s we initialize all lower priority strategies, so $B_1 \upharpoonright s$ is also preserved. Thus $\Theta_e^{\Phi_e^{A_1},B_1}(c)[s]$ is preserved at all further stages and $\Theta_e^{\Phi_e(A_1),B_1} \neq C_1$.

If Case (3) never applies for α after stage s_1 then neither does Case (5). Hence α does not end prematurely any true stage $t > s_1(\alpha)$.

Suppose that there is a stage s_4 such that at all α-true stages $t > s_4$ Case (4) applies for α. It follows that the value of l_α^- does not change after stage s_4 and hence at all $t > s_4$ there is a number $n < l_\alpha^-$ such that $(\Theta_e^{\Phi_e(A_1),B_1}(n))[t] \uparrow$ or $(\Theta_e^{\Phi_e(A_1),B_1}(n))[t] \downarrow\neq C_1(n)[t]$. Hence $\Theta_e^{\Phi_e(A_1),B_1}$ is not total or not equal to C_1 and $\mathcal{R}_e^1$ is satisfied.

Finally suppose that at infinitely many α-true stages Case (6) applies. We will show that if $\Theta_e^{\Phi_e^{A_1},B_1}$ is total then $\Delta_\alpha^{\Phi_e^{A_1}}$ is total and equal to G_1. Fix a natural number n. At every α-true expansionary stage s we have that if $\Delta_\alpha^{\Phi^{A_1}}(n)[s] \downarrow$ then $\Delta_\alpha^{\Phi^{A_1}}(n)[s] = G_1(n)$ or else Case (5) would apply. As G_1 is Δ_2^0 we can fix a stage s_n such that $G_1 \upharpoonright n$ does not change at any further stage. At the least α-true expansionary stage after s_n the final value of the chit $c(n)$ will be defined. As $\Theta_e^{\Phi_e^{A_1},B_1}(c(n)) \downarrow$ there will be a stage $s_c \geq s_n$ such that the computation $\Theta_e^{\Phi_e^{A_1},B_1}(c(n))$ does not change. If at any stage $t > s_c$ the computation $\Delta_\alpha^{\Phi_e^{A_1}}(n)$ becomes undefined then it will be redefined using the correct computation for $c(n)$ and hence will not become undefined at further stages. Thus $\Delta_\alpha^{\Phi_e^{A_1}} = G_1$. □

From Lemma 6.2, Lemma 6.5 and Lemma 6.7 we get immediately the True Path Lemma.

Corollary 6.8. *There is an infinite path f in the tree such that for every n, $f \upharpoonright n$ is visited infinitely often and initialized only finitely often. All $\mathcal{R}$ and $\mathcal{Q}$ requirements are satisfied.*

Proof. Assume that $\mathcal{L}_e$-strategies with index e that is smaller than the index of this particular construction do not require attention after stage s_0. The proof is by induction on n. Suppose that $f \upharpoonright n$ is visited infinitely often and initialized only finitely often. Let o be the true outcome of the strategy $f \upharpoonright n$, i.e. the leftmost outcome visited at infinitely many stages and let $f \upharpoonright (n+1) = (f \upharpoonright n)\hat{\ }o$. The strategy $f \upharpoonright (n+1)$ can be initialized at stages $s > s_0$ such that $\delta[s]$ is of higher priority. As $\mathcal{R}$ and $\mathcal{Q}$ strategies of higher priority along f end finitely many stages prematurely and are not initialized, there will be a stage after which $f \upharpoonright (n+1)$ is also not initialized for this reason. The only other possibility is that $f \upharpoonright (n+1) = \alpha$ is an $\mathcal{R}$-strategy and is initialized by the Ω-strategy, because the Ω-strategy violated α's restraint. By the induction hypothesis higher priority $\mathcal{R}$-strategies are either not visited or do not define new restraints, so let r be the largest restraint requested by a higher priority strategy than α. As G_1 and G_2 are Δ_2^0 by Corollary 6.4, it follows that Ω will stop modifying either set on numbers less than r after a certain stage $s > s_0$. From then on at stages $t > s$ the strategy α will be the highest priority strategy whose restraint can possibly be violated by Ω and Ω will always choose to respect it. □

Lemma 6.9. *$\Omega^{G_1,G_2}(n) = X(n)$.*

Proof. At every stage s the Ω strategy ensures that for every $n < s$, $\Omega^{G_1,G_2}(n)[s]\downarrow = X(n)[s]$. We need to show that Ω is total. Fix a natural number n. Consider $f \upharpoonright n$ and let $s_0 = s_0(f \upharpoonright n)$ be such that $f \upharpoonright n$ is not initialized after s_0 and let s_1 be the stage such that $f \upharpoonright n$ is not restarted after s_1. It follows from Lemma 6.5 and Lemma 6.7 that these stages exist and that $(G_1 \oplus G_2)[s_1] \upharpoonright \omega(s_0)$ does not change after stage s_1. Hence $\Omega^{G_1,G_2}(n)\downarrow$. □

Lemma 6.10. $\Gamma_i^{B_i,G_i} = C_i$

Proof. Let c be a natural number. Suppose that c is enumerated in C_i at stage s by a strategy α. Then c is a chit for a number n and α attacks in relation to a computation defined using (c, τ_Φ, τ_B). At stage s the strategy α defines the value of b_c. By the way that chits are assigned to numbers it follows that every Γ_i-computation defined for c until stage s either uses the fact that $G_i(n) = 1 - G_i(n)[s]$ or uses the fact that $B_1(b_c) = 0$. By Lemma 6.6 the value of $G_i(n)$ is the same at all stages $t \geq s$ hence computations of the first kind are never valid again. At stage s α initializes all lower priority strategies and so they can never again change the value of $B_i(b_c)$. From this point on we maintain that $B_i(b_c) = C_i(c)$. There are finitely many strategies α', the ones of higher priority than α, that can change the value of $B_i(b_c)$ at a stage $t > s$ and each can do this only once when it diagonalizes under Case (6), given that it is not initialized in the interval $[s,t]$. It follows that when such a strategy attacks the value of its parameter $s_0(\alpha')$ is smaller than c. So if α' changes the value of $B_i(b_c)$ then it also ensures $C_i(c) = B_i(b_c)$. After all these finitely many strategies are done $B_i(b_c)$ and $C_i(c)$ stop changing. It follows that B_i and C_i are Δ^0_2 and at all stages $t > s$ if $\Gamma_i^{B_i,G_i}(c)[t]\downarrow$ then $\Gamma_i^{B_i,G_i}(c)[t] = C_i(c)[t] = B_i(b_c)[t]$.

Suppose inductively that $\Gamma_i^{B_i,C_i}(d)\downarrow = C_i(d)$ for all $d < c$. Suppose that s_1 is a stage such that:

(1) All computations for $d < c$ have settled.
(2) $G_i \upharpoonright g_i(n)$ does not change at further stages.
(3) $b_c\uparrow$ or $B_i \upharpoonright b_c$ does not change at further stages.
(4) $C_i \upharpoonright c$ does not change at further stages.

Then at stages $t > s$ the value of $\gamma_i(c)$ does not change. Let $s_2 \geq s_1$ be such that $B_1 \upharpoonright \gamma_i(c)$ does not change at further stages. If $\Gamma_i^{B_i,G_i}(c)[s_2]\downarrow$ and $b_c\uparrow$ then at all stages until s_2 we have $C_i(c) = 0$, so every computation defined until s_2 is of the right value of $C_i(n)$. If $b_c\downarrow$ then by the discussion above $\Gamma_i^{B_i,G_i}(c)[s_2] = B_i(b_c)[s] = C_i(c)[s_2]$. If $\Gamma_i^{B_i,G_i}(c)[s_2]\uparrow$ then we define it for the last time correctly and it remains valid at all further stages. □

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(Slaman) Department of Mathematics, University of California, Berkeley, Berkeley, CA 94720-3840, USA

E-mail address, Slaman: `slaman@math.berkeley.edu`

(Soskova) Faculty of Mathematics and Informatics, Sofia University, 5 James Bourchier Blvd., 1164 Sofia, Bulgaria

E-mail address, Soskova: `msoskova@fmi.uni-sofia.bg`