The Local Structure of the Enumeration Degrees

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The candidate confirms that the work submitted is her own, except where work which has formed part of jointly-authored publications has been included. The contribution of the candidate and the other authors to this work has been explicitly indicated overleaf. The candidate confirms that appropriate credit has been given within the thesis where reference has been made to the work of others.

Jointly authored publications

The author has included as part of her thesis the following jointly authored publications:

"How Enumeration Reducibility Yields Extended Harrington Non-Splitting", Mariya Soskova and S. Barry Cooper, accepted in Journal of Symbolic Logic. The problem and the basic module for the theorem have been suggested by the second author. The first author then completed the construction and the proof under the supervision of the second author. The paper was written by the first author and then revised by the second author. This work is presented in Chapter 2 of this thesis.

"Cupping Δ^0_2 Enumeration Degrees to $0'$", Mariya Soskova and Guohua Wu, *Computation and Logic in the Real World*, S. Cooper, B. Löwe and A. Sorbi, LNCS, **4497**, 727–738. This project was commenced during a visit of the first author to The Nanyang University in Singapore. The main problems discussed in this paper were suggested by the second author. During this visit both authors met every day and gradually worked out together the strategies and the proofs of the theorems included. Sections 1 and 2 were written by the second author. Sections 3 and 4 were written by the first author. The paper was then revised several times by both authors. This work is presented as Theorem 5.0.1 and Theorem 6.0.1.

"Total Degrees and Nonsplitting Properties of Σ^0_2 Enumeration Degrees", Marat M. Arslanov, S. Barry Cooper, Iskander Sh. Kalimullin and Mariya M. Soskova, *Theory and Applications of Models of Computation*, M. Agrawal, D. Du, Z. Duan, A. Li, LNCS, **4978**, 568–578. This project was started by the first three authors. The fourth author was invited to complete the proof of Theorem 3 of this paper, for which some basic modules already existed. The paper only discusses the main ideas behind the construction in Theorem 3. This particular theorem is presented in Chapter 7 with a complete construction and proof.

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Abstract

This thesis discusses properties of the local structure of the enumeration degrees. We begin with some historical background of the subject. We give motivation for investigating the properties of the local structure of the enumeration degrees and discuss the basic concepts and methods used throughout the thesis.

Chapter 2 presents evidence that the study of the structure of enumeration degrees can provide a richer understanding of the structure of the Turing degrees. We prove that there exists a Π^0_1 enumeration degree which is the bottom of a cone within which the Π^0_1 enumeration degrees cannot be cupped to $\mathbf{0}'_e$. As a corollary we obtain a generalization of Harrington's non-splitting theorem for the Δ^0_2 Turing degrees.

Chapters 3 and 4 are dedicated to the study of properties, specific to the properly Σ^0_2 enumeration degrees. In Chapter 3 we construct a properly Σ^0_2 enumeration degree above which there is no splitting of $\mathbf{0}'_e$. Degrees with this property can be used to define a filter in the local structure of the enumeration degrees that consists entirely of properly Σ^0_2 enumeration degrees and $\mathbf{0}'_e$. In Chapter 4 we strengthen the result obtained by Cooper, Li, Sorbi and Yang of the existence of a non-bounding enumeration degree by constructing a 1-generic enumeration degree that does not bound a minimal pair. Degrees with this property can be used to define an ideal consisting of properly Σ^0_2 enumeration degrees and $\mathbf{0}_e$.

Chapters 5 and 6 concern the cupping properties of Δ^0_2 enumeration degrees and the sub-classes of the Δ^0_2 enumeration degrees related to the finite and ω- levels of

the Ershov hierarchy. In Chapter 5 we complement a result by Cooper, Sorbi and Yi by showing that every non-zero Δ^0_2 enumeration degree can be cupped by a partial low Δ^0_2 enumeration degree. On the other hand we show that one cannot computably list a sequence of degrees which contains a cupping partner for every Δ^0_2 enumeration degree. In Chapter 6 we concentrate on the smaller subclasses, where the situation improves. We prove that every non-zero ω-c.e. enumeration degree can be cupped by a 3-c.e. enumeration degree and as the 3-c.e. enumeration degrees are computably enumerable this property constitutes a difference between the Δ^0_2 enumeration degrees and the ω-c.e. enumeration degrees. Furthermore we establish a structural difference between the class of Π^0_1 enumeration degrees and the 3-c.e. enumeration degrees by proving that one cannot find a single Σ^0_2 enumeration degree that cups every non-zero 3-c.e. enumeration degree to $\mathbf{0}'_e$.

Finally in Chapter 7 we show that the structure of the 3-c.e. enumeration degrees is far from trivial as there exists a Lachlan non-splitting pair with top a Π^0_1 enumeration degree and bottom a 3-c.e. enumeration degree.

Contents

Chapter 1

Introduction

The goal of this thesis is to enrich our understanding of the local structure of the enumeration degrees. A goal set in isolation without proper justification of its benefits is useless. We begin therefore with a some historical background to the subject Computability Theory aiming to motivate the research that is described in the following chapters. We will not be able to give a complete account of all significant achievements in this subject, rather we will only point out the ones that have led us to the choice of problems we investigate. For a fuller historical account of Computability Theory we refer the reader to [ASF06], which is also our main source.

Computability Theory begins with a paper by Alan Turing [Tur36] in which he solves the *Entscheidungsproblem*, one of the long standing problems, set by David Hilbert as part of his program to formalize the foundations of mathematics. Among the many significant concepts developed in this article is the notion of relativized computation with oracle Turing machines, which allows us to compare the information content of sets. Through this paper Turing attracts the attention of many distinguished mathematicians, including Kleene and Post who proceed to lay the grounds of the subject Degree Theory in a series of articles and books ([Kle36], [Kle43], [Pos44], [Pos48], [Kle52] and [KP54]).

One of the main topics in Degree Theory is the study of computably enumerable degrees, as these represent unsolvable problems that arise from other fields of mathematics. The work done by Kleene and Post, complemented by the discovery of the priority method by Muĉnik [Muc56] and independently by Friedberg [Fri57], influences the direction in which computability theory develops and allures many young mathematicians into this field. The priority method evolves from finite to infinite injury in the works by Shoenfield [Sho61] and Sacks ([Sac63] and [Sac64]) becoming a powerful tool in the local theory of the Turing degrees and disclosing many nice properties of the structure of the computably enumerable degrees. These lead Shoenfield [Sho65] to conjecture that the structure of the computably enumerable degrees as a partial ordering is rather simple, reminiscent of that of the rational numbers.

This conjecture is quickly refuted through the work of Lachlan [Lac66] and Yates [Yat66] who show independently that there are minimal pairs of computably enumerable degrees. In fact Lachlan brings the priority method to yet another level, the $0'''$ priority method, and proves some very surprising properties of computably enumerable degrees, among which appear the non-diamond theorem [Lac66], the existence of a non-bounding degree [Lac79] and also the non-splitting theorem [Lac75], in which he shows the existence of a pair of c.e. Turing degrees $a < b$ such that b cannot be split in the c.e. Turing degrees above a. In this article, also known as *The Monster Paper* due to its extreme complexity, Lachlan introduces for the first time the use of trees of strategies. Harrington [Har80] is later able to strengthen this result by proving that the top degree can be taken to be $\mathbf{0}'$. The techniques that he uses to achieve this enable him and Shelah [HS82] to prove that the theory of the c.e. Turing degrees is undecidable. Finally Harrington and Slaman prove that the complexity of the theory of computably enumerable degrees is as high as possible, as the theory of first order arithmetic can be interpreted in it. These proofs remain unfortunately unpublished and we refer the reader to Nies, Shore and Slaman [NSS98] for an even more general result on this topic.

Thus the structure of computably enumerable degrees, once assumed to be as simple as that of the rational numbers, turns out to be one of the most complicated structures studied in mathematics. Naturally computability theorists start to search for other means of investigating the information content of sets of natural numbers. One example, probably currently the most fashionable, is the reducibility of sets based on their ability to recognize patterns as is done in *Randomness*, see [DH08], [Nie08] and [BLS08]. Another is the study of *strong reducibilities*, where one considers computations that are restricted in a certain way. Representatives of this approach include truth-table reducibility and the study of the computational complexity of sets, see [Odi89] and [Odi99] for an extensive survey. Each new approach gives a little more insight about the structure of the Turing degrees and suggests problems of its own. As an example consider the famous $P = NP$? problem which arises from the study of polynomial time reducibility.

The approach we have chosen is based on the *enumeration reducibility* between sets. Enumeration reducibility is suggested by Friedberg and Rogers [FR59]. A formal definition will be given in the next section, but for now it is enough to say that a set A is enumeration reducible to a set B if given any enumeration of the set B we can effectively obtain an enumeration of the set A. The sets A and B are enumeration equivalent if each is enumeration reducible to the other. In contrast to Turing reducibility, where the information used from the oracle is total, enumeration reducibility uses partial information. Each operator that compares sets, enumeration reducible to each other, is called an enumeration operator and is closely related to a computably enumerable set. One might argue that this reducibility is more natural than Turing reducibility. Cooper [Coo90] goes as far as to say:

"... enumeration reducibility is *the* fundamental, general concept of relative computability in as much as the nature of the computable universe is intimately bound up with the set of enumeration operators."

By identifying sets that are enumeration equivalent to each other we obtain a degree structure, the structure of the enumeration degrees. It is an upper semi-lattice and Cooper [Coo84] defines a jump operation for it. The local structure of the enumeration degrees is the structure consisting of all degrees less than the jump of the least degree $\mathbf{0}_e$. Cooper [Coo84] proves that the local structure of the enumeration degrees coincides with the structure of all Σ^0_2 enumeration degrees.

The main argument in defense of taking the approach of studying the structure of the enumeration degrees is the fact that the Turing degrees can be embedded in the enumeration degrees using an order theoretic embedding defined by Rogers [Rog68]. This embedding preserves the structure of the Turing degrees, including the order, the least upper bound and the jump operator. Furthermore the computably enumerable Turing degrees embed exactly onto the Π^0_1 enumeration degrees. Both the local and the global structures of the enumeration degrees can therefore be viewed as extensions of the local and global structures of the Turing degrees respectively. We automatically have a lot of information about the enumeration degrees by transferring results obtained in the Turing degrees via Rogers' embedding. On the other hand we may argue that clarifying the properties of the larger structure may give us more information about the smaller structure. In Chapter 2 we give substantial proof of the validity of this argument. We prove that there is an incomplete Π^0_1 enumeration degree $\mathbf{a}$ such that the Π^0_1 enumeration degrees above $\mathbf{a}$ are not cupped by the Σ^0_2 enumeration degrees above $\mathbf{a}$. As an immediate corollary we obtain a generalization of Harrington's non-splitting theorem, namely that there is a computably enumerable incomplete Turing degree $\mathbf{a}$ such that no computably enumerable degree above it is cupped to $\mathbf{0}'$ even by a Δ^0_2 Turing degree above $\mathbf{a}$.

Another interesting feature of the local structure of the Turing degrees requires us to restrict our attention to the Δ^0_2 enumeration degrees. Most of the surprising properties listed above and true of the computably enumerable degrees cease to be true of the extended structure of the Δ^0_2 enumeration degrees. Cooper, Sorbi and Yi [CSY96] prove that every nonzero Δ^0_2 enumeration degree is cuppable in contrast to Cooper and Yates' result of the existence of a nonzero non-cuppable computably enumerable Turing degree, see [Coo73]. Cooper, Li, Sorbi and Yang [CLSY05] prove that every nonzero Δ^0_2 enumeration degree bounds a minimal pair in contrast to Lachlan's non-bounding theorem [Lac79] for the computably enumerable degrees. Arslanov and Sorbi [AS99] prove that there is a Δ^0_2 splitting of $\mathbf{0}'_e$ above every Δ^0_2 enumeration degree in contrast to Harrington's non-splitting theorem for the computably enumerable degrees. Furthermore the Δ^0_2 enumeration degrees are dense by Arslanov, Kalimullin and Sorbi [AKS01] unlike the Δ^0_2 Turing degrees, where Spector [Spe56] proves that minimal degrees exist. All of these results suggest that the local structure of the enumeration degrees is much better organized and even possibly much simpler than the local structure of the Turing degrees.

Unfortunately this is not the case at all when we consider the whole structure of the Σ^0_2 enumeration degrees. Although the density is preserved, as proved by Cooper [Coo84], all of the other strange properties become valid once again. Cooper, Sorbi and Yi [CSY96] show the existence of a non-cuppable nonzero Σ^0_2 enumeration degree. Cooper, Li, Sorbi and Yang [CLSY05] prove the existence of a nonzero non-bounding Σ^0_2 enumeration degree. This result will be the main topic of Chapter 4, where we will prove a generalization of it. We prove that there exists a 1-generic enumeration degree that does not bound a minimal pair in the enumeration degrees. Finally in Chapter 3 we complete this analogy by proving the analog of Harrington's non-splitting theorem for the enumeration degrees. We prove that there exists an incomplete Σ^0_2 enumeration degree such that $\mathbf{0}'_e$ cannot be split in the Σ^0_2 enumeration degrees above it. Thus the local structure of the enumeration degrees seems just as rich in surprises as is that of the computably enumerable degrees. Slaman and Woodin [SW97] prove that its theory is undecidable.

This close connection between the properties of the Σ_2^0 enumeration degrees and the computably enumerable Turing degrees leads naturally to the conjecture, made by Cooper [Coo84], that the two structures are elementary equivalent. Cooper's conjecture is refuted by Ahmad [Ahm91] who proves that the diamond can be embedded in the Σ_2^0 enumeration degrees preserving least and greatest elements and by Lachlan's non-diamond theorem for the computably enumerable degrees the two structures cannot be elementary equivalent. In fact Ahmad and Lachlan [AL98] discover an extraordinary property of the Δ_2^0 enumeration degrees, the existence of nonzero non-splitting Δ_2^0 enumeration degrees, i.e. ones that are not the least upper bound of any two lesser degrees, showing that Sack's splitting theorem fails for the local structure of the enumeration degrees. This result later enables Kent [Ken05] to prove that even the theory of the Δ_2^0 enumeration degrees is undecidable.

In view of the disclosed complexity of the Δ_2^0 enumeration degrees we continue in Chapter 5 our investigation of their properties. We prove that every nonzero Δ_2^0 enumeration degree can be cupped by a partial low Δ_2^0 enumeration degree thus complementing the previous result by Cooper, Sorbi and Yi that every nonzero Δ_2^0 enumeration degree can be cupped by a total Δ_2^0 enumeration degree. We next ask the question whether we can computably list a sequence of Δ_2^0 enumeration degrees containing a cupping partner for every nonzero Δ_2^0 enumeration degree. Here we are faced once again with the intricacy of the structure of the Δ_2^0 enumeration degrees as we find that the answer to the proposed question is negative.

In search for a simpler structure we refine our classification of the Δ_2^0 enumeration degrees. In Chapter 6 we consider classes of enumeration degrees based on the finite and ω- levels of the Ershov hierarchy. Cooper [Coo90] proves that the 2-c.e. enumeration degrees coincide with the Π_1^0 enumeration degrees. The second level in our classification consists therefore of all 3-c.e. enumeration degrees. The positive result we obtain in favor of a possible simplicity of these finer classes is that every nonzero ω-c.e. enumeration degree can be cupped by a 3-c.e. enumeration degree. A further argument in this direction is given by Arslanov, Kalimullin and Sorbi [AKS01], who show that the n-c.e. degrees are downwards dense and by Kalimullin [Kal02] who shows that every nonzero n-c.e. degree has a non-trivial splitting. Once again this apparent simplicity turns out to be elusive as is exhibited in our next discovery. We prove that for every incomplete Σ^0_2 enumeration degree $\mathbf{a}$ there is a nonzero 3-c.e. enumeration degree $\mathbf{b}$ such that $\mathbf{a}$ does not cup $\mathbf{b}$ to $\mathbf{0}'_e$.

The last chapter of this thesis is devoted to providing further evidence of the complexity of the structure of the 3-c.e. degrees. We prove that there exists a Π^0_1 enumeration degree $\mathbf{a}$ and a 3-c.e. enumeration degree $\mathbf{b} < \mathbf{a}$ such that $\mathbf{a}$ cannot be split in the enumeration degrees above $\mathbf{b}$, thus providing an analog of Lachlan's non-splitting theorem for each class of enumerations degrees that we consider and completing the study of the non-splitting properties of the Σ^0_2 enumeration degrees.

1.1 The enumeration degrees

Enumeration reducibility or e-reducibility is a relation between sets of natural numbers. As was mentioned earlier intuitively we say that a set A is enumeration reducible to a set B if given any enumeration of B, we can effectively enumerate the set A. We formalize this idea in the following way:

Definition 1.1.1. *A set A is enumeration reducible ($\leq_e$) to a set B if there is a c.e. set W such that:*

$$n \in A \Leftrightarrow (\exists u)[\langle n, D_u\rangle \in W \wedge D_u \subseteq B],$$

where D_u denotes the finite set with code u under the standard coding of finite sets.

Every c.e. set W_e can be viewed in this sense as corresponding to an operator Φ_e, defined by $\Phi_e^B = \{\ n|\ (\exists u)[\langle n, u\rangle \in W_e \wedge D_u \subseteq B]\}$. We shall call this operator an *enumeration operator*. It is straightforward to see that a set A is enumeration reducible to a set B if and only if there is an enumeration operator Φ_e such that $A = \Phi_e^B$. In our further discussions we shall not distinguish between a c.e. set and its corresponding operator.

Definition 1.1.2. *A set A is enumeration equivalent ($\equiv_e$) to a set B if $A \leq_e B$ and $B \leq_e A$. The equivalence class of A under the relation $\equiv_e$ is the enumeration degree or e-degree $d_e(A)$ of A.*

The structure of the enumeration degrees $\langle \mathcal{D}_e, \leq \rangle$ is the class of all e-degrees with relation $\leq$, defined by $d_e(A) \leq d_e(B)$ if and only if $A \leq_e B$. Similarly to the structure $\mathcal{D}_T$ of the Turing degrees it has a least element $\mathbf{0}_e = d_e(\emptyset)$, we can define a least upper bound, by setting $d_e(A) \vee d_e(B) = d_e(A \oplus B)$ and a jump operator $d_e(A)' = d_e(J_e(A))$. The enumeration jump of a set A, denoted by $J_e(A)$ is defined by Cooper [Coo84] as $\overline{K_A} \oplus A$, where $K_A = \{\ n |\ n \in \Phi_n^A\}$.

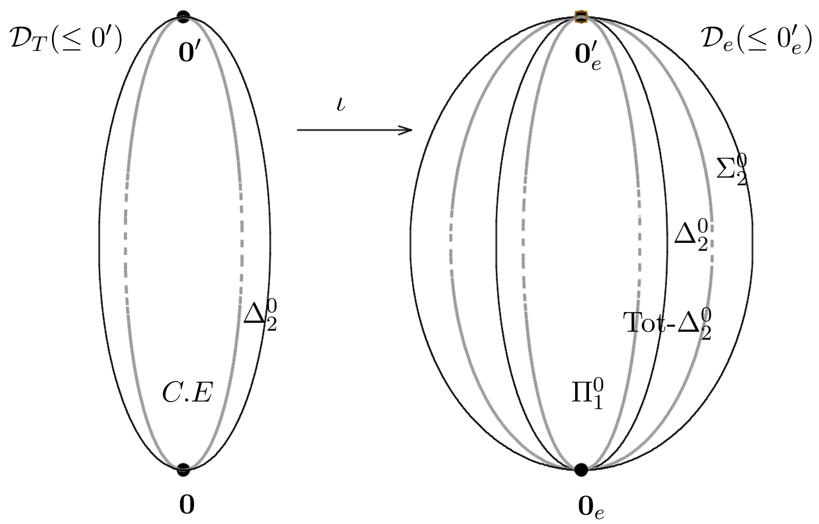
The jump operator gives rise to the local structure of the enumeration degrees, consisting of all enumeration degrees $\mathbf{a} \leq \mathbf{0}_e'$. We shall refer to degrees that contain a Σ_n^0, a Π_n^0 or a Δ_n^0 set as Σ_n^0, Π_n^0 or Δ_n^0 degrees respectively. The bottom degree, $\mathbf{0}_e$, consists of all c.e. sets and is the only Σ_1^0 enumeration degree. Cooper [Coo84] proves that the enumeration degrees $\mathbf{a} \leq \mathbf{0}_e'$ are exactly the Σ_2^0 enumeration degrees. We shall denote the local structure of the enumeration degrees by $\mathcal{D}_e(\leq 0_e')$.

Rogers [Rog68] defines an embedding ι of the Turing degrees into the enumeration degrees. Consider $\iota(d_T(A)) = d_e(A \oplus \overline{A})$. This embedding preserves the order, the least upper bound and the jump operator. The images of the Turing degrees under this embedding are called total degrees.

The local structure of the Turing degrees $\mathcal{D}_T(\leq 0')$ embeds into the local structure of the enumeration degrees $\mathcal{D}_e(\leq 0_e')$. It follows that total enumeration degrees in the local structure are necessarily Δ_2^0 enumeration degrees and the images of the c.e. Turing degrees are exactly the Π_1^0 enumeration degrees.

Definition 1.1.3. *An enumeration degree* $\mathbf{a}$ *is quasi-minimal if for every* $\mathbf{b} \leq \mathbf{a}$, *if* $\mathbf{b}$ *is total then* $\mathbf{b} = \mathbf{0}_e$.

The existence of quasi-minimal degrees, shown by Medvedev [Med55], proves that there are enumeration degrees that are not images of Turing degrees under ι. These enumeration degrees shall be referred to as partial. In fact one can easily construct a Δ^0_2 quasi-minimal degree, see for example [Cop90] or Chapter 5. Thus the local structure of the enumeration degrees contains both total and partial degrees and properly extends the local structure of the Turing degrees. Furthermore Cooper and Copestake [CC88] prove the existence of properly Σ^0_2 enumeration degrees, ones that contain only properly Σ^0_2 sets. This already gives a hierarchy of the enumeration degrees in $\mathcal{D}_e(\leq 0'_e)$ as is illustrated by the following picture.



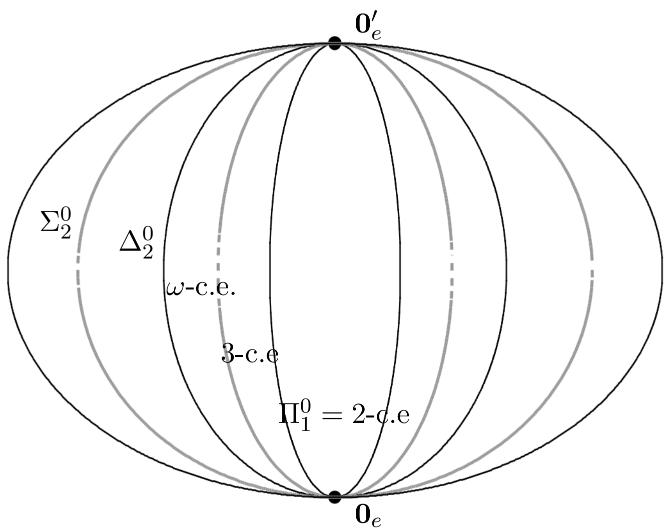
This coarse classification of the degrees in the local structure will be sufficient for the next four chapters. For the last two chapters we will need to consider a finer partition of the Δ^0_2 enumeration degrees that arises from the Ershov hierarchy, [Ers68a] and [Ers68b], also known as the *difference hierarchy.*

Definition 1.1.4. *1. For* $n < \omega$ *a set* A *is* n*-c.e. if there is a total computable function* f *such that for each* x, $f(x,0) = 0$, $|\{s \mid f(x,s) \neq f(x,s+1)\}| \leq n$ *and*

$A(x) = \lim_s f(x,s)$.

2. *A is ω-c.e. if there are two total computable functions $f(x,s)$ and $g(x)$ such that for all x, $f(x,0) = 0$, $|\{s \mid f(x,s) \neq f(x,s+1)\}| \leq g(x)$ and $A(x) = \lim_s f(x,s)$.*

An enumeration degree which contains an n-c.e. set, where $n \leq \omega$, will be called an n-c.e. enumeration degree. Cooper [Coo90] proves that the class of 2-c.e. enumeration degrees coincides with the class of Π^0_1 enumeration degrees. This will be the smallest class in our hierarchy. For every $n > 2$ we have a class consisting of all n-c.e. enumeration degrees. Each of these classes is proper as observed by Cooper [Coo90]. Then follows the class of all ω-c.e. degrees, which is proper subclass of all Δ^0_2 enumeration degrees. We have a well defined hierarchy of classes of enumeration degrees within the local structure $\mathcal{D}_e(\leq 0'_e)$ illustrated in the following picture.



Finally we mention a different approach in classifying the enumeration degrees below $\mathbf{0}'_e$, based on the jump of a degree. The jump operator is monotone and hence the jump of every Σ^0_2 enumeration degree is at least $\mathbf{0}'_e$ and at most $\mathbf{0}''_e$.

Definition 1.1.5. *Let* $\mathbf{a}$ *be a* Σ^0_2 *enumeration degree. We say that* $\mathbf{a}$ *is* low *if* $\mathbf{a}' = \mathbf{0}'_e$. *We say that* $\mathbf{a}$ *is* high *if* $\mathbf{a}' = \mathbf{0}''_e$.

1.2 Computable enumerations

The classification of the Σ^0_2 enumeration degrees is closely connected with the way representative sets of members of each class can be approximated. If $\{A[s]\}_{s<\omega}$ is a Σ^0_1 approximation to A then $A[0] = \emptyset$ and for all $x \notin A$ we have that $x \notin A[s]$ at all stages s, for all $x \in A$ there is a unique t such that $x \notin A[s]$ at stages $s < t$ and $x \in A[s]$ at stages $s \geq t$. If the approximation is Π^0_1 then $A[0] = \mathbb{N}$, the set of all natural numbers, for all $x \in A$ we have that $x \in A[s]$ at all stages s and for all $x \notin A$ there is a unique t such that $x \in A[s]$ at stages $s < t$ and $x \notin A[s]$ at stages $s \geq t$. If the approximation is n-c.e., where $n < \omega$, the number of stages t at which $A(x)[t] \neq A(x)[t+1]$ is bounded by n. If it is ω-c.e. then there is a total computable function g such that for every x the number of stages t at which $A(x)[t] \neq A(x)[t+1]$ is bounded by $g(x)$. In a Δ^0_2 approximation the number of stages at which $A(x)[t] \neq A(x)[t+1]$ is always finite. Finally if the approximation is Σ^0_2 then the number of changes is finite for elements $x \in A$ and unbounded for elements $x \notin A$.

In all cases $n \in A$ if and only if there is a stage t such that $n \in A[s]$ for all $s \geq t$. A useful notion when dealing with Σ^0_2 sets is the *age* of an element. This notion was used first by Nies and Sorbi [NS99] and was given its name by Kent [Ken05].

Definition 1.2.1. *Given a Σ^0_2 approximation $\{A[s]\}_{s<\omega}$ to a set A, a stage s, and an element $n \in A[s]$, we define $a(A, n, s)$, the age of n in A at stage s, to be the least s_n such that for all t, if $s_n \leq t \leq s$ then $n \in A[t]$. The age of a finite set $F \subset A[s]$ at stage s is $a(A, F, s) = \max\{\, a(A, n, s) |\ n \in F\}$.*

An element n belongs to a Σ^0_2 set A if and only if its *age* relative to a fixed Σ^0_2 approximation reaches a finite limit value. We will denote this value by $a(A, n)$ and refer to it as the *limit age*. The *limit age* for a finite subset $F \subseteq A$ is defined, as one might expect, as $\max\{\, a(A, n) |\ n \in F\}$.

In the following chapters we will frequently require a computable enumeration of approximations to all sets of a certain class. The most common example of a class of sets that can be computably enumerated is the class of all c.e. sets.

Definition 1.2.2. *Let C be any of the considered classes of enumeration degrees. A sequence of enumeration degrees $\{a_i\}_{i<\omega}$, where $a_i \in C$ for every i, can be* C-computably enumerated *if there is a computable sequence $\{A_i[s]\}_{i,s<\omega}$ of C approximations to representatives A_i of each degree a_i. The class C is computably enumerable if it is C-computably enumerable.*

Every Π^0_1 set A is the complement of a c.e. set W. A Π^0_1 approximation to A can be obtained from a c.e. approximation $\{W[s]\}_{s<\omega}$ to W by setting $A[s] = \overline{W[s]}$ and a computable enumeration of the class of all Π^0_1 enumeration degrees can be obtained from a computable enumeration of all c.e. sets. We shall see in Section 1.4.2 that the class of all Σ^0_2 enumeration degrees is also computably enumerable. The class of Δ^0_2 enumeration degrees however cannot be computably enumerated. Intuitively this follows from the well known fact that we cannot computably enumerate the set of all total computable functions. A formal argument can be obtained, among many other ways, from the results in Chapter 5.

Every other class of enumeration degrees that we consider is computably enumerable. For $n < \omega$ we use the fact that every n-c.e. set A is a boolean combination of n c.e. sets $W_1 \dots W_n$, i.e. $A = (((W_1 \setminus W_2) \cup W_3) \dots W_n)$. An n-c.e. approximation to the set A can be obtained from c.e. approximations $\{W_i[s]\}_{i<n,s<\omega}$ by setting $A[s] = (((W_1[s] \setminus W_2[s]) \cup W_3[s]) \dots W_n[s])$. We can computably list a sequence of all n-tuples of natural numbers and from it we get a computable enumeration of all n-c.e. sets.

A computable enumeration of all ω-c.e. sets can, for example, be obtained from a computable enumeration of all pairs of partial computable functions. For every pair of partial computable functions f and g we will define an approximation $\{A[s]\}_{s<\omega}$. We will use the symbol $\downarrow$ to denote the phrase *is defined* and the symbol $\uparrow$ to denote the phrase *is undefined.* We set $A[0] = \emptyset$. For every $s > 0$ we set $A(x)[s] = A(x)[s-1]$ if one of the following conditions is true.

1. There is some $y \leq x$ such that $g(y)[s] \uparrow$.

2. $g(y)[s] \downarrow$ for all $y \leq x$ but $|\{t < s \mid A(x)[t] \neq A(x)[t+1]\}| \geq g(x)[s]$.

3. $f(x, 0)[s] \uparrow$.

In all other cases let $t \leq s$ be the largest number such that $f(x, r)[s] \downarrow$ for all $r \leq t$. We set $A(x)[s] = f(x, t)[s]$.

First note that every approximation obtained in this way is an ω-c.e. approximation. If the function g is total then the number of changes at $A(x)[t]$ is bounded by $g(x)$. If the function g is partial then let n be the least element such that $g(n) \uparrow$ and let $\hat{g}$ be the total computable function defined by $\hat{g}(x) = g(x)$ if $x < n$ and otherwise $\hat{g}(x) = 0$. Then the number of changes in $A(x)[t]$ is bounded by $\hat{g}(x)$.

Now consider an ω-c.e. set B defined using the total computable functions f and g and the set A approximated by $\{A[s]\}_{s<\omega}$, the approximation obtained with the described procedure from the functions f and g. We will prove that $A = B$. Fix any number n and let s_n be a stage such that $f(n, t) = B(n)$ for all $t > s_n$. From the definition of an ω-c.e. set it follows that $|\{\ t|\ f(n, t) \neq f(n, t+1)\}| \leq g(n)$. Consider the least stage s such that $g(m)[s] \downarrow$ for all $m \leq n$ and $f(n, t)[s] \downarrow$ for all $t \leq s_n$.

From the construction of the approximation it follows that every change at $A(n)[t]$ corresponds to a change at $f(n, t)$, as we can only change the approximation at n at stage t if we define $A(n)[t]$ under the second case of the construction. Thus at stage s the number of times that the constructed approximation has changed at n has not yet reached its limit $g(n)[s] = g(n)$ and we shall define $A(n)[s] = f(n, s_n)$ using the second case of the construction. As $f(n, t) = f(n, s_n)$ at all further stages $t > s_n$ we shall correspondingly have $A(n)[t] = A(n)[s] = B(n)$ at all stages $t \geq s$.

1.3 The priority method

In every construction that appears in this thesis we use in one form or other the priority method. We shall summarize here the basic concepts that underlie this method and illustrate them with an example. We have chosen the most basic example, historically the first problem solved with this method by Muĉnik [Muc56] and Friedberg [Fri57], but set in the enumeration degrees.

Theorem 1.3.1. *There exist two incomparable Π^0_1 enumeration degrees.*

Problems that are solved by the priority method require us usually to construct a set with certain properties. The first step is to formalize these properties by listing a countable set of requirements that we need to satisfy in the course of our construction in order to guarantee that the constructed set has the required properties. The requirements are usually divided into a finite number of groups, based on their similarity.

In our example we are required to construct two Π^0_1 sets A and B, which are incomparable, i.e. $A \not\leq_e B$ and $B \not\leq_e A$. This can be formalized by the following two groups of requirements:

1. $\mathcal{P}_e : A \neq \Phi_e^B$;

2. $\mathcal{Q}_e : B \neq \Phi_e^A$,

where $\{\Phi_e\}_{e<\omega}$ is a computable enumeration of all enumeration operators or equivalently all c.e. sets.

The construction proceeds in stages at which we construct approximations to the required sets, in our example A and B. These approximations might be required to be of certain type, for example Π^0_1. We shall define the two sets A and B to be the set of all natural numbers at first and then we shall extract certain numbers from the sets in the course of the construction.

To satisfy a requirement we define a computable strategy or module, a set of actions that need to be taken at a particular stage, that will ultimately lead to the satisfaction of the requirement. For all requirements of a certain type the strategies are very similar and differ only in a finite number of parameters, such as their index. A strategy can be considered as a program that keeps record of its own parameters and is activated by us at certain stages, when it is allowed to modify the values of its parameters, to enumerate or extract elements from the constructed sets and to impose certain restrictions on the actions of strategies executed at further stages.

The strategy for satisfying the requirement $\mathcal{P}_e$ is very standard and will be referred to as the *Friedberg-Muĉnik* strategy.

Definition 1.3.1. *Let A be any set of natural numbers and Φ be an enumeration operator. If $n \in \Phi^A$ then we will denote by $use(\Phi, A, n)$ the length l of the least initial segment of A such that $n \in \Phi^{A\restriction l}$.*

The strategy has one parameter, a witness x, which will be undefined initially and proceeds as follows if executed at stage s:

1. If the witness x is not selected, then let x be a fresh number, one that has not appeared in the construction so far.

2. If $x \notin \Phi_e^B[s]$ then do nothing.

3. If $x \in \Phi_e^B[s]$ then extract x from $A[s]$ and restrain $B[s] \restriction use(\Phi_e, B, x)[s]$ in B.

The strategy for a requirement $\mathcal{Q}_e$ is almost the same as that for $\mathcal{P}_e$, but in it the places of A and B are exchanged.

Strategies may have more than one possible method for satisfying their requirement. They choose the correct one based on the situation that they observe when executed. The choice of a particular method gives an outcome of the strategy. In our example the strategies have two outcomes. If every time we execute a particular strategy, working on $\mathcal{P}_e$ say, it stops at step 2 then the requirement will be satisfied as $x \in A \setminus \Phi_e^B$. We denote this outcome by w for 'wait'. If on the other hand the strategy executes step 3 at some stage, then it is successful as $x \in \Phi_e^B \setminus A$. This outcome will be denoted by f for 'finished'. Every time a strategy is executed it will select one of these outcomes, depending on what it currently assumes to be the right method. In our case the strategy will have outcome w at stage s if it ends its actions at step 2 and outcome f at stage s if it executes step 3.

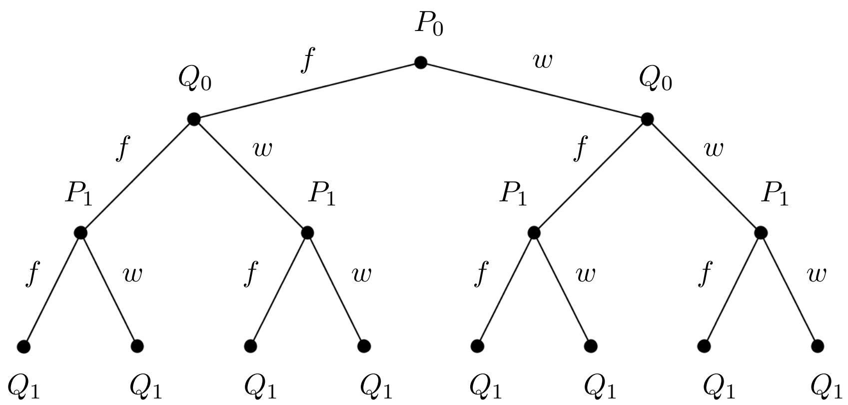
To satisfy a requirement we need to execute the corresponding strategy at infinitely many stages. The main difficulty in this method is that strategies of different types usually contradict each other. For example a $\mathcal{P}_e$-strategy at step 3 would like to keep a certain finite set $B \upharpoonright u(\Phi_e, B, x_1)$ in B. A $\mathcal{Q}_e$-strategy on the other hand might have already chosen a witness $x_2 < use(\Phi_e, B, x_1)$ and upon reaching step 3, would like to extract it from B, thereby injuring the restraint imposed by the first strategy. To resolve conflicts of this sort we order the requirements linearly and give requirements that appear earlier in this ordering higher priority. A possible ordering of the requirements in our example is

$$\mathcal{P}_0 < \mathcal{Q}_0 < \mathcal{P}_1 < \mathcal{Q}_1 < \mathcal{P}_2 ...$$

Strategies of higher priority, i.e. strategies for satisfying a requirement of higher priority, are allowed to ignore restrictions imposed by strategies of lower priority. We call this *to injure* a lower priority strategy. Injury appears usually when a strategy decides to switch to a different method for satisfying its requirement. Strategies of lower priority work under the assumption that the method chosen by higher priority strategies is final. If they are injured, they are initialized and restart their work under new assumptions. As a consequence strategies of lower priority do not injure strategies of higher priority. So the aim of the priority method is to organize the execution of strategies in a way so that a strategy for each requirement is executed at infinitely many stages and initialized (or restarted) only at finitely many stages.

An effective way to organize this process is by using a *tree of strategies*. First we order the outcomes of the strategies. This ordering will be denoted by $<_L$. Outcomes that require the initialization of lower priority strategies appear first. In our example $f <_L w$. Let O denote the set of all outcomes. The set $O^{<\omega}$, consisting of all finite strings in the alphabet O, has a natural lexicographical order $<$ induced by the one defined on O. For $\alpha, \beta \in O^{<\omega}$ we shall also use the relations $\alpha \subset \beta$ if β is an extension of α and $\alpha <_L \beta$ if $\alpha < \beta$ and β is not an extension of α.

Let $\mathcal{R}$ denote the set of all requirements. The tree of strategies is a computable function T with domain $D(T)$ a downwards closed subset of $O^{<\omega}$ and range $R(T) = \mathcal{R}$. The tree of strategies therefore assigns a requirement to every node in its domain. Higher priority requirements are assigned to nodes at higher levels of the tree. If $T(\alpha) = \mathcal{R}_e$ then we shall say that α is an $\mathcal{R}_e$-node or an $\mathcal{R}_e$-strategy and α will be equipped with its own instance of the $\mathcal{R}_e$-module, i.e. it will have its own parameters, whose values it will be allowed to modify when activated. We will use the word *strategy* in two ways: to represent a module, a description of a specific set of actions and parameters, and to represent a node on the tree equipped with a module.



The first few levels of the tree of strategies T in our example are illustrated in the picture above. It will have $D(T) = \{f, w\}^{<\omega}$. Nodes α of even length $2n$ shall be assigned to $\mathcal{P}$-requirements: $T(\alpha) = \mathcal{P}_n$. Nodes β of odd length $2n+1$ shall be assigned to $\mathcal{Q}$-requirements: $T(\beta) = \mathcal{Q}_n$.

An infinite path g in the tree of strategies is a maximal linearly ordered subset of $D(T)$. The tree of strategies shall also have the property that for each infinite path g, $R(T \upharpoonright g) = \mathcal{R}$, i.e. each requirement appears at least once along each infinite path.

The tree of strategies therefore lists all possibilities for the distribution of the outcomes of all strategies in order of their priority. The aim of the construction will be to single out an infinite path trough the tree, which corresponds to the correct distribution, and activate its corresponding nodes at infinitely many stages.

Having defined the tree of strategies we are ready to describe the general construction. At stage 0 all nodes are initialized. At each stage $s > 0$ we construct a finite path $\delta[s]$ of length s through the domain of T starting at the root of the tree. We say that a node α is visited at stage s, also that s is an α-true stage, if $\alpha \subseteq \delta[s]$. At true stages the node α is activated. It will execute its strategy and approximate its outcome o. The next node visited at stage s will be $\alpha\hat{\ }o$. At the end of each stage s we initialize all nodes $\beta > \delta[s]$.

Thus the main focus in every proof will be the existence of an infinite path h in the tree of strategies, called *the true path*, with the following properties:

1. $(\forall n)(\exists^\infty s)[\ h \upharpoonright n \subseteq \delta[s]\ ]$;

2. $(\forall n)(\exists s_i(n))(\forall s > s_i(n))[\ h \upharpoonright n$ is not initialized at stage $s\ ]$.

It follows that if the true path exists then every strategy along it is eventually able to satisfy its requirement.

Following this general recipe we carry out the construction of the required sets A and B in our example. The true path exists and it is the leftmost path of nodes visited at infinitely many stages. To prove that every requirement is satisfied is now an easy task and we will not give further details.

1.4 Approximations

The basic method discussed in the previous sections uses the information obtained from the approximations of a list of given sets. The example illustrated the easiest situation: the given sets were only the enumeration operators. In constructions examined in further chapters we will be required to approximate much more complicated sets and will have to choose our approximations carefully.

If we would like to obtain information about all sets that are enumeration reducible to a certain set A, we might be faced with an impossible task. To illustrate this difficulty suppose we are approximating the set Φ^A, where A is a Σ^0_2 set with a Σ^0_2 approximating sequence $\{A[s]\}_{s<\omega}$ and Φ is an enumeration operator with standard c.e. approximating sequence $\{\Phi[s]\}_{s<\omega}$. We approximate Φ^A by setting $\Phi^A[s] = \Phi[s]^{A[s]}$. The set Φ contains only two axioms for the element n: $\langle n, \{a\}\rangle$ and $\langle n, \{b\}\rangle$. The Σ^0_2 approximation of our set A has the following property: at even stages $2s$ we have that $a \in A[2s]$ and $b \notin A[2s]$; at odd stages $2s+1$ we have $a \notin A[2s+1]$ and $b \in A[2s+1]$. As a consequence both elements a and b do not belong to the set A, hence $n \notin \Phi^A$ as both axioms are invalid. But in our approximating sequence $\{\Phi^A[s]\}_{s<\omega}$ it seems that $n \in \Phi^A$ at all stages s. Thus we obtain no information from this approximating sequence, as we cannot distinguish in any way between elements in and out of the set.

The requirement that A is Σ^0_2 is not essential. The same situation could appear with a Π^0_1 set. Suppose that the operator Φ has infinitely many axioms for n of the form $\langle n, \{a_i\}\rangle$, where $\{a_i\}$ is some sequence of distinct natural numbers. The approximations now have the following property for every stage s, $\langle n, \{a_s\}\rangle \in \Phi[s]$ and $a_s \in A[s] \setminus A[s+1]$. Again at every stage it seems that $n \in \Phi^A[s]$, but every axiom is eventually invalid and ultimately $n \notin \Phi^A$.

So to be able to approximate a set of the form Φ^A we will have to select a 'nicer' approximation to the set A. Cooper [Coo90] suggests a Σ^0_2 approximation, with infinitely many *thin* stages. That every Σ^0_2 set has such an approximation is proved by

Jockusch [Joc68]. Lachlan and Shore [LS92] generalize this notion by defining a *good approximation.*

Definition 1.4.1. *Let* $\{A[s]\}_{s<\omega}$ *be a uniform computable sequence of finite sets. We say that* $\{A[s]\}_{s<\omega}$ *is a* good approximation *to the set* A *if:*

G1: $(\forall n)(\exists s)[\ A\restriction n \subseteq A[s] \subseteq A\ ]$ *and*

G2: $(\forall n)(\exists s)(\forall t > s)[\ A[t] \subseteq A \Rightarrow A\restriction n \subseteq A[s]\ ]$.

Stages s *at which* $A[s] \subseteq A$ *are called good stages.*

The main useful property of good approximations is given by the following proposition, proved in [LS92]:

Propostion 1.4.1. *If* $\{A[s]\}_{s<\omega}$ *is a good approximation to* A*,* G *the set of good stages and* Φ *is any enumeration operator then*

$$\lim_{s\in G}\ \Phi^A[s] = \Phi^A.$$

Proof. First we note that at any good stage s, we have $\Phi^A[s] \subseteq \Phi^A$, i.e. if $n \notin \Phi^A$ then $n \notin \Phi^A[s]$ at all good stages $s \in G$. On the other hand if $n \in \Phi^A$ then there is a valid axiom in Φ, say $\langle n, D\rangle$, that appears in the approximation of Φ at stage s'. Let m be the largest element in D. By $G2$ there is a stage s_m such that for all good stages $t > s_m$, we have $A\restriction m \subseteq A$, and so at all good stages $t > \max(s_m, s')$ the axiom $\langle n, D\rangle$ will be valid and $n \in \Phi^A[t]$. □

Using a good approximation to the set A will make the previously described situations impossible and will enable us to obtain sufficient information about any approximated set Φ^A. For each of type of sets, Π^0_1, Δ^0_2 and Σ^0_2, we will define a specific way to obtain a good approximation and prove some additional properties. In the constructions described in further chapters we will always state explicitly if we require a good approximation to a certain set. In all other cases the standard approximation will be used.

1.4.1 A good approximation for a Π^0_1 or a Δ^0_2 set

We shall give a uniform way to obtain a good approximation to a Π^0_1 set or a Δ^0_2 set from a standard approximation. Suppose we are given an approximation $\{A[s]\}_{s<\omega}$ to the set A. For every s define $ap(s) = \mu n[\ A[s](n) \neq A[s-1](n)\ ]$, if $s > 0$ and $A[s] \neq A[s-1]$, and $ap(s) = s$ otherwise. Now we set $\hat{A}[s] = A[s] \upharpoonright ap(s)$.

Propostion 1.4.2. *If $\{A[s]\}_{s<\omega}$ is a Δ^0_2 approximation to A, then $\{\hat{A}[s]\}_{s<\omega}$ is a good Δ^0_2 approximation to A.*

Proof. Fix n. Let $s > n$ be a stage such that $A[s] \upharpoonright n = A[t] \upharpoonright n$ for all $t \geq s$. It follows that $ap(t) > n$ for all $t > s$. This is enough to guarantee that the approximation $\hat{A}[t]$ changes finitely often on all elements $m < n$ and as n is arbitrary $\{\hat{A}[s]\}_{s<\omega}$ is a Δ^0_2 approximation to A. This property implies G_2, we only need to establish G_1. Now let $k > n$ be the least element such that $(\exists s' > s)[\ A(k)[s'] \neq A(k)[s'-1]\ ]$. Then at such a stage s' we have $ap(s') = k$ and $\hat{A}[s'] = A[s] \upharpoonright k \subseteq A$. □

If we modify a Π^0_1 approximation in this way, will not obtain a Π^0_1 approximation. An element n might be extracted at stage s only because $n > ap(s)$ and later it can re-appear in the approximation. We can show nevertheless that we have a computable way of telling if a certain element has left the approximation for good.

Propostion 1.4.3. *If $\{A[s]\}_{s<\omega}$ is a Π^0_1 approximation to A, then $\{\hat{A}[s]\}_{s<\omega}$ is a good approximation to A. For every element n:*

$$n \notin A \Leftrightarrow (\exists s)[\ \hat{A}[s](n) = 0 \wedge n < ap(s)\ ].$$

Furthermore if $\hat{A}[s](n) = 0$ and $n < ap(s)$, then for all $t > s$, $\hat{A}[t](n) = 0$.

Proof. As every Π^0_1 approximation is a Δ^0_2 approximation the first statement follows from the previous lemma. From G_1 it follows that the value of $ap(s)$ grows unboundedly. If $n \notin A$ then there is a stage s such that $n \notin A[t] \supseteq \hat{A}[t]$ for all $t > s$. On the other hand if $n \notin \hat{A}[s]$ and $n < ap(s)$ then $n \notin A[s]$. By the properties of a Π^0_1 approximation it follows that $n \notin A$ and $n \notin A[t] \supseteq \hat{A}[t]$ for all $t > s$. □

1.4.2 A good approximation for a Σ_2^0 set

The method described in the previous section cannot be applied for Σ_2^0 approximations. Consider again the approximation $\{A[s]\}_{s<\omega}$ of a Σ_2^0 set A, where $a \in A[2s] \setminus A[2s+1]$ and $b \in A[2s+1] \setminus A[2s]$ for every s. Then for every t we will have $ap(t) \leq \max(a, b)$ and the modified approximating sequence $\{\hat{A}[s]\}_{s<\omega}$ approximates at most the finite set $A \upharpoonright \max(a, b)$.

In this section instead of giving an effective method to modify a given Σ_2^0 approximation, we shall define a computable enumeration of good approximations to all Σ_2^0 sets, following a result by Jockusch [Joc68]. The Σ_2^0 sets are exactly the sets that are c.e. in the halting set K. Thus $\{W_e^K\}_{e<\omega}$ is an enumeration of all Σ_2^0 sets. Note that here we are using notions from Turing reducibility, rather than enumeration reducibility. W_e^K denotes the domain of the e-th oracle Turing machine using oracle K.

The first step is to define a *better approximating sequence*, also defined in [LS92], to the characteristic function χ_K of the c.e. set K. This is a uniform computable sequence of finite binary functions $\kappa[s]$ such that:

B1: $(\forall n)(\exists s)[\ \chi_K \upharpoonright n \subseteq \kappa[s] \subseteq \chi_K\ ]$.

B2: $(\forall n)(\exists s)(\forall t > s)[\ \{\ n |\ \kappa[t](n) = 1\ \} \subseteq K \Rightarrow \chi_K \upharpoonright n \subseteq \kappa[t]\ ]$.

A stage s at which $\kappa[s] \subseteq \chi_K$ is called a *better* stage.

Let $\{K[s]\}_{s<\omega}$ be the standard approximating sequence to the c.e. set K. Define $ap(s)$ as in the previous section: $ap(s) = \mu n[\ K[s](n) \neq K[s-1](n)\ ]$, if $s > 0$ and $K[s] \neq K[s-1]$, and $ap(s) = s$ otherwise. Then set $\kappa[0] = \emptyset$ and if $s > 0$

$$\kappa[s](n) = \begin{cases} 1 & \text{if } n \in K[s], \\ 0 & \text{if } n \notin K[s] \text{ and } n < ap(s) \\ \text{not defined} & \text{otherwise.} \end{cases}$$

It is straightforward to check that $\{\kappa[s]\}_{s<\omega}$ is a better approximating sequence to χ_K. Furthermore as for all t we have that $\{n\ |\kappa[t](n) = 1\} \subseteq K$, the second property of a better approximating sequence can be improved:

B2: $(\forall n)(\exists s)(\forall t > s)[\ \chi_K \upharpoonright n \subseteq \kappa[t]\ ]$.

We now approximate the Σ^0_2 set $A = W_e^K$ with the sequence $\{A[s]\}_{s<\omega}$, where $A[s] = W_e[s]^{\kappa[s]}$.

Propostion 1.4.4. *1. Every better stage for the approximating sequence $\{\kappa[s]\}_{s<\omega}$ is a good stage for the approximating sequence $\{A[s]\}_{s<\omega}$.*

2. $\{A[s]\}_{s<\omega}$ is a good Σ^0_2 approximation to A.

Proof. The first property is obvious and follows from the definition of oracle Turing machines and our definition of the approximating sequence $\{A[s]\}_{s<\omega}$. It yields that $\{A[s]\}_{s<\omega}$ has infinitely many good stages.

First we prove that the approximation is Σ^0_2, i.e:

$\Sigma^0_2 : (\forall n)[\ n \in A \Rightarrow (\exists s)(\forall t > s)[\ n \in A[t]\ ]]$.

Let $n \in A = W_e^K$. As every Turing computation is finite there is an m and an s such that $n \in W_e[s]^{K\upharpoonright m}$. By the modified property $B2$ we have that there is a stage s' such that at all stages $t > s'$ ($\chi_K \upharpoonright m \subseteq \kappa[t]$). Hence at all stages $t > \max(s', s)$ we will have $n \in A[t]$.

This property together with the fact that if $n \notin A$ and s is a good stage then $n \notin A[s]$ proves that $\{A[s]\}_{s<\omega}$ has also both properties $G1$ and $G2$. □

Note that by the first property of Proposition 1.4.4 the so defined approximations of Σ^0_2 sets have infinitely many common good stages. This can be used for example to obtain a good approximating sequence to $A \oplus B$, where A and B are Σ^0_2 sets. If $\{A[s]\}_{s<\omega}$ and $\{B[s]\}_{s<\omega}$ are obtained in the described way then $\{A \oplus B[s]\}_{s<\omega}$, where $A \oplus B[s] = A[s] \oplus B[s]$, is a good approximation to $A \oplus B$.

1.5 Structural properties

Throughout the rest of the thesis we will be discussing various properties of the local structure of the enumeration degrees and their relationship to the local structure of the Turing degrees. Both discussed structures are upper semi-lattices with least and greatest element and naturally the structural properties concern the greatest lower bound and the least upper bound of certain elements. We summarize here some basic algebraic relations between elements of any upper semi-lattice with least and greatest element, that will be used frequently in the following chapters.

Let $\langle \mathcal{A}, \mathbf{0}, \mathbf{1}, <, \vee \rangle$ be an upper semi-lattice. With letters $\mathbf{a}, \mathbf{b}, \mathbf{c}$ we shall denote elements of this semi-lattice. The two basic notions are the dual relations *to cup* and *to cap*. Their names stem from the symbols used to denote least upper bound ($\vee$) and greatest lower bound ($\wedge$), respectively.

Definition 1.5.1. *If* $\mathbf{a} \vee \mathbf{b} = \mathbf{c}$ *and* $\mathbf{a}$, $\mathbf{b} < \mathbf{c}$ *then we shall say that* $\mathbf{a}$ *cups* $\mathbf{b}$ *to* $\mathbf{c}$. *We shall also say that the pair* $(\mathbf{a}, \mathbf{b})$ *is a splitting of* $\mathbf{c}$.
In the special case when $\mathbf{c} = \mathbf{1}$, *we shall simply say that* $\mathbf{a}$ *cups* $\mathbf{b}$.

Definition 1.5.2. *If* $\mathbf{a} \wedge \mathbf{b} = \mathbf{c}$ *and* $\mathbf{a}$, $\mathbf{b} > \mathbf{c}$ *then we shall say that* $\mathbf{a}$ *caps* $\mathbf{b}$ *to* $\mathbf{c}$. *We shall also say that* $\mathbf{a}$ *and* $\mathbf{b}$ *form a minimal pair above* $\mathbf{c}$.
In the special case when $\mathbf{c} = \mathbf{0}$, *we shall simply say that* $\mathbf{a}$ *caps* $\mathbf{b}$ *and that* $(\mathbf{a}, \mathbf{b})$ *is a minimal pair.*

Chapter 2

Extended Harrington Non-splitting

We begin with a result meant to give convincing motivation for the investigation of the properties of the local structure of the enumeration degrees. Our aim will be to prove an extension of Harrington's non-splitting theorem for $\mathcal{D}_T(\leq 0')$ using the wider context of the Σ^0_2 enumeration degrees. We start by reviewing previous results about splitting and non-splitting in the local structure of the Turing degrees.

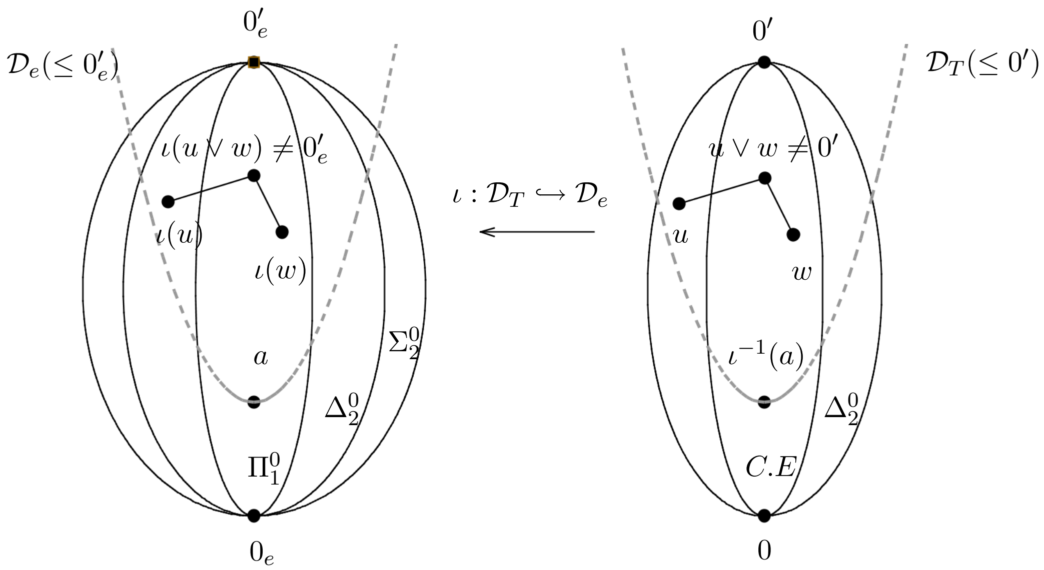
Sacks [Sac63] showed that every nonzero computably enumerable degree has a c.e. splitting. Hence, relativising, every c.e. degree has a Δ^0_2 splitting above each proper predecessor. Arslanov [Ars85] showed furthermore that $\mathbf{0}'$ has a 2-c.e. splitting above each c.e. $\mathbf{a} < \mathbf{0}'$.

It had been commonly believed that one can combine Sacks' splitting theorem [Sac63] with his result about the density of the c.e. degrees, [Sac64]. Lachlan [Lac75] showed that this is not the case by proving the existence of a c.e. $\mathbf{a} > \mathbf{0}$ which has no c.e. splitting above some proper c.e. predecessor. The technique that he used to prove this result, the first instance of a $0'''$-priority method using a tree of strategies, was significantly more complicated than any other known at the time and the article came to be known as "The monster paper". Harrington's work presented as hand-written notes [Har80] led to a better understanding of the technique. He improved the result by showing that one could take $\mathbf{a} = \mathbf{0}'$. This method has been widely used thereafter and has had a number of consequences for definability and elementary equivalence in the Turing degrees below $\mathbf{0}'$. We prove below what appears to be the strongest possible of such non-splitting results.

Theorem 2.0.1. *There exists a computably enumerable degree* $\mathbf{a} < \mathbf{0}'$ *such that there is no nontrivial splitting of* $\mathbf{0}'$ *by a pair of a c.e. degree and a* Δ^0_2 *degree both above* $\mathbf{a}$.

The result is obtained by transferring a structural feature of $\mathcal{D}_e(\leq 0'_e)$ back to $\mathcal{D}_T(\leq 0')$ via the inverse ι^{-1} of the standard embedding of the Turing degrees into the enumeration degrees.

Theorem 2.0.2. *There exists a* Π^0_1 *enumeration degree* $\mathbf{a} < \mathbf{0}'_e$ *such that there exists no nontrivial splitting of* $\mathbf{0}'_e$ *by a pair of a* Π^0_1 *enumeration degree and a* Σ^0_2 *enumeration degree both above* $\mathbf{a}$.



This would appear to be the first example of a structural feature of the Turing degrees obtained via a proof in the wider context of the enumeration degrees (rather than the other way round) and can be viewed as justification of the study of $\mathcal{D}_e(\leq 0'_e)$, suggesting that structural properties of the enumeration degrees can be used to obtain definability of natural relations over the Turing degrees, where this cannot be got just within the Turing degrees. Whether such relations exist is completely open, of course.

The property we prove is stronger than its corollary can express, as the Σ^0_2 enumeration degrees are a proper superclass of the total Δ^0_2 enumeration degrees. This suggests that the local structure of the enumeration degrees is possibly richer than that of the Turing degrees. The proof of Theorem 2.0.2 brings us one step closer to the desired non-splitting result for the Σ^0_2 e-degrees which will be presented in the next chapter.

The work presented in this chapter is joint with S. Barry Cooper and will be published in [SC07].

2.1 Requirements and strategies

To prove Theorem 2.0.2 we shall use the priority method and follow the basic steps outlined in Section 1.3. We start by formalizing the requirements.

We assume standard computable enumerations of all enumeration operators $\{\Psi_i\}_{i<\omega}$ and of all triples $\{(\Theta, U, \overline{W})_i\}_{i<\omega}$ of enumeration operators Θ, Σ^0_2 sets U and Π^0_1 sets $\overline{W}$. We shall denote the elements of the i-th such triple by Θ_i, U_i and $\overline{W}_i$ respectively. We will construct a Π^0_1 set A whose enumeration degree $\mathbf{a}$ will be the one required in Theorem 2.0.2 and an auxiliary Π^0_1 set E to satisfy the following list of requirements:

1. The degree $\mathbf{a}$ should be strictly less than that of $\mathbf{0}'_{\mathbf{e}}$. It will be enough to construct the set A as Σ^0_2-incomplete. We shall use E to witness the incompleteness of A.
$$\mathcal{N}_i : E \neq \Psi_i^A.$$
2. Any pair of an incomplete Σ^0_2 enumeration degree $\mathbf{u}$ and an incomplete Π^0_1 enumeration degree $\mathbf{w}$ above $\mathbf{a}$ should not form a splitting of $\mathbf{0}'_{\mathbf{e}}$. The second group of requirements ensures that either $\mathbf{u} \vee \mathbf{w}$ is incomplete or at least one of the

degrees **u** or **w** is already complete:

$$\mathcal{P}_i : E = \Theta_i^{U_i,\overline{W}_i} \Rightarrow (\exists \Gamma_i, \Lambda_i)[\overline{K} = \Gamma_i^{U_i,A} \vee \overline{K} = \Lambda_i^{\overline{W}_i,A}],$$

where $\overline{K}$ is any Π_1^0 member of the degree $\mathbf{0}'_{\mathbf{e}}$ and $\Gamma_i^{U_i,A}$, for example, denotes an e-operator enumerating relative to the data enumerated from two sources U_i and A or equivalently from their least upper bound $U_i \oplus A$.

The requirements shall be given the following priority ordering:

$$\mathcal{N}_0 < \mathcal{P}_0 < \mathcal{N}_1 < \mathcal{P}_1 < \dots .$$

We shall describe the basic strategies for both types of requirements, the problems that we need to overcome in order to implement them and the conflicts that might arise when we combine them.

An $\mathcal{N}$-requirement could be satisfied by the simple Friedberg-Muĉnik strategy from Section 1.3 which we shall denote in our further discussions by FM.

We are given three options to satisfy a single $\mathcal{P}$-requirement with corresponding parameters Θ, U and $\overline{W}$. The first and simplest one is to provide evidence that $\Theta^{U,\overline{W}} \neq E$. The other two options are to construct enumeration operators Γ or Λ proving that at least one of the sets U or $\overline{W}$ is already too powerful and can reduce $\overline{K}$ by itself without the help of the other.

Definition 2.1.1. *The* length of agreement *between two sets A and B, denoted by $l(A, B)$, is the length of the initial segment on which the sets A and B agree.*

The intent is that we monitor the length of agreement $l(\Theta^{U,\overline{W}}, E)[s]$ at each stage s of the construction. A bounded length of agreement should turn out to be sufficient proof for the inequality between the two sets. Further actions only need to be made at *expansionary stages*, stages at which the length of agreement attains a greater value than it has had at previous stages. Initially we will use a $(\mathcal{P}, \Gamma)$-strategy designed to construct an enumeration operator Γ which reduces the set $\overline{K}$ to the sets U and A. We progressively try to rectify Γ at each stage s by ensuring that $n \in \overline{K}[s] \Leftrightarrow n \in \Gamma^{U,A}[s]$

for each n below $l(\Theta^{U,\overline{W}}, E)[s]$. We will do this by defining markers $u(n)$ and $\gamma(n)$ and enumerating axioms of the form $\langle n, U[s] \upharpoonright u(n), \{\gamma(n)\}\rangle$ for elements $n \in \overline{K}[s]$. If at a later stage n leaves the set $\overline{K}$ then Γ can be rectified via an extraction of the marker $\gamma(n)$ from A.

By using a *good* Σ^0_2 approximation to the sets U and $U \oplus \overline{W}$ we automatically achieve two things:

- For every element u there will be cofinitely many stages t at which $U \upharpoonright u \subset U[t]$ and infinitely many good stages s at which $U \upharpoonright u = U[s] \upharpoonright u$. Hence we will eventually be able to enumerate correct axioms in the constructed operator Γ.
- If $\Theta^{U,\overline{W}} = E$ then it follows from Proposition 1.4.1 that the length of agreement will grow unboundedly at good stages.

2.1.1 Conflicts

A substantial difficulty arises when we consider how to combine the strategies of the two different types. Consider one $\mathcal{N}$-requirement below one $\mathcal{P}$-requirement. $(\mathcal{P}, \Gamma)$ is constructing an operator Γ using markers $u(n)$ and $\gamma(n)$ for the axiom of elements n. The A-restraint of $\mathcal{N}$ following the extraction of the witness x from E is in conflict with the need to rectify the operator Γ at expansionary stages. We try to resolve this by using a modified strategy $(\mathcal{N}, \Gamma)$. It will choose a number d, called a *threshold*, and try to achieve $\gamma(n) > use(\Psi, A, x)$ for all $n \geqslant d$ at a stage previous to the imposition of the restraint. We will need to use a modified version of the use-function.

Definition 2.1.2. *Let Φ be an enumeration operator and A a set. The* generalised use-function *φ is defined as follows:* $\varphi(x) = \max\left\{ use(\Phi, A, y) |\ (y \leq x) \wedge (y \in \Phi^A)\right\}$.

$(\mathcal{N}, \Gamma)$ tries to maintain $\theta(x) < u(d)$ in the hope that after we extract x from E each return of $l(E, \Theta^{U,\overline{W}})$ will produce an extraction from $U \upharpoonright \theta(x)$ which can be used to avoid an A-extraction in moving $\gamma(d)$.

In the event that some such attempt to satisfy $\mathcal{N}$ ends with a $\overline{W} \restriction \theta(x)$-change then we must implement a backup $\mathcal{P}$-strategy, $(\mathcal{P}, \Lambda)$, which is designed to allow lower priority $\mathcal{N}$-strategies to work below the Γ-activity and to construct an operator Λ reducing $\overline{K}$ to $\overline{W}$ and A, using the $\overline{W} \restriction \theta(x)$-changes to move λ-markers. Below $(\mathcal{P}, \Lambda)$ is a backup strategy $(\mathcal{N}, \Lambda)$ designed to take advantage of the improved strategy for $\mathcal{P}$. Both strategies $(\mathcal{N}, \Gamma)$ and $(\mathcal{N}, \Lambda)$ will attack simultaneously at stage s_1 by extracting their witnesses x_1 and $\widehat{x}_1$ from E ensuring that at least one of them has succeeded in providing the necessary U- or $\overline{W}$-change at the next expansionary stage.

2.1.2 Approximations

Consider a triple $(\Theta, U, \overline{W})$ corresponding to a $\mathcal{P}$-requirement. We have already established that we require good Σ^0_2 approximations to the sets U and $U \oplus \overline{W}$ to implement the $\mathcal{P}$-strategies. To implement the backup strategies we will substantially use the fact that $\overline{W}$ is a Π^0_1 set and the changes observed in it are essentially permanent. Therefore we need to define a good Σ^0_2 approximation to U a good Π^0_1 approximation to $\overline{W}$ in the sense of Section 1.4.1 with infinitely many common good stages, so that, when we combine these two approximations, we obtain a good Σ^0_2 approximation to $U \oplus \overline{W}$.

The set $\overline{W}$ is the complement of a c.e. set W. As $W \oplus K \equiv_T K$ and hence U is c.e. in $W \oplus K$, there is some e such that $U = W_e^{W \oplus K}$. In fact we can equivalently enumerate all Σ^0_2 set as $\{W_e^{W \oplus K}\}_{e<\omega}$. Let $\{W_i\}_{i<\omega}$ be a computable enumeration of all c.e. sets. The index of every $\mathcal{P}$-requirement i corresponds to a triple (j, e, a). Then Θ_i will be the j-th enumeration operator W_j in this computable enumeration, $\overline{W}_i$ will be the complement of the a-th c.e. set W_a in this listing, and U_i will be the domain of the e-th oracle Turing machine W_e using oracle $K \oplus W_a$.

To approximate the sets now we proceed as in Section 1.4.2. We define a better approximation $\{\alpha[s]\}_{s<\omega}$ to the characteristic function of the c.e. set $W \oplus K$ together with the corresponding function $ap(s)$. We set $U[s] = W_e[s]^{\alpha[s]}$. We can prove easily as in Proposition 1.4.4 that $\{U[s]\}_{s<\omega}$ is a good Σ_2^0 approximation to U and that every α-better stage is a good stage for $\{U[s]\}_{s<\omega}$.

We approximate $\overline{W}$ using the same better approximation to $W \oplus K$ by setting $\overline{W}[s] = \{\ n|\ \alpha[s](2n) = 0\}$. Notice that this is roughly the same as taking the Π_1^0 approximation $\{\overline{W[s]}\}_{s<\omega}$ and modifying it as in Section 1.4.1. Naturally we have the following property:

Propostion 2.1.1. *$\{\overline{W}[s]\}_{s<\omega}$ is a good approximation to $\overline{W}$ with good stages a superset of the α-better stages. If s is a stage such that $n \notin \overline{W}[s]$ and $2n < ap(s)$ then for all $t > s$ we have that $n \notin \overline{W}[t]$ and hence $n \notin \overline{W}$.*

Proof. If s is an α-better stage and $n \in \overline{W}[s]$ then $\chi_W(n) = \chi_{W\oplus K}(2n) = \alpha(2n) = 0$ and hence $n \in \overline{W}$.

If s is a stage such that $n \notin \overline{W}[s]$ and $2n < ap(s)$ then $\alpha[s](2n) = 1$ hence $2n \in W \oplus K[s] \subset W \oplus K$. It follows that $2n \in W \oplus K[t]$ for all $t > s$ and hence $\alpha[t](2n) = 1$ for all $t > s$, hence $n \notin \overline{W}[t]$ at all $t > s$ and $n \notin \overline{W}$.

That $\{\overline{W}[s]\}_{s<\omega}$ is a good approximation to $\overline{W}$ is now proved easily. □

We have obtained good approximations to the sets U and $\overline{W}$ with infinitely many common good stages. Thus by setting $U \oplus \overline{W}[s] = U[s] \oplus \overline{W}[s]$ we obtain a good Σ_2^0 approximation to the set $U \oplus \overline{W}$. As a consequence if $\Theta^{U,\overline{W}} = E$ and G is the set of all good stages in the approximation to $U \oplus \overline{W}$ then there will be infinitely many expansionary stages, as by Proposition 1.4.1

$$\lim_{s\in G}\ \Theta^{U,\overline{W}}[s] = \Theta^{U,\overline{W}}.$$

Moreover if $n \in \Theta^{U,\overline{W}}$, then there is a stage s such that $(\forall t > s)[n \in \Theta^{U,\overline{W}}[t]]$, and if $n \notin \Theta^{U,\overline{W}}$ then at good stages t we have $n \notin \Theta^{U,\overline{W}}[t]$. Of course, it could happen that the expansionary stages are not necessarily the good stages. And if $\Theta^{U,\overline{W}} \neq E$, we could still have infinitely many expansionary stages.

2.2 The first levels of the tree of strategies

We will describe the modules for each of the strategies and list the parameters that will be related to them. In each of our descriptions of a particular strategy we shall have the context of the tree in mind. The strategy shall be assigned to a particular node δ on the tree (a formal definition of the tree of strategies will be given in Section 2.4.1), the current stage will be denoted by s and the previous δ-true stage by s^- ($s^- = s$ if δ has been initialized since the last stage at which it was visited). All parameters will inherit their values from s^- unless otherwise specified. For this reason we will sometimes omit the indices that specify the stage if it is clear.

The highest priority strategy will be assigned to the root of the tree. It is $\mathcal{N}_0$ and will simply follow the Friedberg- Muĉnik strategy $(\mathcal{N}_0, FM_0)$. The first level of the tree will work on the requirement $\mathcal{P}_0$ with its first possible strategy $(\mathcal{P}_0, \Gamma_0)$.

2.2.1 The $(\mathcal{P}, \Gamma)$-strategy

We have already discussed the main idea for this strategy in Section 2.1. Here we will add details to it and give the formal module. Suppose for definiteness that the $(\mathcal{P}, \Gamma)$-strategy we are discussing is α. The strategy α shall be assigned a distinct infinite computable set A_α from which it will choose the values of its A-markers. Whenever α chooses a fresh marker it will be of value greater than any number appeared so far in the construction. The sets U, $\overline{W}$ and Θ that α works with shall be approximated at α-true stages.

The strategy will have two outcomes $e <_L l$, with which it will distinguish between expansionary and non-expansionary stages. To every element at every stage s we will associate current markers $u(n)[s]$ and $\gamma(n)[s] \in A_\alpha$ and a corresponding current axiom $\langle n, U[t] \upharpoonright u(n)[s], \{\gamma(n)[s]\}\rangle$, where $t \leq s$ is the stage at which $\gamma(n)$ was assigned its current value. An axiom $\langle n, U_n, m\rangle$ is valid at stage s if $U_n \subseteq U[s]$ and $m \in A[s]$.

We will examine the current axiom in Γ for an element $n \in \overline{K}[s]$ if n is below the length of agreement between $E[s]$ and $\Theta^{U,\overline{W}}[s]$, choosing a new axiom as current if the old one is invalid. In this way will be sure to catch the true approximation to the set $U \upharpoonright u(n)$ so that if $u(n)$ remains constant, so will the axiom for n after a certain stage due to the Σ^0_2-property of our approximations. If $n \notin \overline{K}$ then it will be enough to ensure that it does not appear in $\Gamma^{U,A}[s]$ at infinitely many stages s. We choose the expansionary stages for this purpose. During the construction we may enumerate a number of axioms for a particular element. Any enumerated axiom might seem invalid at one stage but turn out to be valid at a later stage. We shall say that an axiom is *potentially applicable* if its A-marker is in A. At expansionary stages s for elements $n \notin \overline{K}[s]$ we shall make sure that there are no valid axioms by extracting the A-markers of any axiom that seems valid at stage s.

At stage s the strategy α acts as follows:

1. If the stage s is not expansionary then $o = l$, otherwise $o = e$.

2. Choose $n < l(\Theta^{U,\overline{W}}, E)[s]$ in turn $(n = 0, 1, \dots)$ and perform following actions:

    - If $u(n) \uparrow$ then define it anew as $u(n) = u(n-1) + 1$ (if $n = 0$ then define $u(n) = 1$). If $u(n)$ is defined, but $ap(s) < 2u(n)$ skip to the next element.
    - If $n \in \overline{K}[s]$:
        - If $\gamma(n) \uparrow$ then define it anew and enumerate the current axiom $\langle n, U[s] \upharpoonright u(n), \{\gamma(n)\}\rangle$ in Γ.
        - If $\gamma(n) \downarrow$ but the current axiom for n is not valid then define the current marker $\gamma(n)$ anew and enumerate the new current axiom $\langle n, U[s] \upharpoonright u(n), \{\gamma(n)\}\rangle$ in Γ.
    - If $n \notin \overline{K}[s]$ but $n \in \Gamma^{U,A}[s]$ and the stage is expansionary then look through all the axioms defined for n, say $\langle n, U_n, m\rangle \in \Gamma[s]$, and extract m for all valid ones.

Note that if $n \notin \overline{K}$ then we will enumerate only finitely many axioms for n in Γ and hence extract only finitely many markers from A.

The third level of the tree will try to satisfy the requirement $\mathcal{N}_1$. The $(\mathcal{P}_0, \Gamma_0)$-strategy will extract markers only at expansionary stages. Hence if its true outcome is l, the strategy will not modify the set A after a certain stage and $\mathcal{N}_1$ can be satisfied via the simple Friedberg-Muchnik strategy proposed initially $(\mathcal{N}_1, FM_0)$. Below the outcome e we will need the more elaborate $(\mathcal{N}_1, \Gamma_0)$.

2.2.2 The $(\mathcal{N}, \Gamma)$-strategy

Suppose the node on which the $(\mathcal{N}, \Gamma)$-strategy acts is labelled by $\beta \supset \alpha$. We shall say that α is the active $\mathcal{P}$-strategy at β. The strategy β shall have four outcomes:

$$g <_L f <_L h <_L w.$$

We start β's activity by performing *Check* first to see whether the threshold is chosen correctly and whether any activity of the active $\mathcal{P}$-strategy for elements below the threshold has injured β's work sofar. If so we restart the module from *Initialization*, otherwise we continue the module from where we left it at the previous β-true stage s^-. If β has been initialized since the last stage at which it was visited or if it has never been visited then β starts from *Initialization* with all parameters undefined.

At *Initialization* the values of the threshold and witness are determined after that the markers for all elements $n \geq d$ are reset so that $(\mathcal{N}, \Gamma)$ will have some control over the current axioms. The third part of the module, called *Honestification*, ensures that a change in U after an attack will be useful. Then $(\mathcal{N}, \Gamma)$ waits for its witness to enter Ψ^A but always checks if Γ has remained honest, defined below. If $x \in \Psi^A$ and the operator is honest, $(\mathcal{N}, \Gamma)$ is ready to start the *Attack*. After the attack comes the evaluation of the *Result*, which will determine whether the backup strategies should be activated or the requirement $\mathcal{N}$ is satisfied for the moment.

- **Check:** If the threshold is not defined, then go to *Initialization*, otherwise:

1. If $d \notin \overline{K}[s]$ then find the least $n > d$, $n \in \overline{K}[s]$ and let that be the new value of the threshold. Cancel the current witness and start from *Initialization*, initializing all strategies below β. Note that the set $\overline{K}$ is infinite, hence we shall eventually find the right threshold.
2. Scan the elements $n \leq d$ such that $n \notin \overline{K}[s]$. If a marker m of n has been extracted from A at this expansionary stage by α then we will cancel the current witness and start from *Initialization*, initializing all strategies below β. This can happen finitely often as long as the threshold remains permanent, as there are finitely many axioms and hence markers that can be extracted from A for elements $n \leq d$, $n \notin \overline{K}[s]$.

- **Initialization:**

1. If a threshold has not yet been defined or is cancelled, choose a fresh threshold $d > l(\Theta^{U,\overline{W}}, E)[s]$.
2. If a witness has not yet been defined or is cancelled, choose a fresh witness $x \in E[s]$, $d < x$, bigger than any witness defined previously.
3. Wait for a stage s such that $x < l(\Theta^{U,\overline{W}}, E)[s]$. Until such a stage is reached the outcome is $(o = w)$.
4. Extract from A all A_α-markers $m(n)$ for potentially applicable axioms of elements n such that $d \leq n < l(\Theta^{U,\overline{W}}, E)[s]$. Cancel the current markers for the elements $n \in \overline{K}[s]$.
5. For every element $y \leq x$, $y \in E[s]$, enumerate in a list *Axioms* the current valid axiom $\langle y, U_y, \overline{W}_y\rangle \in \Theta[s]$, which was valid the longest, i.e. with least *age* $a(U \oplus \overline{W}, U_y \oplus \overline{W}_y, s)$ (See Definition 1.2.1). Here the definition of $\theta(x)$ at stage s will be modified again to capture the greatest element of precisely these axioms currently listed in *Axioms*. Define the current marker $u(d)$ to be greater than $\theta(x)[s]$ and let the outcome be $(o = h)$.

- **Honestification:** Scan the list *Axioms*. If for any element $y \leq x$, $y \in E[s]$, the listed axiom was not valid at some stage t since the last β-true stage then update the list *Axioms*, let $(o = h)$ and

  1. Extract from A all A_α-markers $m(n)$ of potentially applicable axioms for elements n such that $d \leq n$, cancel the current markers for the elements $n \in \overline{K}[s]$ and define $u(d) > \theta(x)$. This ensures the following property: for all elements $n \geq d$, $n \in \overline{K}[s]$ the U-parts of the axioms in Γ include the U-parts of all axioms listed in *Axioms* for elements $y \leq x$, $y \in E[s]$. If $n \notin \overline{K}[s]$ then all its A_α-markers will be extracted from A so that no new extraction of a marker by the active $\mathcal{P}$-strategy α for these elements can surprise us.

  Otherwise we shall say that the operator is *honest* and move on to:

- **Waiting:** Wait for a stage s such that $x \in \Psi^A[s]$ returning at each successive stage to *Honestification*. Let the outcome be $(o = w)$.

- **Attack:**

  1. If $x \in \Psi^A[s]$ and $u(d) > \theta(x)$ then extract x from E and restrain A on $use(\Psi, A, x)[s]$. The outcome is $(o = g)$ starting a *nonactive* stage for the backup strategies. At this stage they cannot perform any actions except for attacking with their own witnesses.

- **Result:** Let $\bar{x} \leq x$ be the least element that has been extracted from E during the stage of the Attack.

  As this is an expansionary stage $\bar{x} \notin \Theta^{U,\overline{W}}[s]$, hence all axioms for $\bar{x}$ in $\Theta[s]$ are not applicable, in particular the one enumerated in *Axioms*, say $\langle \bar{x}, U_{\bar{x}}, \overline{W}_{\bar{x}} \rangle$. At least one element from $U_{\bar{x}}$ or $\overline{W}_{\bar{x}}$ has been extracted from U or $\overline{W}$ respectively.

We will attach to the witness x the necessary information about this attack, namely a parameter $Attack(x) = \langle \bar{x}, U_{\bar{x}}, \overline{W}_{\bar{x}} \rangle$.

If $\overline{W}_{\bar{x}} \subseteq \overline{W}[s]$ then the attack is successful. The A_α-markers of elements $n \geq d$ have been lifted above $use(\Psi, A, x)[s]$, the restraint on A, as all previously enumerated axioms for elements $n \geq d$ will not be valid. Hence if later on we want to ensure that $\Gamma^{U,A}(n) = 0$ we will only need to extract a marker that is already above the restraint. If the change in $U_{\bar{x}}$ is permanent, then this will lead to success for $(\mathcal{N}, \Gamma)$.

If $\overline{W}_{\bar{x}} \not\subseteq \overline{W}[s]$ then the attack is unsuccessful. The plan is to start the backup strategies and then try again with a new witness. In this case we will move the markers $\gamma(n)$ for $n \geq d$, $n \in \overline{K}[s]$, by extracting the current ones and defining the markers anew in order to provide a safe working space for the backup strategy. We will only do this if we can guarantee that the change in $\overline{W}[s]$ is permanent. We will only evaluate the result at stages s at which $ap(s) > 2\theta(x)$. The value of $ap(s)$ grows unboundedly and the active $\mathcal{P}$-strategy α will perform any action on element $n \geq d$ only at stages at which $ap(s) > 2u(n) \geq 2\theta(x)$. Thus this restriction just delays the work of the strategy by a few stages.

1. Wait until the length of agreement has returned and $ap(s) > 2\theta(x)$. The outcome is $(o = w)$ while we wait.
2. *Unsuccessful attack:* If $\overline{W}_{\bar{x}} \not\subseteq \overline{W}[s]$ then extract from A and cancel all A_α-markers for elements $n \geq d$. Remove the restraint on A and cancel the current witness x. Return to Initialization at the next stage. The outcome is $(o = g)$ starting an *active* stage for the backup strategies.
3. *Successful attack :* If $\overline{W}_{\bar{x}} \subseteq \overline{W}[s]$ then the outcome is $(o = f)$. Return to Result at next stage. Note that if it later on turns out that $\overline{W}$ does change, α will re-evaluate the attack as unsuccessful and proceed with a new cycle.

2.2.3 Analysis of outcomes

We shall list the possible outcomes of the defined modules and determine a right boundary R below which successive strategies are allowed to work. The right boundary is relevant only for $\mathcal{N}$-strategies, it tells them that the set A will not be modified below R by higher priority $\mathcal{N}$-strategies and nonactive $\mathcal{P}$-strategies. The right boundary will move off to infinity as the stages grow. So for example the $(\mathcal{N}, FM)$ strategy working below R after selecting a witness x will (2) Wait for $x \in \Psi^A[s]$ with $use(\Psi, A, x)[s] < R[s]$ and (3) extract x from E and restrain A on $(A \upharpoonright use(\Psi, A, x))[s]$.
$(\mathcal{P}, \Gamma)$ has two possible outcomes:

(**l**) There is a stage after which $l(\Theta^{U,\overline{W}}, E)$ remains bounded by its previous expansionary value. Then $\mathcal{P}$ is trivially satisfied. In this case $\mathcal{N}$ will be satisfied by the strategy $(\mathcal{N}, FM)$ working below right boundary $R = \infty$.

(**e**) There are infinitely many expansionary stages. The $(\mathcal{N}, \Gamma)$-strategy is activated.

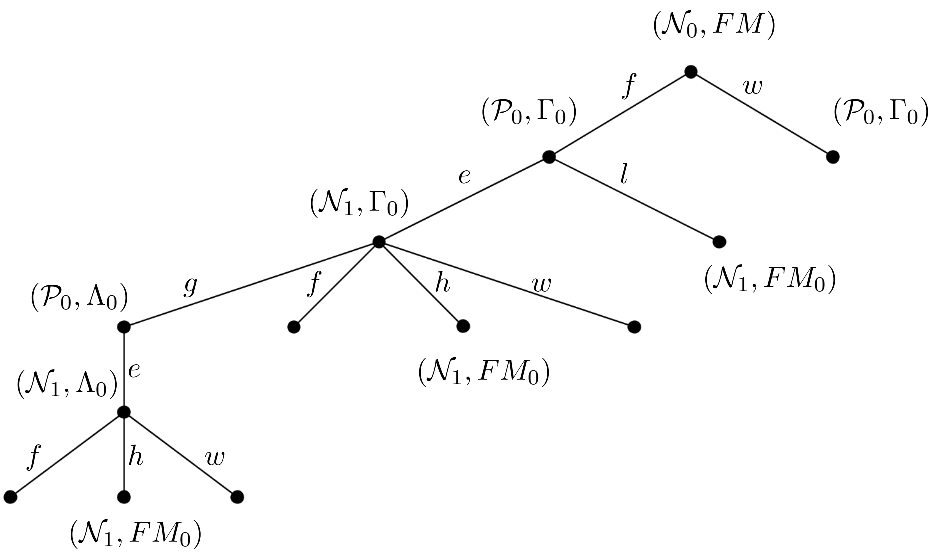
The possible outcomes of $(\mathcal{N}, \Gamma)$ are:

(**w**) There is an infinite wait at *Waiting* for $\Psi^A(x) = 1$ for some witness x. Then $\mathcal{N}$ is satisfied because $E(x) = 1 \neq \Psi^A(x)$ and $(\mathcal{P}, \Gamma)$ remains intact. Successive strategies work below $R = \infty$.

(**f**) There is some witness x with $Attack(x) = \langle \bar{x}, U_{\bar{x}}, \overline{W}_{\bar{x}} \rangle$, for which the attack is permanently successful. Then there is a permanent change in $U_{\bar{x}}$ and the markers of all witnesses are moved above $use(\Psi, A, x)$. At sufficiently large stages $\overline{K} \upharpoonright d$ has its final value. So there is no injury to the strategies below f. $\Psi^A(x) = 1 \neq E(x)$ and $\mathcal{N}$ is satisfied, leaving $(\mathcal{P}, \Gamma)$ intact. $R = \infty$.

(**h**) There are infinitely many occurrences of *Honestification* for some witness x precluding an occurrence of *Attack*. Then there is a permanent witness x which has unbounded $\lim_{sup} \theta(x)$. This means that $\Theta^{U,\overline{W}}(y) = 0$ for some $y \leq x$, $y \in E$, thus $\mathcal{P}$ is satisfied. In this case $(\mathcal{N}, \Gamma)$ destroys the operator Γ, as it changes infinitely often the markers of a fixed element - its threshold. We call this *capricious destruction*. As a consequence we can guarantee that the activity of both $(\mathcal{P}, \Gamma)$ and $(\mathcal{N}, \Gamma)$ will be above $\gamma(d)$ and as the stages grow, the value of $\gamma(d)$ grows unboundedly, providing enough space for lower priority strategies below outcome h to work properly. The requirement $\mathcal{N}$ is satisfied by a second instance of $(\mathcal{N}, FM)$ placed immediately below outcome h working below $R = \gamma(d)$.

(g) We implement the unsuccessful attack step infinitely often. The $(\mathcal{P}, \Gamma)$-strategy is capriciously destroyed in this case as well. As anticipated we must activate the backup strategies. They work below $R = x$.



2.2.4 The backup strategies

The outcome g is visited in two cases: at the beginning of an attack and after an unsuccessful attack. The first case starts a nonactive stage for the subtree below g allowing $\mathcal{N}$-strategies to synchronize their attacks with the one performed by $(\mathcal{N}, \Gamma)$. The second case starts an active stage at which the strategies will do their usual work. Unless otherwise specified the described actions are performed at active stages.

The $(\mathcal{P}, \mathbf{\Lambda})$-strategy

Suppose for definiteness that the $(\mathcal{P}, \Lambda)$-strategy we visit at stage s is $\widehat{\alpha}$. The $(\mathcal{P}, \Lambda)$-strategy acts only at active stages in a similar but less complicated way than the $(\mathcal{P}, \Gamma)$-strategy. The strategy is only visited at expansionary stages. It has only one outcome e.

1. Choose $n < l(\Theta^{U,\overline{W}}, E)$ in turn $(n = 0, 1, \dots)$ and perform following actions:
   - If $w(n) \uparrow$, then define it new as $w(n) = w(n-1)+1$. If $w(n)$ is defined, but $ap(s) < 2w(n)$ skip to the next element.
   - If $n \in \overline{K}[s]$:
     - If $\lambda(n) \uparrow$, then define it anew and define an axiom $\langle n, \overline{W}[s] \upharpoonright w(n), \{\lambda(n)\}\rangle \in \Lambda$.
     - If $\lambda(n) \downarrow$, but $\Lambda^{\overline{W},A}[s](n) = 0$ then define $\lambda(n)$ anew and define an axiom $\langle n, \overline{W}[s] \upharpoonright w(n), \{\lambda(n)\}\rangle \in \Lambda$.
       Note that in this case the old axiom will never be valid again as either the old λ-marker is extracted from A or there is a change in the approximation to $\overline{W}$. In the second case there is some element m used in the old axiom such that $m \in \overline{W}[t] - \overline{W}[s]$, where stage t is when the old axiom was defined. As $ap(s) > 2w(n)$ by Proposition 2.1.1 we have $m \notin \overline{W}[s']$ at all $s' \geq s$.
   - If $n \notin \overline{K}[s]$, but $n \in \Lambda^{\overline{W},A}[s]$ then extract $\lambda(n)$ from A.

The $(\mathcal{N}, \mathbf{\Lambda})$-strategy

Let the $(\mathcal{N}, \Lambda)$-strategy be $\widehat{\beta}$. The actions that $(\mathcal{N}, \Lambda)$ performs are similar to the ones performed by $(\mathcal{N}, \Gamma)$ but are directed at the active $\mathcal{P}$-strategy at $\widehat{\beta}$ which is now $\widehat{\alpha}$. The strategy $\widehat{\beta}$ extracts only $A_{\widehat{\alpha}}$-markers used in the definition of the operator Λ. It has its own threshold $\widehat{d}$, witness $\widehat{x}$. Every attack that this strategy performs will be successful, so the outcomes of this strategy are only

$$f <_L h <_L w.$$

- **Check:** If the threshold is not defined, then go to *Initialization*, otherwise:

  1. If $\widehat{d} \notin \overline{K}[s]$ then find the least $n > \widehat{d}$, $n \in \overline{K}[s]$ and let that be the new value of the threshold. Cancel the current witness and start from *Initialization*, initializing all strategies below $\widehat{\beta}$.

  2. Scan the elements $n \leq \widehat{d}$ such that $n \notin \overline{K}[s]$. If a new $A_{\widehat{\alpha}}$-marker $m(n)$ has been extracted from A at this stage then cancel the current witness and start from *Initialization*, initializing all strategies below $\widehat{\beta}$.

- **Initialization:**

  1. Choose a new threshold $\widehat{d}$, bigger than any defined until now such that $l(\Theta^{U,\overline{W}}, E)[s] < \widehat{d}$.

  2. Choose a new witness $\widehat{x} \in E[s]$ such that $\widehat{d} < \widehat{x}$, bigger than any witness defined until now. Note that when $\widehat{x}$ is chosen β has just started an active backup stage and cancelled its own witness. The next witness that β will use will be defined after this stage and hence will be of value greater than $\widehat{x}$.

  3. Wait for a stage s such that $\widehat{x} < l(\Theta^{U,\overline{W}}, E)[s]$, $(o = w)$.

  4. Extract all $A_{\widehat{\alpha}}$-markers $m(n)$ for elements n such that $\widehat{d} \leq n$ and cancel the current markers for $n \in \overline{K}[s]$.

  5. For every $y \leq \widehat{x}$, $y \in E[s]$, enumerate in the list *Axioms* the current valid axiom from $\Theta[s]$, that has been valid longest. Define $w(\widehat{d}) > \theta(\widehat{x})$, $(o = h)$.

- **Honestification:** If for some $y \leq \widehat{x}$, $y \in E[s]$, the corresponding axiom in *Axioms* was not valid at some stage since the last $\widehat{\beta}$-true stage then update the list and let $(o = h)$ and:

1. Extract all $A_{\widehat{\alpha}}$-markers $m(n)$ for elements n such that $\widehat{d} \leq n$, cancel the current markers for elements $n \in \overline{K}[s]$ and define $w(\widehat{d}) > \theta(\widehat{x})$.

- **Waiting:** Wait for a stage s such that $\widehat{x} \in \Psi^A[s]$ with $use(\Psi, A, \widehat{x})[s] < R[s]$ returning at each successive step to *Honestification*, $(o = w)$. Once this happens go to *Attack*.

- **Attack:**

1. Wait for a nonactive stage, $(o = w)$. This synchronizes the attacks of the two strategies β and $\widehat{\beta}$.
2. If Λ is not honest do nothing and return to *Honestification* at the next active stage. Otherwise extract $\widehat{x}$ from E.

- **Result:** The next stage at which this strategy will be accessible will be an unsuccessful attack for $(\mathcal{N}, \Gamma)$, hence if the strategy does not get initialized due to a $\overline{K} \upharpoonright \widehat{d}$-change, there will be a permanent $\overline{W} \upharpoonright \theta(\widehat{x})$-change and the $(\mathcal{P}, \Lambda)$-strategy will have cleared its λ-markers so that $\widehat{x} \in \Psi^A$ will be preserved:

  The least element that has been extracted during the attack is $\bar{x} \leq \widehat{x}$. This outcome is visited if the attack is unsuccessful, i.e. $\overline{W} \upharpoonright \theta(\bar{x})$ has changed. By the monotonicity of the generalized use function $\theta(\bar{x}) \leq \theta(\widehat{x})$ and we have therefore a change in $\overline{W} \upharpoonright \theta(\widehat{x})$. At the next accessible stage we can simple assume:

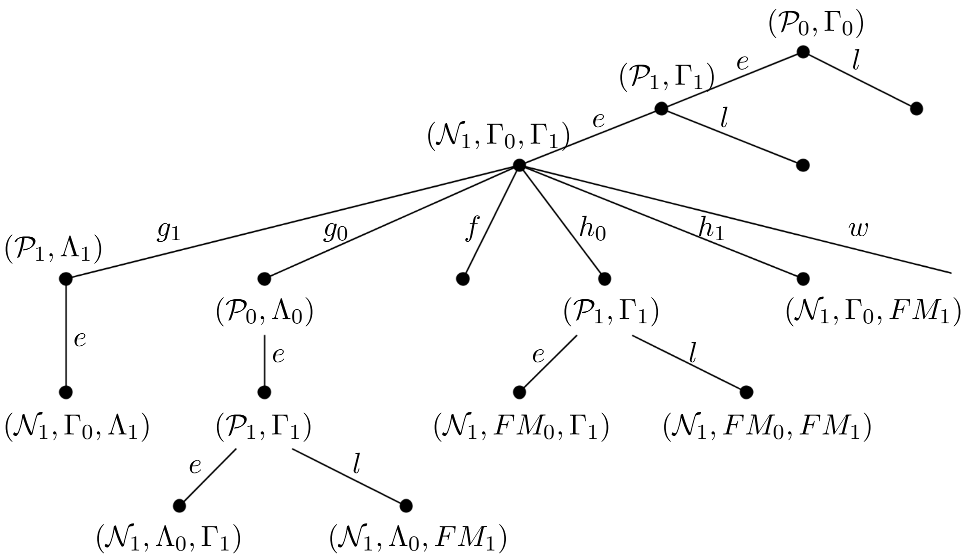
  Successful attack: Return to Result at the next stage, $(o = f)$.

2.3 An $\mathcal{N}$-strategy below two $\mathcal{P}$-strategies

The next levels of the tree of strategies are defined in the expected way. We start up the $(\mathcal{P}, \Gamma)$-strategies for the next $\mathcal{P}$-requirement, $\mathcal{P}_1$. After this follow the strategies for $\mathcal{N}_1$. The most complicated one among them is the $(\mathcal{N}_1, \Gamma_0, \Gamma_1)$-strategy, the one that is placed below the two expansionary outcomes of the two $\mathcal{P}$-strategies working above it. We shall try to give the basic intuition about how this strategy is designed, leaving the formal definition of the various strategies for Section 2.4.2, where a general construction will be given.

As we saw in the previous section an $\mathcal{N}$-strategy can force a $\mathcal{P}$-strategy to change its method to FM below outcome h or to $(\mathcal{P}, \Lambda)$ below outcome g. In both cases the strategy capriciously destroys the operator Γ. Now we need to be more careful as two higher priority $\mathcal{P}$-strategies are involved. As $\mathcal{P}_0 < \mathcal{P}_1$ we will keep in mind when designing the strategy that if the $\mathcal{P}_0$-strategy needs to be changed we can afford to restart the $\mathcal{P}_1$-strategy. If the $\mathcal{P}_1$-strategy is changed though, we must make sure that this does not affect the strategy for $\mathcal{P}_0$.

Let β be the $(\mathcal{N}, \Gamma_0, \Gamma_1)$-strategy with active $\mathcal{P}$-strategies α_0 and α_1. The module of β will be divided in the same submodules: *Check, Initialization, Honestification, Waiting, Attack* and *Result.* Most submodules shall have two copies, one for each active $\mathcal{P}$-strategy. The strategy β will have one witness x but two thresholds $d_2 < d_1$. Some of the outcomes will also come in two copies as can be seen from the following picture:



After the two thresholds and witness are selected at **Initialization** the strategy performs **Honestification** first to Γ_0 with the list $Axioms_0$. If Γ_0 is not honest then β will clear both the A_{α_0}- and A_{α_1}-markers, providing safe working space for strategies below outcome h_0. This will destroy the strategy α_1, therefore below outcome h_0 we shall have a new copy of the $\mathcal{P}_1$-strategy $(\mathcal{P}_1, \Gamma_1)$ starting work from the beginning. If Γ_0 is honest then we will perform $Honestification(1)$. In case Γ_1 is not honest only A_{α_1}-markers will be extracted. If this is the true outcome β shall eventually not extract any A_{α_0}-markers and α_0 will remain intact and still be active for $\mathcal{N}$-strategies below outcome h_1. $\mathcal{N}$-strategies below either outcome h_0 and h_1 will work in the same way as was described in the previous section as they have only one active $\mathcal{P}$-strategy. The only difference is that now they too need to work below a right boundary R set up by $(\mathcal{N}, \Gamma_0, \Gamma_1)$.

Attack is performed once $x \in \Psi^A$ and both operators are honest. There are two sorts of backup strategies: the ones below outcome g_0 and the ones below outcome g_1. A nonactive stage shall be started for strategies below the outcome visited during the previous attack.

Result is performed first for Γ_0. If the attack is 0-unsuccessful then outcome g_0 is visited and capricious destruction is performed on both operators. Again below outcome g_0 we have a copy of the $(\mathcal{P}_1, \Gamma_1)$-strategy starting its work from the beginning. Only if the attack is 0-successful will we examine the result for the second operator Γ_1.

With outcome g_1 we are faced with a difficulty in design. We need to provide safe working space for strategies below this outcome and start a new cycle of the $(\mathcal{N}, \Gamma_0, \Gamma_1)$-strategy. We will only perform capricious destruction on Γ_1. We have seen a 0-successful attack and it looks like the method for satisfying $\mathcal{P}_0$ is correctly chosen. Below outcome g_1 the $\mathcal{P}_0$-active strategy will still be α_0. The difficulty is that when we start a new cycle we will carry on extracting A_{α_0}-markers for d_0 and elements greater than d_0 in order to prepare Γ_0 for the attack with the next witness. If this situation repeats infinitely many times we will have actually destroyed Γ_0 with no advancement on the satisfaction of $\mathcal{P}_0$. We solve this by selecting a new value for the threshold d_0 at every active visit of the outcome g_1. So on each new cycle after an active g_1-visit β will move its activity regarding A_{α_0}, allowing α_0 to remain intact. As a consequence we will need to rethink the **Check** module. We can safely perform *Check* for the Γ_1 as in the previous section, initializing all lower priority strategies should a marker for an element $n < d_1$ be extracted since the last β-true stage. When we perform *Check* for Γ_0 we will only initialize strategies below outcomes which assume that the threshold d_0 is constant. We will not initialize strategies below outcome g_1 as if this is the true outcome, then the value of d_0 grows unboundedly and there is a risk that we might need to initialize at infinitely many stages.

Now we are ready to proceed to the main construction and the proof that it works.

2.4 All requirements

We will start by describing the different strategies connected with each requirement and the outcomes of each strategy. Each $\mathcal{P}$-requirement has two types of strategies. An $\mathcal{N}$-requirement has many types of strategies, depending on the number of $\mathcal{P}$-requirements of higher priority.

For every $\mathcal{P}$-requirement $\mathcal{P}_i$ there is a $(\mathcal{P}_i, \Gamma_i)$-strategy with outcomes $e <_L l$ and a $(\mathcal{P}_i, \Lambda_i)$-strategy with one outcome e.

The requirement $\mathcal{N}_0$ has one strategy $(\mathcal{N}_0, FM)$. For every $\mathcal{N}$-requirement $\mathcal{N}_i$, where $i > 0$, we have strategies of the form $(\mathcal{N}_i, S_0, \dots, S_{i-1})$, where $S_j \in \{\Gamma_j, \Lambda_j, FM_j\}$. The outcomes are f, w and for each $j < i$ if $S_j \in \{\Gamma_j, \Lambda_j\}$ there is an outcome h_j, if $S_j = \Gamma_j$, there is an outcome g_j. They are ordered according to the following rules:

1. For all j_1 and j_2, $g_{j_1} <_L f <_L h_{j_2} <_L w$.
2. If $j_1 < j_2$ then $g_{j_2} <_L g_{j_1}$ and $h_{j_1} <_L h_{j_2}$.

Let $\mathbb{O}$ be the set of all possible outcomes and $\mathbb{S}$ be the set of all possible strategies.

2.4.1 The tree of strategies

The tree of strategies is a computable function $T : D(T) \subset \mathbb{O}^{<\omega} \to \mathbb{S}$ which has the following properties:

1. If $T(\alpha) = S$ and O_S is the set of outcomes for the strategy S then for every $o \in O_S$, $\alpha\hat{}o \in D(T)$.

2. $T(\emptyset) = (\mathcal{N}_0, FM)$ and $\emptyset$ has no active $\mathcal{P}$-nodes.

3. If $T(\alpha) = (\mathcal{N}_i, S_0, S_1, \dots, S_{i-1})$ with active $\mathcal{P}$-nodes $\alpha_1, \dots, \alpha_{i-1}$ then

Below outcome $\mathbf{g_j}$: $T(\alpha\hat{}g_j) = (\mathcal{P}_j, \Lambda_j)$ and $T(\alpha\hat{}g_j\hat{}e) = (\mathcal{P}_{j+1}, \Gamma_{j+1})$, $\dots$, $T(\alpha\hat{}g_j\hat{}e\hat{}o_{j+1}\hat{}o_{i-2}) = (\mathcal{P}_{i-1}, \Gamma_{i-1})$, where $o_k \in \{e_k, l_k\}$ for $j+1 \leq k \leq i-2$.
$T(\alpha\hat{}g_j\hat{}e\hat{}o_{j+1}\hat{}o_{i-1}) = (\mathcal{N}_i, S_0, \dots, \Lambda_j, S'_{j+1} \dots, S'_{i-1})$, where $S'_k = \Gamma_k$ if $o_k = e$ and $S'_k = FM_k$ if $o_k = l$ for every k such that $j < k < i$. The active $\mathcal{P}_k$-nodes at this node are α_k for $k < j$, $\alpha\hat{}g_j$ for $k = j$, if $o_{j+1} = e$ then $\alpha\hat{}g_j\hat{}e$ for $j+1$, if $o_k = e$ then $\alpha\hat{}g_j\hat{}e \dots \hat{}o_{k-1}$ for $k > j+1$. In all other cases there is no active $\mathcal{P}_k$-node.

Below outcome $\mathbf{h_j}$: $T(\alpha\hat{}h_j) = (\mathcal{P}_{j+1}, \Gamma_{j+1}), \dots, T(\alpha\hat{}h_j\hat{}o_{j+1}\hat{}o_{i-2}) = (\mathcal{P}_{i-1}, \Gamma_{i-1})$, where $o_k \in \{e_k, l_k\}$ for $j+1 \leq k \leq i-2$.
$T(\alpha\hat{}h_j\hat{}o_{j+1}\hat{}o_{i-1}) = (\mathcal{N}_i, S_0, \dots, FM_j, S'_{j+1}, \dots, S'_{i-1})$, where $S'_k = \Gamma_k$ if $o_k = e_k$ and $S'_k = FM_k$ if $o_k = l_k$ for every k such that $j < k < i$. There is no active $\mathcal{P}_j$-node at this node, the rest of the active $\mathcal{P}_k$-nodes are defined as below outcome g_j.

Below outcome f: $T(\alpha\hat{}f) = (\mathcal{P}_i, \Gamma_i)$. Then $T(\alpha\hat{}f\hat{}e) = (\mathcal{N}_{i+1}, S_0, \dots, S_{i-1}, \Gamma_i)$ with active $\mathcal{P}$-nodes $\alpha_0, \dots, \alpha_k, \alpha\hat{}f\hat{}e$ and $T(\alpha\hat{}f\hat{}l) = (\mathcal{N}_{i+1}, S_0, \dots, S_{i-1}, FM_i)$ with active $\mathcal{P}$-nodes $\alpha_0, \dots, \alpha_k$ and no active $\mathcal{P}_i$-node.

Below outcome w: $T(\alpha\hat{}w) = (\mathcal{P}_i, \Gamma_i)$. Then $T(\alpha\hat{}w\hat{}e) = (\mathcal{N}_{i+1}, S_0, \dots, S_{i-1}, \Gamma_i)$, with active $\mathcal{P}$-nodes $\alpha_0, \dots, \alpha_k, \alpha\hat{}w\hat{}e$ and $T(\alpha\hat{}w\hat{}l) = (\mathcal{N}_{i+1}, S_0, \dots, S_{i-1}, FM_i)$ with active $\mathcal{P}$-nodes $\alpha_0, \dots, \alpha_k$ and no active $\mathcal{P}_i$-node.

2.4.2 Construction

Following the basic rules from Section 1.3 at each stage s we shall construct a finite path through the tree of strategies $\delta[s]$ of length s starting from the root. The nodes that are visited at stage s shall perform activities as described below and modify their parameters. Each $\mathcal{N}$-node α shall have a right boundary R_α which will also be defined below. $R_\emptyset = \infty$. After the stage is completed, all nodes to the right of the constructed $\delta[s]$ will be initialized, their parameters will be cancelled or set to their initial value $\emptyset$.

An $\mathcal{N}$-strategy on node α works with respect to the active $\mathcal{P}$-strategies at α. It also synchronizes its work with some of the higher priority $\mathcal{N}$-strategies. It will be useful to define a notion of dependency between the different $\mathcal{N}$-strategies.

Definition 2.4.1. *A node α with $T(\alpha) = (\mathcal{N}_i, S_0, S_1, \ldots, S_{i-1})$ depends on node $\beta \subset \alpha$, if $\alpha \supseteq \beta\hat{\ }g_j$ and $S_j = \Lambda_j$ for some j. The node α is independent if it is not dependent on any node $\beta \subset \alpha$.*

If α is dependent it might depend on many of its initial segments. The biggest(closest) node on which α depends will be called the instigator of α, denoted by $ins(\alpha)$. The strategy α must time its attacks with the attacks performed by $ins(\alpha)$, i.e. whenever α is ready to attack, it waits for an $ins(\alpha)$-nonactive stage and attacks on that stage. All the rest of the activity by α is performed only at active stages. We define a stage s to be *nonactive* if a strategy $\sigma \subset \delta[s]$ starts an attack at stage s. Stage s is also σ-nonactive. A stage is active if it is not nonactive. Note that if $\beta\hat{\ }g_j$ is on the true path then there will be infinitely many β-nonactive stages at which $\beta\hat{\ }g_j$ is visited. In fact every $\beta\hat{\ }g_j$-true active stage is followed by a $\beta\hat{\ }g_j$-nonactive stage before the next $\beta\hat{\ }g_j$-true active stage.

In our further discussions we shall denote with M_α, m_α, Z_α and z_α: Γ_α, γ_α, U_α and u_α respectively if α is a $(\mathcal{P}, \Gamma)$-strategy and Λ_α, λ_α, $\overline{W}_\alpha$ and w_α respectively if α is a $(\mathcal{P}, \Lambda)$-strategy. We will denote by s^- the previous α-true stage and by o^- the outcome it had at that stage. If α has been initialized since its previous true stage or if it has never before been visited then $s^- = s$ and o^- is the rightmost outcome.

Suppose we have constructed $\delta[s] \upharpoonright n = \alpha$. If $n = s$ then the stage is finished and we move on to stage $s + 1$. If $n < s$ then α is visited and the actions that α performs are as follows:

(I.) $T(\alpha) = (\mathcal{P}_i, \Gamma_i)$. This strategy is responsible for approximating the sets $U_i, \overline{W}_i$ and Θ_i. It considers the next approximation only at active stages and will define the function $ap_\alpha(s)$ accordingly. At these we perform the actions as stated in the main module in Section 2.2.1. $\delta[s](n + 1) = l$ at non-expansionary stages. At expansionary stages $\delta[s](n+1) = e$. At nonactive stages no actions are performed and $\delta[s](n+1) = o^-$.

(II.) $T(\alpha) = (\mathcal{P}_i, \Lambda_i)$. At active stages we perform the actions as stated in the main module in Section 2.2.4. $\delta[s](n+1) = e$. At nonactive stages no actions are performed, $\delta[s](n + 1) = e$.

(III.) $T(\alpha) = (\mathcal{N}_i, S_0, \dots, S_{i-1})$ with active $\mathcal{P}$-nodes $\alpha_0, \dots, \alpha_{i-1}$. At active stages we perform *Check* first. If it doesn't instruct us otherwise then we carry on with the module from where it was left at the previous α-true stage s^- (from *Initialization* if $s^- = s$). At nonactive stages α may only attack, otherwise it has outcome $o = o^-$.

- **Check:** If a threshold is not defined, then the strategy α goes to *Initialization*, otherwise it performs $Check(j)$ for $j = i - 1, i - 2, \dots, 1$.

  Check($i-1$): Scan all $n \leq d_{i-1}$. If an A_{α_k}-marker for n has been extracted from A by α_k, the active $\mathcal{P}_k$-strategy at α, for $k \leq i - 1$ at a stage t: $s^- < t \leq s$ then we will initialize all strategies below α and start from *Initialization*.

  Check(j): Scan all $d_{j+1} < n \leq d_j$. If an A_{α_k}-marker for n has been extracted from A by α_k for $k \leq j$, at a stage t: $s^- < t \leq s$ then all successors of α, that assume that d_j does not change infinitely often, are initialized. These are strategies γ such that $\gamma \supseteq \alpha\hat{\ } g_k$ for $k \leq j$ or $\gamma \supseteq \alpha\hat{\ } o$, where $o \in \{ h_l, f, w | \ l < i\}$, hence all strategies below and to the right of outcome g_j.

If α is evaluating *Result* and the last active g-outcome was g_l and $l < j$ then α continues from the Initialization step. Otherwise α continues to evaluate *Result*.

If a threshold d_j is extracted from $\overline{K}[s]$ then it is shifted to the next possible value, i.e. to the least $n > d_j, n \in \overline{K}[s]$. If this injures the order between thresholds then the other thresholds are shifted as well.

- **Initialization:** Each strategy $S_j \neq FM_j$ picks a threshold if it is not already defined. The different thresholds must be in the following order:

$$d_{i-1} < d_{i-2} < \cdots < d_0.$$

The strategy picks its j-threshold as a fresh number such that its marker has not yet been defined by the active $\mathcal{P}_j$-strategy α_j. Then α picks a witness $x \in E[s]$ again as a fresh number.

If $l(\Theta_j^{U_j,\overline{W}_j}, E)[s] \leq x$ for some $j < i$ then $\delta(n+1)[s] = w$, working below $R = R_\alpha[s]$.

If $l(E, \Theta_j^{U_j,\overline{W}_j})[s] > x$ for all $j < i$ then α extracts from A all A_{α_j}-markers for all axioms for all elements $n \geq d_j$. Then cancels all current α_j-markers for $n \geq d_j$ and $n \in \overline{K}[s]$ and defines $z_{\alpha_j}(d_j) > \theta_j(x)[s]$.

For every element $y \leq x$, $y \in E[s]$, α enumerates in the list $Axioms_j$ the current valid axiom from $\Theta_j[s]$ that has been valid longest. The next stage will start from *Honestification.* $\delta[s](n+1) = h$, working below $R = R_\alpha[s]$.

- **Honestification:** The strategy α performs $Honestification(0)$, which suggests an outcome. Depending on what this outcome is the strategy may or may not execute $Honestificaion(1)$:

Honestification(j): If $S_j = FM_j$ then $(o = w)$. Otherwise:

1. Scan the list $Axioms_j$. If for some element $y \leq x$, $y \in E[s]$, the listed axiom was not valid at some stage t since the last α-true stage s^- then update the list $Axioms_j$, let (o $= h$) and go to (2) otherwise let (o $= w$).

2. Extract all A_{α_j}-markers of potentially applicable axioms for elements n such that $d_j \leq n < l(\Theta_j^{U_j,\overline{W}_j}, E)[s]$. Cancel the current α_j-markers for elements $n \in \overline{K}[s]$ and define $z_{\alpha_j}(d_j) > \theta_j(x)[s]$.

If the outcome of $Honestification(j)$ is w then α performs $Honestification(j{+}1)$ if $j{+}1 < i$ and goes to *Waiting* if $j{+}1 = i$. If the outcome is h then α extracts all A_{α_k}-markers of potentially applicable axioms for elements $n \geq d_k$ for all $k > j$. Then cancels the current α_k-markers for element $n \in \overline{K}[s]$. The outcome is $\delta[s](n+1) = h_j$ working below $R = \min(R_\alpha[s], m_{\alpha_j}(d_j))$. At the next stage α starts from *Honestification.*

- **Waiting:** If all outcomes of all $Honestification(j)$-modules are w, i.e all enumeration operators are honest then α checks if $x \in \Psi_i^A[s]$ with $use(\Psi, A, x)[s] < R_\alpha[s]$. If not then the outcome is $\delta[s](n{+}1) = w$, working below $R = R_\alpha[s]$. At the next stage α returns to $Honestification.$ If $x \in \Psi_i^A[s]$ with $use(\Psi, A, x)[s] < R_\alpha[s]$ then α goes to *Attack*.

- **Attack:** If α is dependent then it waits for an $ins(\alpha)$-nonactive stage.

  $\delta[s](n+1) = w$, working below $R = R_\alpha[s]$.

  If the stage is $ins(\alpha)$-nonactive, $x \in \Psi^A[s]$, $use(\Psi, A, x)[s] < R_\alpha$ and all operators are honest (i.e. the axioms recorded in the lists $Axioms_j$, $j < i$, have remained valid at all stages since s^-) then α extracts x from E and restrains A on $u(\Psi, A, x)[s]$. This starts an α-nonactive stage for the strategies below the most recently visited outcome g_j (if none has been visited until now then below the leftmost g-outcome) working below the boundaries they worked before.

  Otherwise it will return to *Honestification* at the next active stage.

- **Result:** If at stage s for some $j < i$ we have that $ap_{\alpha_j}(s) < 2\theta_j(x)$ then set $\delta[s](n+1) = w$, working below $R = R_\alpha[s]$.

  Otherwise let $\bar{x}$ be the least element extracted from E during the attack. It has a corresponding entry $\langle \bar{x}, U_{\bar{x},j}, \overline{W}_{\bar{x},j} \rangle$ in $Axioms_j$. Define $Attack(x) = \langle \bar{x}, U_{\bar{x},0}, \overline{W}_{\bar{x},0}, \ldots, U_{\bar{x},i-1}, \overline{W}_{\bar{x},i-1} \rangle$. We will denote by $Attack(x)[j]$ the pair $(U_{\bar{x},j}, \overline{W}_{\bar{x},j})$. Let L be the largest restraint imposed on A during the attack and preform $Result(0)$.

  Result(j):

  If $S_j = FM_j$ or $S_j = \Lambda_j$, then go to $Result(j+1)$. Otherwise if $S_j = \Gamma_j$ and one of the following two conditions is true:

  1. There was a change in $\overline{W}_{\bar{x},j}$, i.e $\overline{W}_{\bar{x},j} \not\subseteq \overline{W}_j[s]$.

  2. For some $k < j$ an A_{α_k}-marker $m_k(n)$ of an element n, where $n < d_k$, was extracted by α_k after the stage of the attack and $m_k(n) < L$.

  Then extract and cancel all A_{α_k}-markers for all elements $n \geq d_k$, where $k \geq j$. Cancel the witness and all thresholds d_l, where $l < j$. Remove any restraint on A and start from *Initialization* at the next stage. $\delta(n+1) = g_j$, working below $R = \min(x, R_\alpha[s])$. Otherwise the attack is j-successful. Go to $Result(j+1)$.

  Result(i): is reached only in case all attacks were successful. Then $\delta(n+1) = f$, working below $R = R_\alpha[s]$. Return to $Result(0)$ at the next stage.

2.5 Proof

2.5.1 The true path

We define *the true path* h to be the leftmost path of nodes on the tree that are visited at infinitely many stages. Such a path exists because the tree is finitely branching. It already has the following properties:

1. $(\forall n)(\exists^\infty s)[h \restriction n \subseteq \delta[s]]$;

2. $(\forall n)(\exists s_l(n))(\forall s > s_l(n))[\delta[s] \not<_L h \upharpoonright n]$.

As outlined in Section 1.3 we will prove that the strategies along the true path satisfy their requirements. To do this we have to first establish that these nodes are initialized at finitely many stages. The leftmost property of the true path deals with initialization at the end of each stage. There is one more case that we have to consider, initialization which occurs during *Check*. So we prove the following lemma.

Lemma 2.5.1. *For every n there is a stage $s_i(n)$ such that $h \upharpoonright n$ does not get initialized after stage $s_i(n)$.*

Proof. We will prove this by induction on the number n. The first case $n = 0$ is trivial, as the root of the tree is never initialized.

Assume that the statement is true for numbers $m \leq n$. Let $s > \max(s_i(n), s_l(n+1))$ be a stage such that at stages $t > s$, $h \upharpoonright n$ is not initialized and $\delta_t \not<_L h \upharpoonright (n+1)$. We will consider the different cases depending on the type of the strategy $h \upharpoonright n$.

I. $T(h \upharpoonright n) = (\mathcal{P}_i, S_i)$, where $S_i \in \{\Gamma_i, \Lambda_i\}$. In this case the strategy $h \upharpoonright (n+1)$ will not be initialized at further stages and $s_i(n+1) = s$.

II. $T(h \upharpoonright n) = (\mathcal{N}_i, S_0, \ldots, S_{i-1})$. Now there are different cases depending on the outcome o along the true path. Starting from the leftmost, we will examine each.

- If $o = g_{i-1}$, then after stage s the threshold d_{i-1} can only change if $d_{i-1} \notin \overline{K}$, and in this case we will shift the value of d_{i-1} to the next element that is currently in the approximation of $\overline{K}$. As $\overline{K}$ is infinite there will be a stage $s_0 \geq s$ after which d_{i-1} will remain fixed. Let $s_1 \geq s_0$ be a stage after which $\overline{K} \upharpoonright (d_{i-1} + 1)$ remains unchanged, i.e. no numbers $z \leq d_{i-1}$ leave $\overline{K}$ after stage s_1. At this stage there are finitely many axioms enumerated in Γ_i for elements $z \leq d_{i-1}$, $z \notin \overline{K}$. And at further stages no new axioms are enumerated for these elements. The markers of these finitely many axioms are the only ones whose extraction from A will force $(h \upharpoonright n)\hat{\ } g_{i-1}$ to be initialized. Let $s_2 \geq s_1$ be a stage by which all of the finitely many markers that get extracted from A are already extracted. Then at stages $t > s_i(n+1) = s_2$, the node $h \upharpoonright (n+1)$ will not be initialized.

- If $o = g_j$, where $0 \leq j < i-1$, then similarly after some stage $s_0 \geq s$ all thresholds d_k for $k \geq j$ will remain fixed, as in order to cancel d_k we need to pass through an outcome g_l with $l > k$ and hence to the left of g_j. But this will not happen according to our choice of stage s. Let $s_1 \geq s_0$ be a stage after which $\overline{K} \upharpoonright (\max(d_j, \ldots, d_{i-1})+1)$ does not change. Again, by that time there are finitely many axioms enumerated in each of the operators S_k, $j \leq k \leq i-1$, for elements $z \notin \overline{K}$. Hence there are finitely many markers whose extraction from A could initialize $h \upharpoonright (n+1)$. Let $s_2 \geq s_1$ be a stage by which all of these finitely many markers that ever get extracted from A are already extracted. Then after stage $s_i(n+1) = s_2$, we have that $h \upharpoonright (n+1)$ will not be initialized.

- If $o \in \{\, w, s, h_j | \; j < i\}$, then after a stage $s_0 \geq s$ all thresholds remain unchanged and after stage $s_1 \geq s_0$ the initial segment $\overline{K} \upharpoonright (\max(d_0, \ldots, d_{i-1})+1)$ remains unchanged. Finally there is a stage $s_2 \geq s_1$, after which no more markers for elements less than or equal to d_j will be extracted from A. Then after stage $s_i(n+1) = s_2$, the node $h \upharpoonright (n+1)$ will not be initialized.

□

From now on for every node $\alpha \subset h$ we will denote by $s_i(\alpha)$ the last stage at which α is initialized. We will prove formally one more property of the true path, one that we have already claimed in the previous sections concerning the distribution of active and nonactive stages.

Propostion 2.5.1. *Suppose $\alpha \hat{\ } g_j \subseteq \beta \subset h$. Then β is visited at infinitely many active and at infinitely many α-nonactive stages.*

Proof. We will prove this with induction on the distance d between α and β.

If the distance is 1 then $\beta = \alpha\hat{\ }g_j$. The g-outcome that α has during an attack is determined by α's previous active g-outcome. β is visited infinitely often, hence it is visited at infinitely many active stages and after each β is visited at an α non-active stage.

Suppose the distance is greater than 1. If there are no nodes σ such that $\alpha\hat{\ }g_j \subset \sigma\hat{\ }g_k \subseteq \beta$ then the same argument proves that β will be visited at an active stage followed by an α-nonactive stage, as at nonactive stages the strategies between α and β will have the same outcome as at the previous active stage. If there is such a σ then induction hypothesis gives us the lemma for α and σ: σ is visited at infinitely many active stages each followed by an α-nonactive visit. By the induction hypothesis again but now for σ and β the strategy β will be visited on infinitely many active stages each followed by a σ-nonactive visit. The only thing left to note is that any σ-nonactive stage is also α-nonactive (although not every α-nonactive stage will be σ-nonactive). □

2.5.2 Satisfaction of the $\mathcal{P}$-requirements

We turn our attention to the $\mathcal{P}$-requirements. First we establish that $\mathcal{P}$-strategies along the true path will succeed in finding a true axiom for each of the elements $n \in \overline{K}$.

Propostion 2.5.2. *Suppose $\Theta_j^{U_j,\overline{W}_j} = E$ and let $\alpha \subset h$.*

1. Suppose $\alpha = (\mathcal{P}_j, \Gamma_j)$ and for some element $n \in \overline{K}$ the current U_α-marker and the γ_α-marker for each $m \leq n$ is not changed by any other strategy after stage s_0. Then α will stop changing the current marker eventually and $n \in \Gamma_j^{U_j,A}$.

2. Suppose $\alpha = (\mathcal{P}_j, \Lambda_j)$. And suppose that for some element $n \in \overline{K}$ the current $\overline{W}_\alpha$-marker and the λ_α-marker for all $m \leq n$ are not changed by any other strategy after stage s_0. Then α will stop changing the current marker eventually and then $n \in \Lambda_j^{\overline{W}_j,A}$.

Proof. The proof is by induction on n. We omit various indices as we only discuss α's parameters. Suppose the lemma is true for all $m < n$. Then:

1. Suppose $u(n)$ remains the same after stage s_0 and the axioms for elements $m < n$ do not change anymore. We will use what we know about the approximation to the set U, namely that it is good and Σ_2. Let G denote the set of good stages. There will be a stage $s_1 \geq s_0$ such that:

   Good: $(\forall t > s_1)[s \in G \Rightarrow U \upharpoonright u(n) = U[t] \upharpoonright u(n)]$;

   Σ_2: $(\forall t > s_1)[U \upharpoonright u(n) \subseteq U[s]]$.

   After stage s_1 the strategy α will examine n. The value of the function $ap_\alpha(t)$ grows unboundedly as the stages progress, so eventually $ap_\alpha(t) \geq 2u(n)$ at all stages $t \geq s_2$. On the other hand the length of agreement $l(\Theta^{U,\overline{W}}, E)[t]$ grows unboundedly at good stages for the approximation of $U \oplus \overline{W}$. As $U \oplus \overline{W}$-good stages are U-good stages, it follows that in fact there will be infinitely U-many good stages on which α examines n.

   Let $s_2 \geq s_1$ be a U-good stage at which α examines n. Let $\langle n, U_n, \{m\}\rangle$ be the current axiom for n at stage s_2. If this axiom is valid at stage s_2 then $U_n \subseteq U[s_2] = U \upharpoonright u(n)$. And hence at all stages $t > s_2$, we have $U_n \subset U[t]$. If the axiom is not valid at stage s_2, then we will enumerate a new axiom $\langle n, U[s_2] \upharpoonright u(n), \{\gamma(n)[s_2]\}\rangle$ in Γ, and for this axiom we will have that at all stages $t > s_2$, $U[s_2] \upharpoonright u(n) \subset U[t]$. In both cases the marker $\gamma(n)$ will not be moved at any later stage $t > s_2$ and the axiom remains valid forever, hence $n \in \Gamma^{U,A}$.

2. Suppose $w(n)$ remains constant after stage s_0 and the axioms for elements $m < n$ do not change anymore. Similarly to the first case we can find a stage $s_1 > s_0$ such that:

   Good: $(\forall t > s_1)[t \in G \Rightarrow \overline{W} \upharpoonright w(n) = \overline{W}[s] \upharpoonright w(n)]$;

   Stable: $(\forall t > s_1)[ap_\alpha(t) > 2w(n)]$.

In fact after stage s_1 the approximation to $\overline{W} \upharpoonright w(n)$ will remain constant. Then at the next α-true stage $s_2 \geq s_1$ we will examine the current axiom for n in Λ, say $\langle n, \overline{W}_n, \{m\}\rangle$. If it is valid at stage s_2 then it will be valid forever. If it is not valid at stage s_2 then we will enumerate a new axiom $\langle n, \overline{W}[s_2] \upharpoonright w(n), \{\lambda(n)\}\rangle$, and this axiom will remain valid forever.

□

We now need to establish that if a $\mathcal{N}$-strategy capriciously destroys a $\mathcal{P}$-strategy along the true path then the $\mathcal{P}$-strategy is either satisfied trivially by $E \neq \Theta^{U,\overline{W}}$ or there is a backup $\mathcal{P}$-strategy also on the true path.

Propostion 2.5.3. *1. Let $\alpha \subset h$ be the biggest $(\mathcal{P}_j, \Gamma)$-strategy. Suppose for some number n the value of the marker $\gamma_\alpha(n)$ grows unboundedly. If $\Theta_j^{U_j,\overline{W}_j} = E$ then there is an $\mathcal{N}$-strategy β such that $\alpha \subset \beta\hat{\ } g_j \subset h$.*

2. Let $\alpha \subset h$ be the biggest $\mathcal{P}_j$-strategy. It builds an operator M_j with A_α-markers denoted by m_α. If for some number n the value of the marker $m_\alpha(n)$ grows unboundedly then $\Theta_j^{U_j,\overline{W}_j} \neq E$.

Proof. 1. Suppose $\Theta_j^{U_j,\overline{W}_j} = E$ and there is some number with unbounded γ_α-marker. Let n be the least such number and let s be a stage after which the A_α-markers for $n' < n$ do not change and are already extracted from A, if they ever get extracted.

If $n \notin \overline{K}$ then there will be a stage s_0 at which n exits $\overline{K}$, i.e. $n \notin \overline{K}[t]$ for all $t > s_0$. After stage s_0 the marker $\gamma_\alpha(n)$ will remain constant. Hence $n \in \overline{K}$.

At any stage t there are finitely many thresholds $d_j[t] \leq n$ assigned to nodes in the tree. If $\gamma_\alpha(n)$ grows unboundedly then it follows from Proposition 2.5.2 that there is an $\mathcal{N}$-strategy $\beta \supseteq \alpha\hat{\ }e$ with active $\mathcal{P}_j$-strategy α and a constant threshold $d_j \leq n$, whose γ_α-marker also grows unboundedly. According to our choice of n

as the least element with unbounded γ_α-marker, $n = d_j$. Furthermore β is on the true path. Strategies to the left of the true path are accessible at finitely many stages hence change the markers of their thresholds finitely often. A strategy to the right of the true path is initialized infinitely often and at initialization a new value for its threshold is chosen.

Suppose that β is a $(\mathcal{N}_i, S_0, \ldots, S_{i-1})$-strategy. The true outcome of β cannot be g_k with $k > j$, because then d_j would change its value at infinitely many stages. Outcomes g_k with $k < j$ and h_k for $k < j$ are followed by a new $(\mathcal{P}_j, \Gamma_j)$-strategy and hence are also impossible according to our assumption. Outcomes f and w and h_k for $k > j$ do not move $\gamma_\alpha(d_j)$ infinitely often, hence $\gamma_\alpha(d_j)$ would be bounded. Suppose that h_j is β's true outcome. And let x be β's witness at stages $t > s_l(|\beta| + 1)$, i.e stages after which β does not have outcomes to the left of h_j. Then the entry for x in $Axioms_j^\beta$ changes at infinitely many stages and hence $x \notin \Theta_j^{U_j, \overline{W}_j}$. On the other hand $x \in E$ as β never attacks with this witness. This contradicts the assumption that $\Theta_j^{U_j, \overline{W}_j} = E$.

Thus the only possible outcome is g_j.

2. Towards a contradiction assume that $\Theta_j^{U_j, \overline{W}_j} = E$, but there is some number with unbounded γ_α-marker. Let n be the least such number. If $M_j = \Gamma_j$, then by the previous case there will be a strategy β with $\alpha \subset \beta\hat{\ } g_j$ along the true path, followed by another $\mathcal{P}_j$-strategy. This contradicts α being the biggest one.

   Hence $M_j = \Lambda_j$. Let s be a stage after which the markers for $n' < n$ do not change and are extracted from A, if they ever get extracted.

   Again if $n \notin \overline{K}$ then there will be a stage s_0 at which n exits $\overline{K}$ and after which $\lambda_\alpha(n)$ remains the same. Hence $n \in \overline{K}$.

   As in the previous case it is clear that $n = d_j$ for some threshold and some $(\mathcal{N}_i, S_0, \ldots, \Lambda_j, \ldots, S_{i-1})$-strategy $\beta \supset \alpha$ along the true path. The true outcome of the strategy β cannot be g_k with $k > j$, because then the value of d_j would grow unboundedly. Outcomes g_k with $k < j$ and h_k for $k < j$ are followed by a new $\mathcal{P}_j$-strategy and hence are impossible as well. Outcomes f and w and h_k for $k > j$ do not move $\lambda_\alpha(d_j)$ infinitely often, hence $\lambda_\alpha(d_j)$ would be bounded.

As β does not have outcome g_j, the only possible outcome is h_j. With a similar argument as in case 1, this yields $E \neq \Theta_j^{U_j,\overline{W}_j}$, giving us the desired contradiction.

□

Corollary 2.5.1. *Every requirement $\mathcal{P}_j$ is satisfied.*

Proof. If $\Theta_j^{U_j,\overline{W}_j} \neq E$ then the requirement is trivially satisfied. Suppose we have $\Theta_j^{U_j,\overline{W}_j} = E$. Let $\alpha \subset h$ be the biggest $(\mathcal{P}_j, M_j)$-strategy along the true path. Then α has infinitely many expansionary stages and by Propositions 2.5.2 and 2.5.3 all A_α-markers used to build the operator M_j are bounded.

For each n we prove that $\overline{K}(n) = M_j^{Z_j,A}(n)$, where $Z_j = U_j$ if $M_j = \Gamma_j$ and $Z_j = \overline{W}_j$ if $M_j = \Lambda_j$.

If $n \notin \overline{K}$ then n is extracted from $M_j^{Z_j,A}$ at least once at every α-true expansionary stage t at which $ap_\alpha(t) > 2z_j(n)$. By Proposition 2.5.2 there are infinitely many such stages, thus $n \notin M_j^{Z_j,A}$.

If $n \in \overline{K}$, then Proposition 2.5.2 proves that $n \in M_j^{Z_j,A}$. □

2.5.3 Satisfaction of the $\mathcal{N}$-requirements

We turn our attention to the $\mathcal{N}$-requirements, examining first the interactions between them.

Lemma 2.5.2. *Let $\alpha \subset h$ be an $\mathcal{N}_i$ requirement along the true path.*

1. *For every α-true stage $s > s_i(\alpha)$ and every stage $t > s$ none of the nodes to the right or to the left of α extract elements from $A[t]$ that are less than $R_\alpha[s]$.*

2. *For every α-true stage $s > s_i(\alpha)$ and every stage $t > s$ none of the $\mathcal{N}_j$-nodes above α extract elements from $A[t]$ that are less than $R_\alpha[s]$.*

3. *Suppose $\beta \subset \alpha$ is a $\mathcal{P}_j$-node which is not the active $\mathcal{P}_j$-node at α. Then for every α-true stage $s > s_i(\alpha)$ and every stage $t > s$ the strategy β does not extract elements from $A[t]$ that are less than $R_\alpha[s]$.*

Hence after stage $s_i(\alpha)$ the only strategies above α that extract elements from A that are less than the right boundary are the active $\mathcal{P}$-strategies at α.

Proof. 1. The nodes to the left of α are not accessible at stages $t > s_i(\alpha)$ and do not extract any elements at all. Nodes to the right are initialized at every α-true stage s. The $\mathcal{P}$-strategies visited at stage $t > s$ will choose their markers to be bigger than the current $R_\alpha[s]$ and $\mathcal{N}$-strategies work with new thresholds whose markers are defined after this and are bigger than $R_\alpha[s]$.

2. We prove this case with induction on the length of α using the fact that if $\beta \subset \alpha$ then $R_\alpha \leq R_\beta$. The first case when $l(\alpha) = 0$ is trivial.

   Let α be of length $n > 0$ and let β be the greatest $\mathcal{N}$-node above α, β is an $(N_j, S_0, \ldots, S_{j-1})$-strategy with $j \leq i$. Let $s > s_i(\alpha)$ be an α-true stage.

   By the induction hypothesis at stages $t > s$ none of the $\mathcal{N}$-nodes above β extract elements less than $R_\beta[s]$, hence they do not extract element less than $R_\alpha[s] \leq R_\beta[s]$. Thus we only need to prove that β also respects this boundary $R_\alpha[s]$ at stages $t > s$. As β is the largest $\mathcal{N}$-strategy above α, β is the strategy that will determine $R_\alpha[s]$ at stage s. We have a few cases depending on the true outcome of β.

   Case $\beta\hat{\ }g_l \subset \alpha$, where $l < j$: then $R_\alpha[s] = \min(x, R_\beta)[s]$. At stages $t > s_i(\alpha)$ outcomes to the left will not be accessible. At stage s the thresholds d_k, for $k < l$ are cancelled and then redefined at the next β-true stage to be bigger than

$R_\alpha[s]$. Their A-markers at the active $\mathcal{P}_k$-strategy, and hence the corresponding A-markers of all element $n \geq d_k$, are chosen at a later stage and hence are bigger than $R_\alpha[s]$. For $r \geq l$ all A-markers at the active $\mathcal{P}_r$-strategy for $n \geq d_r$ are cancelled and extracted from $A[s]$. Any new A-marker defined after stage s will be bigger than $R_\alpha[s]$. Thus at stages $t > s$ the strategy β will not extract elements less than $R_\alpha[s]$.

Case $\beta\hat{\ }h_l \subset \alpha$, $l \geq 0$: then $R_\alpha[s] = \min(m(d_l)[s], R_\beta[s])$. At stages $t > s_i(\alpha)$ the outcomes to the left are not accessible and for every $k < l$ the A-markers at the $\mathcal{P}_k$-active strategy for elements $n \geq d_k$ are not extracted by β. For every $r \geq l$ the A-markers at the active $\mathcal{P}_r$ strategy for $n \geq d_r$ are cancelled at stage s and later redefined to be bigger than $R_\alpha[s]$.

Case $\beta\hat{\ }f \subset \alpha$ or $\beta\hat{\ }w \subset \alpha$: then β does not extract elements at any stage $t > s_i(\alpha)$.

This completes the proof of the induction step and hence the statement.

3. If β is not the active $\mathcal{P}_j$-node at α then either $\beta\hat{\ }l \subset h$ in which case β dose not extract any markers after stage $s_i(\alpha)$ or there is an $\mathcal{N}_k$-strategy γ with $j < k \leq i$, $\beta \subset \gamma \subset \alpha$ and true outcome $o \in \{\ h_l, g_l |\ l \leq j\}$.

   Whatever γ's true outcome is its j-threshold d_j remains unchanged after stage $s_i(\alpha)$. At every α-true stage s we visit $\gamma\hat{\ }o$ and the A_β-markers for $n \geq d_j$ are cancelled and extracted from A. The values of the new A_β-markers will be bigger than $R_\alpha[s]$. Hence if β extracts an element at stage t of value less than $R_\alpha[s]$ then it must be a marker of an element $n < d_j$. In this case at the next γ-true stage $t' \geq t$ during $Check(j)$ the strategy α would be initialized.

$\square$

We claimed that the right boundary R moves off to infinity. Here we give a formal proof.

Propostion 2.5.4. *For every node α along the true path* $\lim_s R_\alpha[s] = \infty$.

Proof. We prove this statement by induction on the length $|\alpha|$ of the node $\alpha \subset h$. The case $|\alpha| = 0$ is trivial because then $T(\beta) = \mathcal{N}_0$, and $R_\alpha = \infty$. Suppose the statement is true for α, we will prove it for its successor on the true path $\alpha\hat{\ }o$.

If α is a $\mathcal{P}$-strategy or α is an $\mathcal{N}$-strategy with $o \in \{w, f\}$ then $R_{\alpha\hat{\ }o}[s] = R_\alpha[s]$ for every $\alpha\hat{\ }o$-true s.

If $o = g_j$, where $j < i$, then the boundary is $R_{\alpha\hat{\ }o}[s] = \min(x[s], R_\alpha[s])$ at every $\alpha\hat{\ }o$-true stage s, where $x[s]$ is α's witness at stage s. The witness $x[s]$ is cancelled at such stages s and later redefined to be bigger. Hence $\lim_s x[s] = \infty$ and from this and the induction hypothesis it follows that $R_{\alpha\hat{\ }o}$ grows unboundedly.

If $o = h_j$, where $j < i$, then $R_{\alpha\hat{\ }o}[s] = \min(m_j(d_j)[s], R_\alpha[s])$, where $m_j(d_j)$ is the current A-marker of α's threshold d_j defined by the active $\mathcal{P}_j$-strategy at α. At every $\alpha\hat{\ }o$-true stage s this marker is cancelled and later redefined bigger, hence again $R_{\alpha\hat{\ }o}$ grows unboundedly. □

We prove two technical, but rather easy properties of the construction. We have claimed these properties already and state them here for completeness.

Propostion 2.5.5. *Let $\alpha \subset h$ be an $\mathcal{N}$-strategy with $ins(\alpha) = \beta$. Suppose $\alpha \supset \beta\hat{\ }g_j$ and α attacks with a witness $\widehat{x}$ at stage t together with an attack of β with x. Then $Attack(\widehat{x})[k] = Attack(x)[k]$ for all $k \leq j$.*

Proof. Let $T(\alpha) = (\mathcal{N}_i, S_0, \ldots, S_{i-1})$. It follows from the definition of an instigator that $S_j = \Lambda_j$ and both strategies α and β are dealing with the same approximations of the sets Θ_j, U_j, V_j controlled by the active $\mathcal{P}_j$-strategy at β. First we will prove that the active $\mathcal{P}_k$-strategies at α for $k < j$ are the active $\mathcal{P}_k$-strategies at β. Suppose this is not true. Then some $\mathcal{P}_k$ active strategy at β was destroyed by some $\mathcal{N}$-strategy σ such that $\beta\hat{\ }g_j \subseteq \sigma \subset \alpha$. If σ has a g-outcome then it would be the instigator of α. Hence it had outcome h_k, where $k < j$. But then $\mathcal{P}_j$ starts from $(\mathcal{P}_j, \Gamma_j)$ below $\sigma\hat{\ }h_k$

and can only change back to Λ_j if a second strategy σ' such that $\sigma\hat{\ }h_k \subseteq \sigma' \subset \alpha$ has outcome g_j in which case σ' would be the instigator of α. Hence for all $k \leq j$ both α and β are dealing with the same approximations to the sets Θ_k, U_k and V_k.

By Proposition 2.5.1 α is visited at β-active stages, followed by β-nonactive stage. Stage t is a β-nonactive stage, let t^- be the previous β-active α-true stage. At this stage α had in its $Axioms_k^\alpha$ for $k \leq j$ a list of axioms for all elements $y \leq \widehat{x}$, which were valid the longest. After stage t^- the strategy β chooses its witness $x > \widehat{x}$ and fills in the corresponding lists $Axioms_k^\beta$. For elements $y \leq \widehat{x}$ these are the same axioms that α recorded. If during $\beta's$ work, one of the list changes its entry for an element $y \leq \widehat{x}$ then at stage t the strategy α would not attack but go back to *Honestification* instead and wait for an active stage at which to modify its own lists. Hence the entry in all $Axioms_k^\beta$ for elements $y \leq \widehat{x}$ is the same as the entry $Axioms_k^\alpha$ for all $k \leq j$ and in particular the entries are the same for the least element extracted during the attack at stage t, say $\bar{x} \leq \widehat{x} < x$. Hence $Attack(\widehat{x})[k] = Attack(x)[k]$ for all $k \leq j$. □

Propostion 2.5.6. *Suppose an $\mathcal{N}_i$-strategy $\alpha \subset h$ with active $\mathcal{P}_j$-strategy β_j starts an attack with witness x at stage s. Let $y \leq x$, $y \in E$ be a number with entry $\langle y, U_{y,j}, \overline{W}_{y,j}\rangle$ in $Axioms_j[s]$. Let $n \geq d_j$ be a number with a potentially applicable axiom $\langle n, Z_{n,j}, \{m\}\rangle$ at stage s at β_j. Then $Z_{n,j} \supseteq U_{y,j}$ if β_j is a $(\mathcal{P}, \Gamma)$-strategy and $Z_{n,j} \supseteq \overline{W}_{y,j}$ if β_j is a $(\mathcal{P}, \Lambda)$-strategy.*

Proof. We consider the case when β_j is a $(\mathcal{P}, \Gamma)$-strategy. During $\alpha's$ cycle with x, the strategy completes *Initialization* at stage s_0, say. Let $s_1 \geq s_0$ be the last stage at which α performs $Honestification(k)$ for $k \leq j$. At stage s_1 the strategy extracts the markers of all potentially applicable axioms for n and cancels its marker $u_j(n)$. Thus at stages $t > s_1$ the marker $u_j(n)[t]$ is defined to be bigger than $u(d_j)(n)[s_1] \geq \theta(x)[s_1] \geq \max(U_{y,j})$. The axiom $\langle n, Z_{n,j}, \{m\}\rangle$ is enumerated in Γ_j at a stage $s_2 > s_1$ and $Z_{n,j} = U[s_2] \upharpoonright u_j(n)[s_2]$. As α attacks at stage s, the operator Γ_j is honest at stages t with $s_1 < t \leq s$, in particular it is honest at stage s_2, i.e. $U_{y,j} \subseteq U[s_2]$, and hence $Z_{n,j} \supseteq U_{y,j}$. □

Given an $\mathcal{N}_i$-strategy $\alpha = (\mathcal{N}_i, S_0, \ldots, S_{i-1})$ along the true path with true outcome f, it is easy to verify that α is j-successful if $S_j = \Gamma_j$ or $S_j = FM_j$. If $S_j = \Lambda_j$ though the success depends on the actions of the instigator of α. We prove that our construction ensures j-success for the last witness x of α.

Lemma 2.5.3. *Suppose $\alpha \subset h$, $T(\alpha) = (\mathcal{N}_i, S_0, \ldots, S_{i-1})$ and $S_j = \Lambda$. Suppose α begins an attack with witness x at stage $s > s_i(\alpha)$. Then at the next stage, at which α is accessible, either there is a $\overline{W}_j \restriction \theta_j(x)$-change or α has a g-outcome or else α is restarted during $Check$.*

Proof. Unfortunately to prove this lemma we need to consider all j such that $S_j = \Lambda_j$. So suppose that $i_0 < i_1 < \cdots < i_r$ are the indices among $0, 1, \ldots, i-1$ such that $S_{i_j} = \Lambda_{i_1}$. Then there are strategies $\alpha_0, \ldots, \alpha_r$ such that $\alpha_0\hat{\ }g_{i_0} \subset \alpha_1\hat{\ }g_{i_1} \ldots \alpha_r\hat{\ }g_{i_r} \subset \alpha$. Furthermore $ins(\alpha) = \alpha_r$, …, $ins(\alpha_1) = \alpha_0$.

Suppose the active $\mathcal{P}$-strategies at α are $\beta_0, \ldots, \beta_{i-1}$. Then β_k for $k < i_0$ are the active $\mathcal{P}$-strategies at α_l, $l \geq 0$; the strategies β_k for $k < i_1$ are active at α_l, $l \geq 1$; …; the strategies β_k for $k < i_r$ are active at α_{i_r}.

At stage s all strategies $\alpha_0, \ldots, \alpha_r$ attack with their own witnesses $x_0, \ldots, x_r$. As α will be visited again at stage s^+ each of the strategies α_j, $j \leq r$ will have a g-outcome at a least stage $s_j > s$.

When each of these strategies is visited for the first time after stage s it records an entry in its parameter $Attack$. The first entry of all parameters $Attack^{\alpha}(x)$ and $Attack^{\alpha_j}(x_j)$, $j \leq r$ is the same: $\bar{x}$, the least element extracted from E during the attack at stage s. By Proposition 2.5.5 we have that $Attack^{\alpha}(x)[k] = Attack^{\alpha_j}(x_j)[k]$ for all $k \leq j$ and all $j \leq r$. Furthermore the value of the parameter L, the largest element restrained in A during the attack at stage s, is the same for all α and α_j,

$j \leq r$.

We will prove by induction on j that if α is not restarted and does not have a g-outcome at stage s^+, then there is a permanent $\overline{W}_j$-change. Assume that the statement is true for $k < j$.

At stage $s_j > s$ the strategy α_j is visited and has outcome g_k. It follows that $k \leq j$ as α is not initialized at stage $s_j > s_i(\alpha)$.

Thus at stage s_j the strategy α_j evaluates $Result(k)$ with $Attack(x_j)[k] = (U_{\bar{x},k}, \overline{W}_{\bar{x},k})$ and one of the two clauses is valid:

(1) There was a change in $\overline{W}_{\bar{x},k}$, i.e. $\overline{W}_{\bar{x},k} \not\subseteq \overline{W}_k[s_j]$.

(2) For some $l < k$ an A_{β_l}-marker $m_l(n)$ of an element n, where $n < d_l^\beta$, was extracted by β_l after the stage of the attack and $m_l(n) < L$.

Suppose (1) is valid. As the result is evaluated at a stage s_j only if $ap_{\beta_k}(s_j) > 2\theta_k(x) > 2\theta_k(\bar{x})$, this change in $\overline{W}$ is permanent. So at stage s^+ when we visit α and it evaluates its result we will have $\overline{W}_{\bar{x},k} \not\subseteq \overline{W}_k[s^+]$.

Suppose $k < j$. If at stage s_j, α reaches $Result(k)$ without having a g-outcome for $l < k$, then (1) will be valid as well for α's evaluation of $Result(k)$. Thus α will have a g-outcome contradicting our assumptions.

If $k = j$ then α has the desired $\overline{W}_j$-change.

Suppose that (2) is valid. Then β_l, for some $l < k$ extracted a marker $m_l(n) < L^\beta[s_j] = L^\alpha[s^+]$ for an element $n < d_l^{\alpha_j}$. This marker was defined before stage s and the corresponding axiom was potentially applicable at stage s. If $n < d_l^\alpha$ then clause (2) will be valid for α's evaluation of $Result(k)$ at stage s^+ and if α is not restarted during $Check$ it will have a g-outcome, contradicting our assumption again.

Suppose $d_l(\alpha) < n$ and let us consider what happened with the attack regarding $\Theta_l^{U_l, \overline{W}_l}$. If β_l is an $(\mathcal{P}_l, \Lambda_l)$-strategy then $i_p = l$, where $p < j$. By induction for $p < j$ we have the necessary $\overline{W}_l$-change at stage s_p before β_l is accessible again after stage s. This change is permanent and by Proposition 2.5.6 all potentially applicable axioms for n at β_l are invalid at stages $t \geq s_p$. It follows that β_l will not extract $m_l(n)$.

This leaves us with the only possibility that β_l is a $(\mathcal{P}, \Gamma)$-strategy. Thus at an expansionary β_l-true stage $s' > s$ the strategy β_l sees that the potentially applicable at stage s axiom for n in Γ_l, say $\langle n, U_{l,n}, \{m_l(n)\}\rangle$, is valid. Thus by Proposition 2.5.6 $U_{\bar{x},l} \subseteq U_{l,n} \subseteq U[s']$ and as s' is expansionary it must be the case that $\overline{W}_{\bar{x},l} \not\subseteq \overline{W}[s']$. Finally as $ap_{\beta_l}(s') > 2u_l(n) > 2\theta(\bar{x})$ we have that this change is permanent. Then at stage s^+ when α evaluates *Result*(l) clause (1) will be valid and α will have outcome g_l again contradicting the assumption. Thus α has the desired $\overline{W}_j$-change which completes the induction step and the proof of the lemma. □

Corollary 2.5.2. *Every requirement $\mathcal{N}_i$ is satisfied.*

Proof. Let α be the last $\mathcal{N}_i$-strategy along the true path. We will prove that it satisfies $\mathcal{N}_i$. The strategy α has true outcome w or f, otherwise there will be a successive $\mathcal{N}_i$-strategy along the true path.

In the first case there is a stage s_1 after which α has only outcome w without passing through any other outcome. Let $s_2 \geq s_1$ be a stage after which $\alpha\hat{\ }w$ is not initialized or restarted. Then the thresholds remain constant after stage s_2 and so does the witness x. The strategy α waits forever for $\Psi_i^A(x) = 1$ with use below R_α. The right boundary R_α grows unboundedly by Proposition 2.5.4, giving $\Psi_i^A(x) = 0$ and $E(x) = 1$.

Let the true outcome be f. Then the strategy α has a permanent witness x at stages $t > s_i(\alpha\hat{\ }f)$ with which it has attacked at a previous stage, as outcome f is only accessible after a successful attack. Let s be the stage of the attack with x. Then $x \in \Psi_i^A[s]$ with $use(\Psi_i, A, x)[s] < R_\alpha[s]$. Hence by Lemma 2.5.2 none of the strategies to the left, right and above, except for the active $\mathcal{P}$-strategies at α, will extract elements $n < R_\alpha[s]$ from A at stages $t > s$.

The active $\mathcal{P}$-strategies at α do not extract any markers below the restraint either. Indeed if a marker $m(n) < use(\Psi_i, A, x)$ for some element $n < d_j$, $j < i$, is extracted from A, then the witness x would be cancelled. On the other hand after α attacks with x at stage s, it receives all required permissions. This is clear for $S_j = \Gamma_j$: the permission is correct and permanent, as otherwise α would have a g-outcome to the left of f. By Lemma 2.5.3 if $S_j = \Lambda_j$ the permission is correct as well, otherwise the witness would be cancelled. By Proposition 2.5.6 the potentially applicable axioms at stage s for element $n \geq d_j$, $j < i$, at the corresponding $\mathcal{P}_j$-active strategy β_j will not be valid at any stage β_j-true stage $t > s$.

Strategies below $\alpha\hat{\ }f$ are accessible for the first time after their initialization at the stage of the attack. All their A-markers are defined after the stage of the attack and would be greater than the restraint on A. Every new threshold is bigger than the thresholds used by α, and their markers will be defined above the restraint. Hence strategies below $\alpha\hat{\ }f$ will never injure $x \in \Psi_i^A$.

Thus $A[s] \upharpoonright use(\Psi_i, A, x)[s] \subseteq A$ and $n \in \Psi_i^A \setminus E$. This completes the proof of the lemma and of Theorem 2.0.2. $\square$

Chapter 3

Non-splitting in the Σ^0_2 Enumeration Degrees

We are ready to break the thread that ties our proofs to the known techniques created for problems of the Turing unverse and move on to the far more complicated world of the Σ^0_2 enumeration degrees.

We already have partial knowledge of the non-splitting properties of the Σ^0_2 enumeration degrees. We have shown the existence of an incomplete Π^0_1 enumeration degree such that there is no nontrivial splitting of $0'_e$ by a pair of a Σ^0_2 enumeration degree and Π^0_1 enumeration degree above it. The restriction of the second element in the considered pairs to the class of the Π^0_1 enumeration degrees is essential. Arslanov and Sorbi [AS99] have shown that there is a Δ^0_2-splitting of $0'_e$ above every incomplete Δ^0_2 enumeration degree. The question that remains to be answered is whether or not $0'_e$ can be split above every incomplete Σ^0_2 enumeration degree. In this chapter we present the result of our investigations, an analog of Harrington's non-splitting theorem for $\mathcal{D}_e(\leq 0'_e)$:

Theorem 3.0.1. *There is a Σ^0_2 enumeration degree $\mathbf{a} < \mathbf{0'_e}$ such that $\mathbf{0'_e}$ cannot be split in the enumeration degrees above the degree $\mathbf{a}$.*

This provides some insight to the structure of properly Σ^0_2 enumeration degrees within $\mathcal{D}_e(\leq 0'_e)$. Cooper and Copestake [CC88] prove the existence of properly Σ^0_2 enumeration degrees that are incomparable with any nonzero incomplete Δ^0_2 enumeration degree. An enumeration degree with a non-splitting property of Theorem 3.0.1 is properly Σ^0_2 and does not have any Δ^0_2 enumeration degrees above it by the result of Arslanov and Sorbi, [AS99]. Kalimullin [Kal03] shows that $0'_e$ and the enumeration jump are definable in the partial ordering $\mathcal{D}_e$. This allows us to define a nonempty set of enumeration degrees $\mathcal{F}$ consisting entirely of properly Σ^0_2 enumeration degrees in $\mathcal{D}_e$:

$$\mathcal{F} = \{\ a|\ a < 0'_e \wedge (\forall u, v)[a \leq u < 0'_e \wedge a \leq v < 0'_e \Rightarrow u \vee v \neq 0'_e]\}.$$

The set $\mathcal{F}$ is upwards closed in the properly Σ^0_2 enumeration degrees, thus $\mathcal{F} \cup \{0'_e\}$ is a filter in $\mathcal{D}_e(\leq 0'_e)$. Furthermore by the density of the Σ^0_2-enumeration degrees, [Coo84], $\mathcal{F}$ is an infinite set of properly Σ^0_2 enumeration degrees.

The work presented in this chapter will be published in [Sos08a].

3.1 Requirements and strategies

The statement of Theorem 3.0.1 is very similar to the main theorem of Chapter 2. As we shall see the requirements for example are almost the same. The main difference is that in the $\mathcal{P}$-requirements we are dealing with a pair of Σ^0_2 enumeration sets instead of a pair of a Σ^0_2 and a Π^0_1 set. We are also less restricted in the construction of the required set, which is now allowed to be a Σ^0_2 set instead of a Π^0_1 set. Naturally our strategies will follow the main ideas from Chapter 2. In this section we shall explain some further difficulties that we will need to consider when designing the strategies.

We assume a standard listing of all enumeration operators $\{\Psi_i\}_{i<\omega}$ and of all triples $\{(\Theta, U, V)_i\}_{i<\omega}$ of enumeration operators Θ, Σ^0_2 sets U and V. We will construct a Σ^0_2 set A whose enumeration degree $\mathbf{a}$ will be the one required in Theorem 3.0.1 and an auxiliary Π^0_1 set E to satisfy the following list of requirements:

1. The degree **a** should be incomplete. We shall use the set E to witness the incompleteness of A.
$$\mathcal{N}_i : E \neq \Psi_i^A.$$

2. Any pair of incomplete Σ_2^0 enumeration degrees **u** and **v** above **a** should not form a splitting of $\mathbf{0}'_e$.
$$\mathcal{P}_i : E = \Theta_i^{U_i,V_i} \Rightarrow (\exists \Gamma_i, \Lambda_i)[\overline{K} = \Gamma_i^{U_i,A} \vee \overline{K} = \Lambda_i^{V_i,A}].$$

The requirements shall be given the same priority ordering:
$$\mathcal{N}_0 < \mathcal{P}_0 < \mathcal{N}_1 < \mathcal{P}_1 < \dots.$$

The first $\mathcal{N}$-requirement can be satisfied by the simple Friedberg-Muchnik strategy $(\mathcal{N}, FM)$.

To satisfy a single $\mathcal{P}$-requirement, with corresponding triple (Θ, U, V) we are given again three options. The first and simplest one is to provide some proof that $\Theta^{U,V} \neq E$. The other two options are to construct enumeration operators Γ or Λ proving that at least one of the sets U or V is already too powerful and can reduce $\overline{K}$ by itself without the help of the other.

We will use good Σ_2^0 approximations to the sets U_i and V_i as defined in Section 1.4.2 for every $i < \omega$. We already saw that by setting $U_i \oplus V_i[s] = U_i[s] \oplus V_i[s]$ we obtain a good Σ_2^0 approximation to the set $U_i \oplus V_i$. Using this approximation we can implement the same techniques that we used in Chapter 2. First we use a $(\mathcal{P}, \Gamma)$-strategy with two outcomes $e <_L l$, that will monitor the length of agreement $l(\Theta^{U,V}, E)[s]$ at each stage s of the construction. A bounded length of agreement proves that $\Theta^{U,V} \neq E$ and further actions only need to be made at expansionary stages. The strategy will attempt to reduce the set $\overline{K}$ to the sets U and A via an enumeration operator Γ. At every stage s we ensure that $n \in \overline{K}[s] \Leftrightarrow n \in \Gamma^{U,A}[s]$ for each n below $l(\Theta^{U,V}, E)[s]$. We will do this by defining markers $u(n)$ and $\gamma(n)$ and enumerating axioms of the form $\langle n, U[s] \upharpoonright u(n), \{\gamma(n)\}\rangle$ for elements $n \in \overline{K}[s]$. If at a later stage n leaves the set $\overline{K}$

then Γ can be rectified via an extraction of the marker $\gamma(n)$ from A.

Below outcome e we need a more elaborate $\mathcal{N}$-strategy $(\mathcal{N}, \Gamma)$. At this point it seems reasonable to assume that we can use the same strategy as we used in Chapter 2. To deal with conflicts with the higher priority $(\mathcal{P}, \Gamma)$-strategy it chooses a threshold d and tries to achieve $\gamma(n) > use(\Psi, A, x)$ for all $n \geqslant d$ at a stage previous to the imposition of a restraint. It tries to maintain $\theta(x) < u(d)$ in the hope that after we extract x from E each return of $l(E, \Theta^{U,V})$ will produce an extraction from $U \upharpoonright \theta(x)$ which can be used to avoid an A-extraction in moving $\gamma(d)$. We use the *generalised* use function $\theta(x)$, as defined in Section 2.1.1.

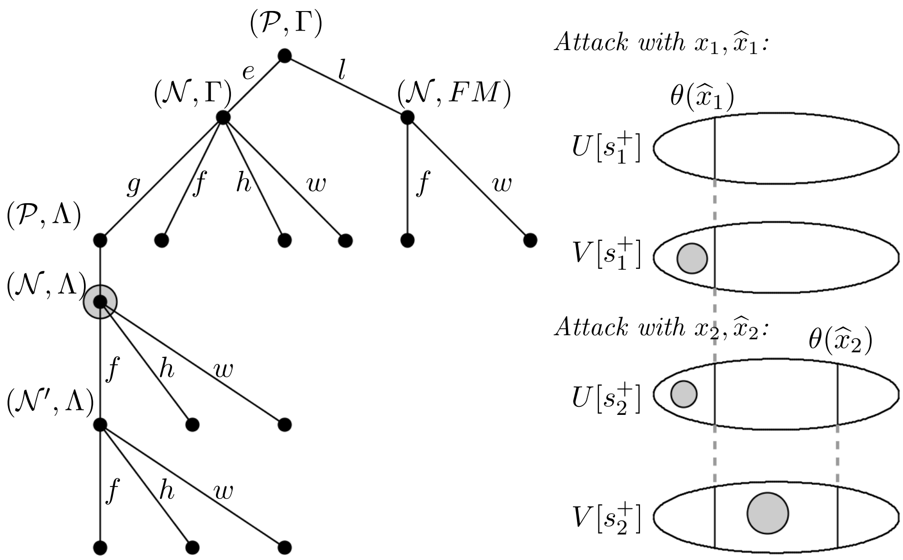
$(\mathcal{N}, \Gamma)$ will have an extra outcome g which shall be visited in the event that some such attempt to satisfy $\mathcal{N}$ ends with a $V \upharpoonright \theta(x)$-change. Below this outcome we implement a backup $\mathcal{P}$-strategy, $(\mathcal{P}, \Lambda)$, which is designed to allow lower priority $\mathcal{N}$-strategies to work below the Γ-activity and to construct an operator Λ reducing $\overline{K}$ to V and A, using the $V \upharpoonright \theta(x)$-changes to move λ-markers.

The need for an even more sophisticated $(\mathcal{N}, \Gamma)$-strategy arises only after we consider the backup strategy $(\mathcal{N}, \Lambda)$, placed immediately after the strategy $(\mathcal{P}, \Lambda)$.

3.1.1 Conflicts

Both strategies $(\mathcal{N}, \Gamma)$ and $(\mathcal{N}, \Lambda)$ will attack simultaneously at stage s_1 by extracting their witnesses x_1 and $\widehat{x}_1$ from E ensuring that at least one of them will succeed in providing the necessary U- or V-change at the next expansionary stage s_1^+. Here $\widehat{x}_1 < x_1$, thus any change in U or V below $\theta(\widehat{x}_1)$ will be a change in U or V below $\theta(x_1)$.

If $(\mathcal{N}, \Lambda)$ turns out to be successful then $(\mathcal{N}, \Gamma)$ will clear the working space for the backup strategies by extracting the current markers of its threshold forcefully, performing *capricious destruction*, choose a new witness x_2 and start a new cycle timing its next attack with the next $\mathcal{N}$-strategy $(\mathcal{N}', \Lambda)$ below $(\mathcal{P}, \Lambda)$. The strategy $(\mathcal{N}', \Lambda)$ chooses its witness $\widehat{x}_2 > \widehat{x}_1$ at stage s_2. The success of $(\mathcal{N}, \Lambda)$ depends on the assumption that the change in the set $V \upharpoonright \theta(\widehat{x}_1)$ observed at stage s_1^+ is permanent. This assumption would be true if V were a Π^0_1 set as in Theorem 2.0.2. Unfortunately we are now dealing with a Σ^0_2 set and there is no guarantee that this change will remain in the approximation at further stages. So it could happen that at stage s_2^+ after the simultaneous attack by $(\mathcal{N}, \Gamma)$ and $(\mathcal{N}', \Lambda)$ there is a further V-change below $\theta(\widehat{x}_2)$ at an element greater than $\theta(\widehat{x}_1)$ making us visit the backup strategies, but the old V-change below $\theta(\widehat{x}_1)$ has moved to the set U. The change we observed at stage s_1^+ in V has disappeared and $V \upharpoonright \theta(\widehat{x}_1)[s_1] = V \upharpoonright \theta(\widehat{x}_1)[s_2^+]$. This will result in an irreparable injury to the strategy $(\mathcal{N}, \Lambda)$. The figure below illustrates this situation, the grey circles represent the changes in the sets U an V as they appear after each attack.



Fortunately we are constructing a Σ^0_2 set as well and are thus allowed to extract its elements any finite number of times without consequence to its characteristic function. $(\mathcal{N}, \Gamma)$ will keep track of its old witnesses. If the change associated with the old witness x_1 moves to the set U, as in the example described above, $(\mathcal{N}, \Gamma)$ will not activate the backup strategies. It will instead restore A as it was during the attack with x_1 at stage s_1 and use the newly appeared U-change for success. Only after changes in V for each of the old witnesses have been observed will the backup strategies be visited.

3.2 The basic modules

The set A will be constructed as a Σ_2^0 set in the following way. At each stage s, the set A will be initially approximated by the set of all natural numbers $\mathbb{N}$. Then we will activate strategies that will extract elements from A. The resulting set after all extractions have been made will be the approximation to A at stage s. We define $n \in A$ if and only if there is a stage s such that $(\forall t > s)[n \in A[t]]$. This will ensure that essentially only the strategies along the true path will be responsible for the elements extracted from A. This is an important feature of the construction that distinguishes it from Harrington's original proof of the non-splitting theorem in the Turing degrees and the proof of Theorem 2.0.2.

We will describe the modules for each of the strategies and list the parameters that will be related to them. We will describe the strategies with the context of the tree in mind. A strategy will be assigned to a particular node δ on the tree, the current stage will be denoted by s and the previous δ-true stage by s^- ($s^- = s$ if δ has been initialized since the last stage at which it was visited). All parameters will inherit their values from s^- unless otherwise specified. For this reason we will sometimes omit the indices that specify the stage if the stage is clear.

3.2.1 The $(\mathcal{P}, \mathbf{\Gamma})$-strategy

This strategy is almost the same as the one described in Section 2.2.1. To every element at every stage s we will again associate current markers $u(n)[s]$ and $\gamma(n)[s]$ and a corresponding current axiom. We examine the element n if it is below the length of agreement $l(\Theta^{U,V}, E)[s]$, updating its parameters if necessary. If $n \notin \overline{K}$ then it will be enough to ensure that it does not appear in $\Gamma^{U,A}[s]$ at infinitely many stages s, the expansionary stages.

Each $\mathcal{P}$-strategy α shall be assigned a distinct infinite computable set A_α from which it will choose the values of its A-markers. Whenever a strategy chooses a fresh marker it will be of value greater than any number appeared so far in the construction.

Suppose for definiteness that the $(\mathcal{P}, \Gamma)$-strategy we visit at stage s is α.

1. If the stage is not expansionary then $o = l$, otherwise $o = e$.

2. Choose $n < l(\Theta^{U,V}, E)[s]$ in turn ($n = 0, 1, \dots$) and perform following actions:

   - If $u(n) \uparrow$ then define it anew as $u(n) = u(n-1) + 1$ (if $n = 0$ then define $u(n) = 1$).
   - If $n \in \overline{K}[s]$
     - If $\gamma(n) \uparrow$ then define it anew and enumerate the current axiom $\langle n, U[s] \upharpoonright u(n), \{\gamma(n)\}\rangle$ in Γ.
     - If $\gamma(n) \downarrow$ but the current axiom for n is not valid then define the current marker $\gamma(n)$ anew and enumerate the new current axiom $\langle n, U[s] \upharpoonright u(n), \{\gamma(n)\}\rangle$ in Γ.
   - If $n \notin \overline{K}[s]$ but $n \in \Gamma^{U,A}[s]$ and the stage is expansionary then look through all the axioms defined for n, say $\langle n, U_n, m\rangle \in \Gamma[s]$, and extract m for all valid ones.

Note that this strategy will extract markers only at expansionary stages. Hence if the true outcome is l, the strategy will not modify the set A and $\mathcal{N}$ can be satisfied via $(\mathcal{N}, FM)$.

3.2.2 The $(\mathcal{N}, \boldsymbol{\Gamma})$-strategy

Suppose the node on which the $(\mathcal{N}, \Gamma)$-strategy acts is labelled by $\beta \supset \alpha$. We shall say that α is the active $\mathcal{P}$-strategy at β.

Some of the parameters that β will have are the same as the ones in Section 2.2.2. It will use a threshold d, a natural number that determines the beginning of the influence of β on the set A. Furthermore β is equipped with a list of witnesses that it has used so far in its attempts to satisfy $\mathcal{N}$ denoted by Wit. One of the witnesses is called the current witness, denoted by x, and plays a special role.

The main feature of the construction, the way we approximate A, clashes with the idea that a certain strategy progressively acts towards satisfying its requirement. Any influence it has tried to inflict on the set A by extracting some element from it will be lost unless the element is extracted again and again infinitely often. This is why each strategy will keep track of all elements it has previously extracted in course of its work and extract these elements at every stage at which the strategy is active. So if a strategy remains inactive, to the left of the true path, it will not have any influence on the set A. If it is on the true path then it will restore its previous work at the beginning of every true stage and build onto that work during the stage.

We will have three different groups of parameters responsible for elements extracted by β during its activities. The first will be the set of markers extracted for elements less than the threshold in O_d by the active $\mathcal{P}$-strategy. Note that the valid axioms whose markers are extracted at expansionary stages need not be the same at every stage. We need to provide some stability for β: if a marker that was extracted from A at a previous β-true stage is not extracted at this one, β will extract it nevertheless and keep track of such elements in O_d. The second group, O_β, will consist of markers extracted during the activity of β. The third group will be markers extracted due to *capricious destruction* after an attack that seems unsuccessful, kept in a parameter O_w for each witness w. These can be later re-enumerated in A (i.e. not extracted from A at β-true stages) if the attack is re-evaluated as successful.

The $(\mathcal{N}, \Gamma)$-module is divided into the same sub-modules as were used in Section 2.2.2: *Check*, *Initialization*, *Honestification*, *Waiting*, *Attack* and *Result*. The first five sub-modules are implemented differently, to incorporate the use of the parameters O_d, O_β and O_w but follow essentially the same ideas. The evaluation of the *Result* is significantly modified as anticipated in Section 3.1.1.

- **Check:** If the threshold is not defined, then go to *Initialization*, otherwise:

  1. If $d \notin \overline{K}[s]$, i.e the threshold has just been extracted from $\overline{K}$, then find the least $n > d$, $n \in \overline{K}[s]$ and let that be the new value of the threshold. Empty Wit, cancel the current witness and start from *Initialization*, initializing all strategies below β. Note that the set $\overline{K}$ is infinite, hence we shall eventually find the correct threshold.

  2. Scan the elements $n \leq d$ such that $n \notin \overline{K}[s]$. If a marker m of n, $m \notin O_\beta \cup O_d$, has been extracted from A at this expansionary stage by α then we will enumerate it in O_d, empty Wit, cancel the current witness and start from *Initialization*, initializing all strategies below β. Note that this can happen finitely often as long as the threshold remains permanent, as there are finitely many axioms and hence markers that can be extracted from A for elements $n \leq d$, $n \notin \overline{K}$.

  3. Extract from A: $Out_\beta = O_\beta \cup O_d \cup \bigcup_{w \in Wit, w<x} O_w$.

- **Initialization:**

  1. If a threshold has not yet been defined or is cancelled, choose a fresh threshold $d > l(\Theta^{U,V}, E)[s]$.

  2. If a witness has not yet been defined or is cancelled, choose a new witness $x \in E[s]$, $d < x$, bigger than any witness defined until now. Enumerate $x \in Wit$.

  3. Wait for a stage s such that $x < l(\Theta^{U,V}, E)[s]$. $(o = w)$

  4. Extract from A and enumerate in O_β all A_α-markers $m(n)$ of potentially applicable axioms for elements n such that $d \leq n < l(\Theta^{U,V}, E)[s]$. An axiom

is *potentially applicable*, if its A_α-marker is not already extracted from A and enumerated in Out_β. Cancel the current markers for these elements.

5. For every element $y \leq x$, $y \in E[s]$, enumerate in the list $Axioms$ the current valid axiom from $\Theta[s]$, which was valid the longest, i.e. with least *age* $a(U \oplus \overline{W}, U_y \oplus \overline{W}_y, s)$ (See Definition 1.2.1). Here the definition of $\theta(x)$ at stage s will be modified as well to capture the greatest element of precisely these axioms currently listed in the list $Axioms$. Let the outcome be $(o = h)$.

- **Honestification:** Scan the list $Axioms$. If for any element $y \leq x$, $y \in E[s]$, the listed axiom was not valid at any stage t since the last β-true stage s^- then update the list $Axioms$, let (o = h) and

  1. Extract and enumerate in O_β all A_α-markers $m(n)$ of potentially applicable axioms for elements n such that $d \leq n$, cancel the current markers for these elements and define $u(d) > \theta(x)$.

  Otherwise go to:

- **Waiting:** Wait for a stage s such that $x \in \Psi^A[s]$ returning at each successive stage to *Honestification*, $(o = w)$.

- **Attack:**

  1. If $x \in \Psi^A[s]$ and $u(d) > \theta(x)$ then extract x from E. The outcome is $(o = g)$ starting a nonactive stage for the backup strategies. Define O_x to be the set of all A_α-markers of potentially applicable axioms for elements n such that $d \leq n$ and set $In(x) = (A \upharpoonright use(\Psi, A, x))[s]$. At the next true stage go to *Result*.

- **Result:** Let $\bar{x} \leq x$ be the least element that has been extracted from E during the stage of the attack. As this is an expansionary stage $\bar{x} \notin \Theta^{U,V}[s]$, hence all axioms for $\bar{x}$ in $\Theta[s]$ are not applicable, in particular the one enumerated in *Axioms*, say $\langle \bar{x}, U_{\bar{x}}, V_{\bar{x}} \rangle$. At least one element from $U_{\bar{x}}$ or $V_{\bar{x}}$ has been extracted from U or V respectively. We will attach to the witness x the necessary information about this attack, namely a parameter $Attack(x) = \langle \bar{x}, U_{\bar{x}}, V_{\bar{x}} \rangle$.

If $V_{\bar{x}} \subseteq V[t]$ at all stages t since the attack then the attack is successful. The A_α-markers of elements $n \geq d$ have been lifted above $use(\Psi, A, x)$ as all previously enumerated axioms for elements $n \geq d$ will not be valid. Hence if later on we want to ensure that $\Gamma^{U,A}(n) = 0$ we will only need to extract a marker that is already above the restraint.

If the attack was unsuccessful then we had a change in V. The plan is to start the backup strategies and then try again with a new witness. In this case we will move the markers $\gamma(n)$ for $n \geq d$, $n \in \overline{K}[s]$, by extracting the current ones, which are already enumerated in the set O_x, and defining the markers anew in order to provide a safe working space for the backup strategy. At any later stage when we activate the backup strategy we would like to have all changes in V for all unsuccessful witnesses that have already been used. As we already discussed in Section 3.1.1 the Σ_2^0 nature of the sets U and V can trick us to believe that a certain witness is unsuccessful, where in fact after finitely many changes in V it turns out to be successful. We would like to be able to restore the old situation as it were during the attack with this old witness and use it to satisfy the requirement. This is where the parameter O_x comes into use. Every time we reach this step of the module we will stop and look back at what has happened with the previous witnesses recorded in the list Wit. If it turns out that we have a permanent U-change useful for some $w \in Wit$ then we can re-enumerate the corresponding O_w in A and satisfy the requirement $\mathcal{N}$ with this witness. Otherwise as the stage is expansionary and hence $w \notin \Theta^{U,V}[s]$ we have the necessary change in V and can activate the backup strategy.

Thus we scan all $w \in Wit$.

1. Let $Attack(w) = \langle \bar{w}, U_{\bar{w}}, V_{\bar{w}} \rangle$. If there was a change in $V_{\bar{w}}$ since this witness was last examined, i.e. there is a stage t such that t is bigger than the stage of the last attack and $V_{\bar{w}} \not\subseteq V[t]$ then extract O_w from A and go to the next witness.
2. Otherwise w is successful, the outcome is $(o = f_w)$. We set the current witness to be w so that O_w is not extracted from A during *Check*. Return to *Result* at the next stage. We say that β restrains the elements $In(w)$ in A.
3. If all witnesses are scanned and all are unsuccessful then cancel the last witness together with the current markers of the elements $n \in \overline{K}[s]$, $d \leq n$ and let the outcome be $o = g$ starting an active stage for the backup strategies. Return to *Initialization* at the next stage, choosing a new witness. Note that *capricious destruction* will be preformed in the sense that O_x will be extracted at further β-true stages during *Check*.

3.2.3 Analysis of outcomes

We shall again define a right boundary R below which successive $\mathcal{N}$-strategies are allowed to work, assuming that the set A shall not change below R due to the activity of higher priority $\mathcal{N}$-strategies or nonactive $\mathcal{P}$-strategy.

$(\mathcal{P}, \Gamma)$ has two possible outcomes:

(l) There is a stage after which $l(\Theta^{U,V}, E)$ remains bounded by its previous expansionary value. Then $\mathcal{P}$ is trivially satisfied. In this case $\mathcal{N}$ will be satisfied by the strategy $(\mathcal{N}, FM)$ working below right boundary $R = \infty$.

(e) There are infinitely many expansionary stages. The $(\mathcal{N}, \Gamma)$-strategy β is activated.

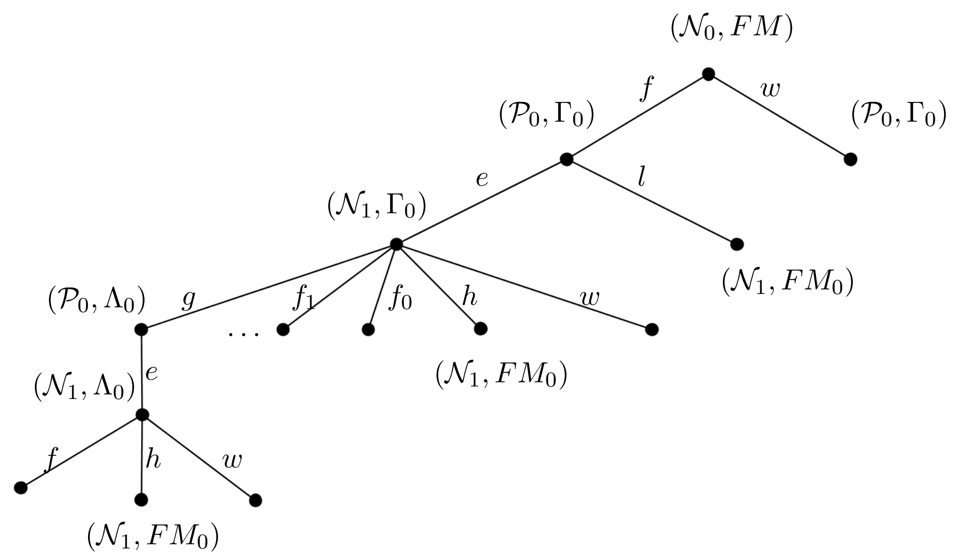
The possible outcomes of $(\mathcal{N}, \Gamma)$ are:

(w) There is an infinite wait at *Waiting* for $\Psi^A(x) = 1$ for some witness x. Then $\mathcal{N}$ is satisfied because $E(x) = 1 \neq \Psi^A(x)$ and $(\mathcal{P}, \Gamma)$ remains intact. Successive strategies work below $R = \infty$.

($\mathbf{f_x}$) There is a stage after which some witness x with $Attack(x) = \langle \bar{x}, U_{\bar{x}}, V_{\bar{x}} \rangle$ never gets its $V_{\bar{x}}$-change. Then there is a permanent change in $U_{\bar{x}}$ and the markers of all witnesses are moved above $use(\Psi, A, x)$. At sufficiently large stages $\overline{K} \upharpoonright d$ has its final value. So there is no injury to the strategies below f_x. $\Psi^A(x) = 1 \neq E(x)$ and $\mathcal{N}$ is satisfied, leaving $(\mathcal{P}, \Gamma)$ intact. Successive strategies work below $R = \infty$.

(h) There are infinitely many occurrences of *Honestification* for some witness x precluding an occurrence of *Attack*. Then there is a permanent witness x which has unbounded $\lim_{sup} \theta(x)$. This means that $\Theta^{U,V}(y) = 0$ for some $y \leq x$, $y \in E$, thus $\mathcal{P}$ is satisfied. In this case $\mathcal{N}$ is satisfied by a second instance of $(\mathcal{N}, FM)$ working below $R = \gamma(d)$.

(g) We implement the unsuccessful attack step infinitely often. As anticipated we must activate the backup strategies. They work below $R = x$.



3.2.4 The backup strategies

The backup strategies will be visited at active stages when they perform their usual activities and at non-active stages, when the $\mathcal{N}$-strategies can attack. In this section as well the basic ideas follow the ones from Section 2.2.4. The $(\mathcal{N}, \Lambda)$-strategy has a threshold $\widehat{d}$ and witness $\widehat{x}$, with $d[s] < \widehat{d}[s] < \widehat{x}[s] < x[s]$ at every non-active stage s. Its actions are directed at its active $\mathcal{P}$-strategy, $(\mathcal{P}, \Lambda)$. After an attack it will be visited only if $(\mathcal{N}, \Gamma)$ has observed a stage at which there was an extraction from V below $\theta(\widehat{x})$. We will implement the $(\mathcal{P}, \Lambda)$-strategy so that this will enough to guarantee the safety of $(\mathcal{N}, \Lambda)$. Thus the $(\mathcal{N}, \Lambda)$-strategy will not have a g-outcome as if it is visited after an attack, it will be successful. In this section we will only implement the $(\mathcal{P}, \Lambda)$-strategy, as the module for $(\mathcal{N}, \Lambda)$ does not use any new ideas. It is obtained from the $(\mathcal{N}, \Gamma)$-module by simplifying the sub-module *Result*. Note that the backup strategy will as well need to keep track of the elements it has extracted in a parameter $Out = O_{\widehat{d}} \cup O$, to be able to build on its previous work.

The $(\mathcal{P}, \Lambda)$-strategy is quite similar to the $(\mathcal{P}, \Gamma)$-strategy, with the difference that it needs to be extra careful in order to catch the true approximations of the initial segments of V as it is only visited at expansionary stages, not necessarily true ones. It has only one outcome e. Suppose it is assigned to the node $\widehat{\alpha}$ and is visited at an active stage s. Let s^- be the previous active visit of $\widehat{\alpha}$.

1. Choose $n < l(\Theta^{U,V}, E)[s]$ in turn $(n = 0, 1, \dots)$ and perform following actions:

   - If $v(n)\uparrow$ then define it anew as $v(n) = v(n-1) + 1$.
   - If $n \in \overline{K}[s]$:
     - If $\lambda(n)\uparrow$ then define it anew and enumerate the current axiom $\langle n, V[s] \restriction v(n) + 1, \{\lambda(n)\}\rangle$ in Λ.
     - If $\lambda(n)\downarrow$ but the current axiom was not valid at some stage t: $s^- < t \leq s$. Then define $\lambda(n)$ anew and let V_n be the finite set chosen out of all

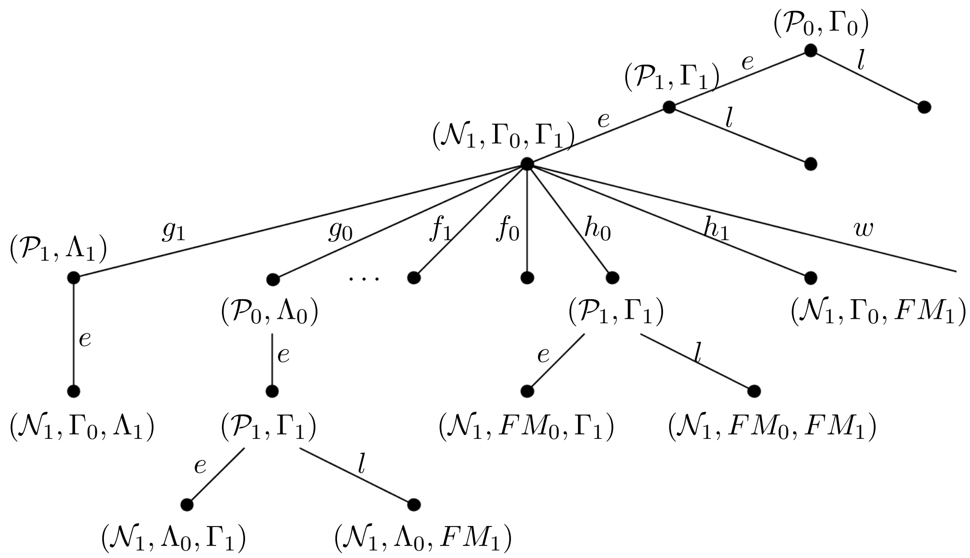
$V[p] \upharpoonright v(n)$ for $s^- < p \leq s$ as the one with least age $a(V, V[p] \upharpoonright v(n), s)$, see Definition 1.2.1. Define the current axiom to be $\langle n, V_n, \{\lambda(n)\}\rangle$ and enumerate it in Λ.

- If $n \notin \overline{K}[s]$ but $n \in \Lambda^{V,A}[s]$ then extract from A all $A_{\widehat{\alpha}}$-markers of axioms for n with V-part V_n such that $(\forall t)[s^- < t \leq s \Rightarrow V_n \subseteq V[t]]$.

3.3 An $\mathcal{N}$-strategy below two $\mathcal{P}$-strategies

The consideration of an $\mathcal{N}$-strategy below two $\mathcal{P}$-strategies reveals further difficulties. We will discuss intuitively the most general case: an $(\mathcal{N}, \Gamma_0, \Gamma_1)$-strategy β working below the expansionary outcomes of two $(\mathcal{P}, \Gamma)$-strategies $\alpha_0 < \alpha_1$. We leave the formal definition of the various strategies for Section 3.4.2, where a general construction regarding all requirements will be given.

As in Section 2.3 the strategy β will perform most of its modules twice, once for each active $\mathcal{P}$-strategy. It will have two g-outcomes and two h-outcomes. It will restart $(\mathcal{P}_1, \Gamma)$ on a successor node if it sees the need to change the method for satisfying $\mathcal{P}_0$ and perform *capricious destruction* on both α_0 and α_1. If the method for $\mathcal{P}_1$ needs to be changed then β shall be careful to leave the strategy at α_0 intact.



There will be two thresholds $d_1 < d_0$ and one current witness x defined at **Initialization**. Each new witness is enumerated in Wit_0. The first witness used is enumerated in Wit_1 as well. Any further witness will be enumerated in Wit_1 only if the attack performed with it will be timed with the backup strategies below outcome g_1, that is if after the previous attack we visited actively outcome g_1.

Honestification is performed first to Γ_0 with the list $Axioms_0$. If Γ_0 is not honest then β will clear both the A_{α_0}- and A_{α_1}-markers, providing safe working space for strategies below outcome h_0. This will destroy the strategy α_1, therefore below outcome h_0 we have a new copy of the $\mathcal{P}_1$-strategy $(\mathcal{P}_1, \Gamma_1)$ starting work from the beginning. If Γ_0 is honest then we will perform $Honestification(1)$. In case Γ_1 is not honest only A_{α_1}-markers will be extracted. If this is the true outcome β shall not extract any A_{α_0}-markers and α_0 will remain intact and still be active for $\mathcal{N}$-strategies below outcome h_1.

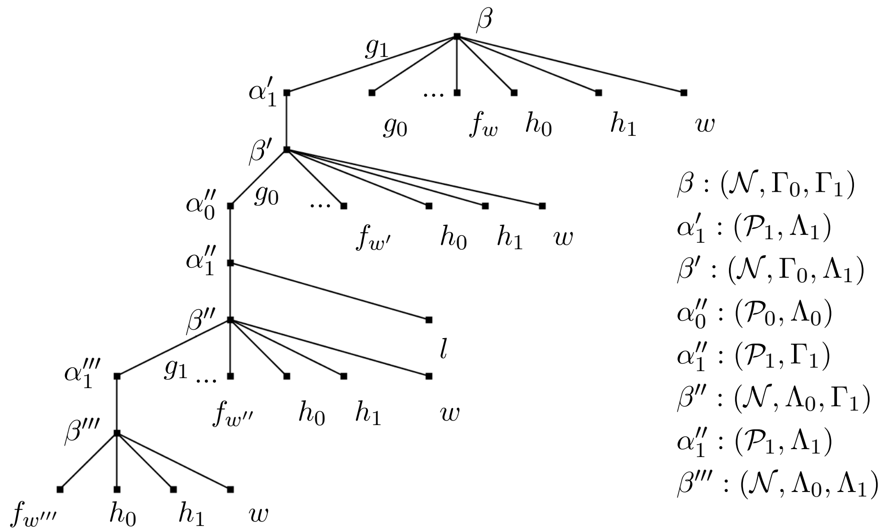
Attack is performed once $x \in \Psi^A$ and both operators are honest. There are two sorts of backup strategies: the ones below outcome g_0 and the ones below outcome g_1. A nonactive stage shall be started for strategies below the outcome visited during the previous attack.

Result is performed first for Γ_0. If the attack is 0-unsuccessful then outcome g_0 is visited and capricious destruction is performed on both operators. Again below outcome g_0 we have a copy of the $(\mathcal{P}_1, \Gamma_1)$-strategy starting its work from the beginning. The outcome g_0 can be visited after many consecutive attacks so the witnesses used will be collected in Wit_0. Only after we see a successful witness $w \in Wit_0$ will we examine the result for the second operator Γ_1.

The witnesses in Wit_1 are then examined one by one. In this case we are not able to guarantee a V_1-change for each of the witnesses to the backup strategies below outcome g_1. Instead a witness w with $Attack(w) = \langle \bar{w}, U_{\bar{w},0}, V_{\bar{w},0}, U_{\bar{w},1}, V_{\bar{w},1} \rangle$, where $\bar{w}$ is the least witness extracted by some strategy during the attack with w, is considered 1-

unsuccessful if there is a $V_{\bar{w},0}$-change or a $V_{\bar{w},1}$-change. If all witnesses are 1-unsuccessful the outcome g_1 is visited.

To explain how an attack works we will need to consider the backup strategies below outcome g_1 as well. We have β' which is the $(\mathcal{N},\Gamma_0,\Lambda_1)$-strategy, then below it β'' which is the $(\mathcal{N},\Lambda_0,\Gamma_1)$-strategy, finally β''' is the $(\mathcal{N},\Lambda_0,\Lambda_1)$-strategy.



All strategies attack together with $w''' < w'' < w' < w$ and hence the least element extracted during the attack is w'''. The strategy α''_1 restarts the approximations to U_1 and V_1 and so we have the following connections between the entries recorded in the corresponding *Attack*-parameters:

$(U_{\bar{w},0},V_{\bar{w},0}) = (U_{\bar{w}',0},V_{\bar{w}',0}) = (U_{\bar{w}'',0},V_{\bar{w}'',0}) = (U_{\bar{w}''',0},V_{\bar{w}''',0})$, i.e. *Attack*[0] is the same for all strategies. Furthermore $(U_{\bar{w},1},V_{\bar{w},1}) = (U_{\bar{w}',1},V_{\bar{w}',1})$ and $(U_{\bar{w}'',1},V_{\bar{w}'',1}) = (U_{\bar{w}''',1},V_{\bar{w}''',1})$, i.e. the pairs (β,β') and (β'',β''') share the same entry in *Attack*[1].

Suppose that after we evaluate the result of β it has outcome g_1. This means that there is a change in $V_{\bar{w},0}$ or $V_{\bar{w},1}$. Then β' evaluates the result of the attack. If the change was not in $V_{\bar{w},0} = V_{\bar{w}',0}$ then β' is successful as it has the desired $(U_{\bar{w}',0},V_{\bar{w}',1})$-change. Otherwise we have a $V_{\bar{w}',0} = V_{\bar{w},0}$-change and β' will have outcome g_0. Now β'' has its desired $V_{\bar{w}'',0} = V_{\bar{w}',0}$-change. If there is a $U_{\bar{w}'',1}$-change then it is successful, otherwise there is a $V_{\bar{w}'',1} = V_{\bar{w}''',1}$-change. β'' will have outcome g_1 and the strategy β''' will be successful. Thus at least one of the strategies along the tree will be successful regardless of the distribution of the changes.

As discussed in Section 2.3, to keep the strategy α_0 intact in the case of infinitely many 1-unsuccessful attacks performed by β, a new value of the threshold d_0 will be chosen at every active visit of the outcome g_1, the set O_{d_0} will be emptied. So on each new cycle after an active g_1-visit β will move its activity regarding A_{α_0}, allowing α_0 to remain intact.

Again we will need to modify the *Check*(0) submodule. It should not be allowed the initialize all strategies below β should an A_{α_0}-marker of an element less than d_0 be extracted by the active $\mathcal{P}_0$-strategy and enumerated in O_{d_0}. If the true outcome is g_1 then the value of d_0 will grow unboundedly and we might initialize all strategies β infinitely often. *Check*(0) shall instead be only allowed to initialize strategies that believe the threshold d_0 is constant, that is all except for the ones below g_1.

The strategy β' working below outcome g_1 has the same active $\mathcal{P}_0$-strategy. It has threshold $d_0' < d_0$ and prepares its attack by extracting A_{α_0}-markers. As we saw in the proof of Lemma 2.5.3 this preparation is useful for β as it ensures that α_0 will not extract markers for elements $n \geq d_0'$ if the attack is 0-successful.

If we neglect this preparation the following situation might happen: Suppose during β''s preparation with a witness w', it extracts and enumerates in $O_{\beta'}$ an α_0-marker m for an element $d_0' \leq n$. Then β chooses a new threshold $d_0 > n$ and the two strategies attack with w and w'.

While we are evaluating β's *Result* the marker m for the element n with $d_0' \leq n \leq d_0$ is extracted by α_0. The strategy β has never before seen this marker extracted, it might injure its restraint on A, thus β's module *Check*(0) would like to restart β from initialization. In this case the witness w will be discarded and the attack with w' will be neglected. If we visit β' again we will not be able to guarantee the expected changes regarding w'. The strategy β' cannot handle a $(U_{w',0}, U_{w',1})$-change.

We will therefore incorporate the preparation provided by β'. If a new marker $m < use(\Psi, A, w)$ for an element $d_0' \leq n < d_0$ is extracted after the attack with w and w', then we can argue that due to the actions of β' this means that w' is 1-unsuccessful, as a $U_{0,\bar{w}} = U_{0,\bar{w}'}$ change will ensure that no markers for elements $n \geq d_0'$ will be extracted by α_0. For this reason the parameter O_w will appear in two ways: $O_{w,own}$ will have the same definition as in the first case, it will include the markers that we extract during capricious destruction, $O_{w,else}$ will contain markers extracted by backup strategies during their preparation for an attack that will be performed together with β's attack with w. It will be enough for β to extract the set $O_{w,else}$ only during the preparation for attack with w, while it is performing $Honestification$ and $Waiting$. This will guarantee that markers in O_{else} do not appear in the axiom for w in Ψ^A. After the attack we will not extract this set any longer as this might interfere with the elements that previous witnesses need to keep in A for their own success.

Now we are ready to proceed to the main construction and the proof that it works.

3.4 All requirements

The set of different strategies is the same as in Section 2.4. For every $\mathcal{P}$-requirement $\mathcal{P}_i$ we have two different strategies: $(\mathcal{P}_i, \Gamma_i)$ with outcomes $e <_L l$ and $(\mathcal{P}_i, \Lambda_i)$ with one outcome e.

The requirement $\mathcal{N}_0$ has one strategy $(\mathcal{N}_0, FM)$. For every $\mathcal{N}$-requirement $\mathcal{N}_i$, where $i > 0$, we have strategies of the form $(\mathcal{N}_i, S_0, \dots, S_{i-1})$, where $S_j \in \{\Gamma_j, \Lambda_j, FM_j\}$. The outcomes are f_x, where x is a natural number, w and for each $j < i$ if $S_j \in \{\Gamma_j, \Lambda_j\}$ there is an outcome h_j, if $S_j = \Gamma_j$, there is an outcome g_j. They are ordered according to the following rules:

1. For all j_1 and j_2, $g_{j_1} <_L \dots f_n <_L f_{n-1} <_L \cdots <_L f_0 <_L h_{j_2} <_L w$.
2. If $j_1 < j_2$ then $g_{j_2} <_L g_{j_1}$ and $h_{j_1} <_L h_{j_2}$.

3.4.1 The tree of strategies

Let $\mathbb{O}$ be the set of all possible outcomes and $\mathbb{S}$ be the set of all possible strategies. The tree of strategies is a computable function $T : D(T) \subset \mathbb{O}^* \to \mathbb{S}$. It resembles the tree defined in Section 2.4.1, but an $\mathcal{N}$-node has, instead of one outcome f, infinitely many outcomes f_x with order type ω^* - the order type of the negative integers. The tree T has the following properties:

1. If $T(\alpha) = S$ and O_S is the set of outcomes for the strategy S then for every $o \in O_S$, $\alpha\hat{\ }o \in D(T)$.

2. $T(\emptyset) = (\mathcal{N}_0, FM)$.

3. If $S = (\mathcal{N}_i, S_0, S_1, \ldots, S_{i-1})$ then

Below outcome $\mathbf{g_j}$: $T(\alpha\hat{\ }g_j) = (\mathcal{P}_j, \Lambda_j)$ and $T(\alpha\hat{\ }g_j\hat{\ }e) = (\mathcal{P}_{j+1}, \Gamma_{j+1})$, $\ldots$, $T(\alpha\hat{\ }g_j\hat{\ }e\hat{\ }o_{j+1}\hat{\ }o_{i-2}) = (\mathcal{P}_{i-1}, \Gamma_{i-1})$, where $o_k \in \{e_k, l_k\}$ for $j+1 \leq k \leq i-2$.
$T(\alpha\hat{\ }g_j\hat{\ }e\hat{\ }o_{j+1}\hat{\ }o_{i-1}) = (\mathcal{N}_i, S_0, \ldots, \Lambda_j, S'_{j+1}, \ldots, S'_{i-1})$, where $S'_k = \Gamma_k$ if $o_k = e_k$ and $S'_k = FM_k$ if $o_k = l_k$ for every k such that $j < k < i$.

Below outcome $\mathbf{h_j}$: $T(\alpha\hat{\ }h_j) = (\mathcal{P}_{j+1}, \Gamma_{j+1}), \ldots, T(\alpha\hat{\ }h_j\hat{\ }o_{j+1}\hat{\ }o_{i-2}) = (\mathcal{P}_{i-1}, \Gamma_{i-1})$, where $o_k \in \{e_k, l_k\}$ for $j+1 \leq k \leq i-2$.
$T(\alpha\hat{\ }h_j\hat{\ }o_{j+1}\hat{\ }o_{i-1}) = (\mathcal{N}_i, S_0, \ldots, FM_j, S'_{j+1}, \ldots, S'_{i-1})$, where $S'_k = \Gamma_k$ if $o_k = e_k$ and $S'_k = FM_k$ if $o_k = l_k$ for every k such that $j < k < i$.

Below outcome $\mathbf{f_x}$: $T(\alpha\hat{\ }f_x) = (\mathcal{P}_i, \Gamma_i)$. Then $T(\alpha\hat{\ }f_x\hat{\ }e) = (\mathcal{N}_{i+1}, S_0, \ldots, S_{i-1}, \Gamma_i)$, $T(\alpha\hat{\ }f_x\hat{\ }l) = (\mathcal{N}_{i+1}, S_0, \ldots, S_{i-1}, FM_i)$.

Below outcome w: $T(\alpha\hat{\ }w) = (\mathcal{P}_i, \Gamma_i)$. Then $T(\alpha\hat{\ }w\hat{\ }e) = (\mathcal{N}_{i+1}, S_0, \ldots, S_{i-1}, \Gamma_i)$, $T(\alpha\hat{\ }w\hat{\ }l) = (\mathcal{N}_{i+1}, S_0, \ldots, S_{i-1}, FM_i)$.

The active $\mathcal{P}$-strategies for each node are defined as in Section 2.4.1.

3.4.2 Construction

Following the basic rules from Section 1.3 again at each stage s we shall construct a finite path through the tree of strategies $\delta[s]$ of length s starting from the root. Each

$\mathcal{N}$-node α shall have a right boundary R_α, defined below. $R_\emptyset[s] = \infty$. After the stage is completed, all nodes to the right of the constructed $\delta[s]$ will be initialized and their parameters will be cancelled or set to their initial value $\emptyset$.

We shall use the notion of *dependence* between strategies and an *instigator* defined in 2.4.1. In our further discussions we shall denote with M_α, m_α, Z_α and z_α: Γ_α, γ_α, U_α and u_α respectively if α is a $(\mathcal{P},\Gamma)$-strategy and Λ_α, λ_α, V_α and v_α respectively if α is a $(\mathcal{P},\Lambda)$-strategy. We will denote by s^- the previous α-true stage and by o^- the outcome it had at that stage. If α has been initialized since its previous true stage or if it has never before been visited then $s^- = s$ and o^- is the rightmost outcome.

Suppose we have constructed $\delta[s] \upharpoonright n = \alpha$. If $n = s$ then the stage is finished and we move on to stage $s + 1$. If $n < s$ then α is visited and the actions that α performs are as follows:

(I.) $T(\alpha) = (\mathcal{P}_i, \Gamma_i)$. This strategy is responsible for approximating the sets U_i, V_i and Θ_i. It will consider the next approximation only at active stages. At these we perform the actions as stated in the main module in Section 3.2.1, $\delta[s](n + 1) = l$ at non-expansionary stages. At expansionary stages $\delta[s](n + 1) = e$. At nonactive stages no actions are performed. The outcome is $o = o^-$.

(II.) $T(\alpha) = (\mathcal{P}_i, \Lambda_i)$. At active stages we perform the actions as stated in the main module in Section 3.2.4, $\delta[s](n+1) = e$. At nonactive stages no actions are performed, $\delta[s](n + 1) = e$.

(III.) $T(\alpha) = (\mathcal{N}_i, S_0, \dots, S_{i-1})$ with active $\mathcal{P}$-nodes $\alpha_0, \dots, \alpha_{i-1}$. At active stages we perform *Check* first. If it doesn't instruct us otherwise then we carry on with the module from where it was left at the previous α-true stage s^- (from *Initialization* if $s^- = s$). At nonactive stages α may only attack, otherwise is has outcome $o = o^-$.

- **Check:** If one of the thresholds is not defined or cancelled, then the strategy α goes to *Initialization*. Otherwise let $Out_\alpha = \bigcup_{j<i} O_{d_j} \cup O_\alpha \cup O_{x,else}$ $\bigcup_{w \in Wit_j, w<x, j<i} O_{w,own}$. The strategy performs $Check(j)$ for $j = i-1, i-2, \dots, 1$.

Check($i-1$): Scan all $n \leq d_{i-1}$. If an A_{α_k}-marker for n, $m_k(n) \notin Out_\alpha$, has been extracted from A by α_k, the active $\mathcal{P}_k$-strategy at α, for $k \leq i-1$ at a stage t: $s^- < t \leq s$ then we will enumerate it in $O_{d_{i-1}}$ and empty Wit_j, $j < i$, initialize all strategies below α and start from initialization.

Check(j): Scan all $d_{j+1} < n \leq d_j$. If an A_{α_k}-marker for n, $m_k(n) \notin Out_\alpha$, has been extracted from A by α_k for $k \leq j$, at a stage t: $s^- < t \leq s$ then we will enumerate it in O_{d_j}. Then all successors of α, that assume that d_j does not change infinitely often, are initialized. These are strategies γ such that $\gamma \supseteq \alpha\hat{\ }g_k$ for $k \leq j$ or $\gamma \supseteq \alpha\hat{\ }o$, where $o \in \{\, h_l, f_x, w |\ l < i, x \in \omega\}$, hence all strategies below and to the right of outcome g_j. Then we will empty Wit_k for $k \leq j$ and leave only the current witness x in them.

If α is evaluating *Result* and the last active g-outcome was g_k and $k \leq j$ then α continues from the Initialization step. Otherwise α continues to evaluate *Result*.

If a threshold d_j is extracted from $\overline{K}[s]$ then it is shifted to the next possible value, i.e to the least $n > d_j, n \in \overline{K}[s]$. If this injures the order between thresholds then the other thresholds are shifted as well. In this case the strategy resets its work in the same way as described in $Check(j)$ when an element enters O_{d_j}.

Extract Out_α from A.

- **Initialization:** Each strategy $S_j \neq FM_j$ picks a threshold if it is not already defined. The different thresholds must be in the following order:

$$d_{i-1} < d_{i-2} < \dots < d_0.$$

Strategy S_j picks its threshold as a fresh number such that its marker has not yet been defined by the active $\mathcal{P}_j$-strategy α_j. Then α picks a witness $x \in E$ again as a fresh number.

If this is the first witness that α picks after it was initialized then x is enumerated in all Wit_j $j < i$, where $S_j = \Gamma_j$ and $O_{x,else} = \emptyset$.

If the previous active g-outcome was g_j at stage s^- then x is enumerated in Wit_k, for $k \leq j$ such that $S_k = \Gamma_k$. Then $O_{x,else}$ is the set of all A_{α_k}-markers m_k of potentially applicable axioms defined by the current active $\mathcal{P}_k$-strategy for elements $n < d_k$, for $k < j$, that were extracted from A at stage s^-.

If $l(\Theta_j^{U_j,V_j}, E)[s] \leq x$ for some $j < i$ then $\delta(n+1)[s] = w$, working below $R = R_\alpha[s]$.

If $l(E, \Theta_j^{U_j,V_j})[s] > x$ for all $j < i$ then α extracts from A and enumerates in O_α all A_{α_j}-markers for all potentially applicable axioms for all elements $n \geq d_j$ from all active operators S_j. Then cancels all current j-markers for $n \geq d_j$ and defines $z_{\alpha_j}(d_j) > \theta_j(x)[s]$.

For every element $y \leq x$, $y \in E[s]$, α enumerates in the list $Axioms_j$ the current valid axiom from $\Theta_j[s]$ that has been valid longest as defined in Section 3.2.2. The next stage will start from *Honestification*. $\delta[s](n+1) = w$, working below $R = R_\alpha[s]$.

- **Honestification:** The strategy α performs $Honestification(0)$.

  Honestification(j): If $S_j = FM_j$ then $(o = w)$. Otherwise:

  1. Scan the list $Axioms_j$. If for any element $y \leq x$, $y \in E[s]$, the listed axiom was not valid at any stage t since the last α-true stage then update the list $Axioms_j$, let (o $= h$) and go to the next step, otherwise let (o $= w$).

  2. Extract and enumerate in O_α all A_{α_j}-markers $m_j(n)$ of potentially applicable axioms for elements n such that $d_j \leq n < l(\Theta_j^{U_j,V_j}, E)[s]$. Cancel their current j-markers. For the elements $n \in \overline{K}[s]$ define $z_{\alpha_j}(n) > \theta_j(x)$.

  If the outcome of $Honestification(j)$ is w then α performs $Honestification(j+1)$

if $j+1 < i$ and goes to waiting if $j+1 = i$. If the outcome is h then α extracts all A_{α_k}-markers of potentially applicable axioms for elements $n \geq d_k$, enumerating them in O_α for all $k > j$. Then cancels their current A_{α_k}-markers. The outcome is $\delta[s](n+1) = h_j$ working below $R = \min(R_\alpha[s], m_{\alpha_j}(d_j))$. At the next stage α starts from $Honestification$.

- **Waiting:** If all outcomes of all $Honestification(j)$-modules are w, i.e. all enumeration operators are honest then α checks if $x \in \Psi_i^A[s]$ with $use(\Psi, A, x)[s] < R_\alpha[s]$. If not then the outcome is $\delta[s](n+1) = w$, working below $R = R_\alpha[s]$. At the next stage α returns to $Honestification$. If $x \in \Psi_i^A[s]$ with $use(\Psi, A, x) < R_\alpha[s]$ then α goes to *Attack*.

- **Attack:** If α is dependent then it waits for an $ins(\alpha)$-nonactive stage. $\delta[s](n+1) = w$, working below $R = R_\alpha[s]$.

  If the stage is $ins(\alpha)$-nonactive, $x \in \Psi^A[s]$, $use(\Psi, A, x) < R_\alpha[s]$ and all operators are honest, (the axioms recorded in the lists $Axioms_j$, $j < i$, have remained valid at all stages since s^-) then α extracts x from E. Define $O_{x,own}$ to be the set of all potentially applicable axioms in the active $\mathcal{P}_j$-operators for elements $n \geq d_j$ and $j < i$. Let $t_x = s$ and $L(x) = use(\Psi, A, x)$ and $In(x) = A \upharpoonright L(x)$. This starts an α-nonactive stage for the strategies below the most recently visited outcome g_j (if none has been visited until now then below the leftmost g-outcome) working below the boundaries they worked before.

  Otherwise it will return to *Honestification* at the next active stage.

- **Result:** Let $\bar{x}$ be the least element extracted from E during the attack. It has a corresponding entry $\langle \bar{x}, U_{\bar{x},j}, V_{\bar{x},j} \rangle$ in $Axioms_j$. Define $Attack(x) = \langle \bar{x}, U_{\bar{x},0}, V_{\bar{x},0}, \ldots, U_{\bar{x},i-1}, V_{\bar{x},i-1} \rangle$. We will denote by $Attack(x)[j]$ the pair $(U_{\bar{x},j}, V_{\bar{x},j})$. Redefine $L(x)$ to be the maximum of all $L(y)$ for all elements y that were extracted during the attack. Empty $O_{x,else}$ as it has done its job. The strategy α performs

Result(0).

Result(j): If $S_j = FM_j$ or $S_j = \Lambda_j$ then go to *Result*$(j+1)$. Otherwise scan all witnesses w in Wit_j. Let *Attack*$(w)[k] = (U_{\bar{w},k}, V_{\bar{w},k})$ for $k \leq j$. If one of the following two conditions is true for any $k \leq j$:

1. $S_k = \Gamma_k$ and there was a change in $V_{\bar{w},k}$ since this witness was last examined, i.e. there is a stage t such that t is bigger than the stage at which this witness was last examined such that $V_{\bar{w},k} \nsubseteq V_k[t]$.

2. An A_{α_k}-marker $m_k < L(w)$ of an element $n < d_k[t_w]$ such hat $m_k \in A[t_w]$ was enumerated in O_{d_k} for $k < j$.

Then extract $O_{w,own}$ from A and go to the next witness.

Otherwise w is k-successful for all $k \leq j$ then go to *Result*$(j+1)$.

If all witnesses are scanned then cancel the last witness cancel the current A_{α_k}-markers for elements $n \geq d_k$, $k > j$. Empty Wit_l and cancel d_l together with O_{d_l} for $l < j$. Return to *Initialization* at the next stage, choosing a new witness and thresholds. The outcome is $\delta[s](n+1) = g_j$. boundary is $R = \min(x, R_\alpha[s])$.

Result(i): We reach this result only in case we have found some witness w that is j-successful for all $j < i$. Then let the current witness be w. Restrain $In(w)$ in A. Let the outcome be $o = f_w$, working below $R = R_\alpha[s]$. At the next stage go back to *Result*(q), where q is the greatest index of a Γ-strategy among $(S_0, \ldots, S_{i-1})$ and $q = i$ if the are no Γ-strategies.

3.5 Proof

We shall now prove that the construction described in Section 3.4.2 works. We shall start by defining a true path in the tree of strategies. Using this path we shall then prove some basic properties of the construction. This will enable us to prove that all

$\mathcal{P}$-requirements are satisfied. Finally we will turn our attention to the $\mathcal{N}$-requirements.

3.5.1 The true path

In this construction for the first time we have an infinitely branching tree of strategies. It is not straightforward that there exists leftmost infinite path of nodes visited at infinitely many stages. Fortunately in this case we are able to prove the following lemma:

Lemma 3.5.1. *There is an infinite path h in our tree of strategies with the following properties:*

1. $(\forall n)(\exists^{\infty} s)[h \upharpoonright n \subseteq \delta[s]]$.
2. $(\forall n)(\exists s_l(n))(\forall s > s_l(n))[\delta[s] \not<_L h \upharpoonright n]$.
3. $(\forall n)(\exists s_i(n))(\forall s > s_i(n))[h \upharpoonright n$ *is not initialized at stage* $s]$.

Proof. We will define the true path with induction on n and prove that it has the properties needed. The case $n = 0$ is trivial: $h \upharpoonright 0 = \emptyset$ is visited at every stage of the construction and is never initialized, $s_l(0) = s_i(0) = 0$.

Suppose we have constructed $h \upharpoonright n = \alpha$ with the required properties. We shall define $h \upharpoonright (n+1)$. We have three cases depending on the type of α.

If α is a $(\mathcal{P}_i, \Gamma)$-strategy then it has two possible outcomes $e <_L l$. If outcome e is visited infinitely often then let $h \upharpoonright (n+1) = \alpha\hat{\ }e$. It has the infinite visit property and being the leftmost possible outcome has $s_l(n+1) = s_l(n)$. Otherwise there is a stage t after which whenever we visit α, we visit also $\alpha\hat{\ }l$. Then $h \upharpoonright (n+1) = \alpha\hat{\ }l$ is visited infinitely often with $s_l(n+1) = \max(s_l(n), t)$. In both cases $s_i(n+1) = \max(s_i(n), s_l(n+1))$, as α does not initialize its successors.

If α is a $(\mathcal{P}_i, \Lambda_i)$-strategy then it has only one outcome $o = e$ visited at every α-true stage, hence $h \upharpoonright (n+1) = \alpha\hat{\ }e$ has the needed properties with $s_l(n+1) = s_l(n)$. α does not initialize its successors, hence $s_i(n+1) = \max(s_i(n), s_l(n+1))$.

Let α be an $(\mathcal{N}_i, S_0, \ldots, S_{i-1})$-strategy, where $S \in \{\Gamma, \Lambda, FM\}$. After a stage $t > s_i(n)$, α has a permanent threshold $d_{i-1} \in \overline{K}$. If we assume otherwise this would mean that $\overline{K}$ is finite and hence computable which is not true. There are finitely many elements $n \notin \overline{K}$, $n < d_{i-1}$, with finitely many axioms defined for them in the corresponding operators $S_0, \ldots, S_{i-1}$, as once an element exits $\overline{K}$ no more axioms are enumerated for it in any operator. Hence there are finitely many markers, which can initialize all nodes below α each only once, on its entry in $O_{d_{i-1}}$, which is not emptied after stage $s_i(n)$. Hence there is a stage $t_1 > t$ after which no more markers enter $O_{d_{i-1}}$. If α has an outcome g_{i-1} (and hence $S_{i-1} = \Gamma_{i-1}$) that is visited infinitely often, then let $h \upharpoonright (n+1) = \alpha\hat{\ } g_{i-1}$ with $s_l(n+1) = s_l(n)$ and $s_i(n+1) = \max(t_1, s_l(n+1))$.

In general suppose g_j is the leftmost outcome that is visited infinitely often. Then there is a stage $t > s_i(n)$ such that for all α-true stages $s > t$ no outcome g_k for $j < k < i$ is visited again. In this case the thresholds $d_{i-1}, \ldots, d_j$ are never cancelled and the corresponding sets $O_{d_{i-1}}, \ldots, O_{d_j}$ are never emptied after stage t. Eventually the thresholds stop shifting, as $\overline{K}$ is infinite. There are finitely many elements $n < d_j$ such that $n \notin \overline{K}$ with finitely many axioms defined in each of the operators $S_0, \ldots, S_{i-1}$. There will be a stage $t_1 > t$ after which no new marker enters O_{d_k}, for $j \leq k < i$. After this stage the outcome g_j will not be initialized. Hence we can define $h \upharpoonright (n+1) = \alpha\hat{\ } g_j$ with $s_l(n+1) = t$ and $s_i(n+1) = \max(t_1, s_l(n+1))$.

If no g-outcome is visited at infinitely many stages then there is a stage $t > s_i(n)$ such that for all α-true stages $s > t$ no g-outcome g_k for $k < i$ is visited again. In this case none of the thresholds are ever cancelled again and none of the sets O_{d_j} are emptied after stage t. Similarly to the previous case we get a stage $t_1 > t$ such that at stages $s > t$ no new markers enter any of the sets O_{d_j} and α does not initialize any of its successors during *Check*.

If the last time we visited a g-outcome it was at an active stage, if *Check* restarts α after stage t or if we never visited any g-outcome then the only possible outcome accessible at stages $s > t$ are w and h_j for $j < i$.

If h_0 is visited infinitely often then let $h \restriction (n+1) = \alpha\hat{}h_0$ with $s_l(n+1) = \max(s_i(n), t)$ and $s_i(n+1) = \max(t_1, s_l(n+1))$.

In general let h_j be the leftmost h-outcome visited infinitely often. Then there is a stage $t_2 > t$ after which no other h-outcome is visited again and then we can define $h \restriction (n+1) = \alpha\hat{}h_j$ with $s_l(n+1) = \max(s_i(n), t_2)$ and $s_i(n+1) = \max(t_2, s_l(n+1))$.

If none of the h-outcomes are visited infinitely often then there is a stage $t_2 > t$ after which h_j for $j < i$ is never visited again. Then $h \restriction (n+1) = \alpha\hat{}w$ with $s_l(n+1) = \max(s_i(n), t_2)$ and $s_i(n+1) = \max(t_2, s_l(n+1))$.

Suppose the last time we visited a g-outcome it was at a non-active stage and α is not restarted after stage t. Then after stage t no more witnesses will be defined as in order to cancel a witness and choose a new one we pass through a g-outcome. Hence at stages $s > t$ we have $Wit_j[s] = Wit_j[t]$ for all $j < i$. The only accessible outcomes after stage t are finitely many: f_x for $x \in Wit_{i-1}$. Denote them by $f_{x_k} <_L \cdots <_L f_{x_1}$.

Suppose outcome f_{x_p} is visited at a stage $s > t$. Then the only outcomes that can be accessible at later stages will be f_{x_q} with $q \geq p$. In order to reach outcomes w, h or f_{x_r} with $r < p$ we need to pass through a g-outcome again, which we know does not happen. Then choose the biggest p such that there is a stage $t_1 > t$ at which we pass through outcome f_{x_p}. It follows that after this stage we will always pass through f_{x_p} whenever we visit α. Hence $h \restriction (n+1) = \alpha\hat{}f_{x_p}$ with $s_l(n+1) = \max(s_l(n), t_1)$ and $s_i(n+1) = s_l(n+1)$. □

Some of the properties of this construction are the same as the properties of the construction defined in Chapter 2. We will state without proof several such properties. The first one concerns the distribution of active and nonactive stages, see Proposition 2.5.1.

Propostion 3.5.1. *Suppose $\alpha\hat{}g_j \subseteq \beta \subset h$. Then β is visited at infinitely many active and at infinitely many α-nonactive stages.*

3.5.2 Satisfaction of the $\mathcal{P}$-requirements

Propostion 3.5.2. *Suppose* $\Theta_i^{U_i,V_i} = E$ *and* $\alpha \subset h$ *is a* $\mathcal{P}_i$*-strategy.*

1. *Suppose* $T(\alpha) = (\mathcal{P}_i, \Gamma_i)$ *and for some element* $n \in \overline{K}$ *the current* U_α*-marker and the* A_α*-marker are not changed by any other strategy after stage* t*. Then* α *will stop changing the current marker eventually and* $n \in \Gamma_\alpha^{U_i,A}$*.*

2. *Suppose* $T(\alpha) = (\mathcal{P}_i, \Lambda_i)$ *and for some element* $n \in \overline{K}$ *the current* V_α*-marker and the* A_α*-marker are not changed by any other strategy after stage* t*. Then* α *will stop changing the current markers eventually and* $n \in \Lambda_\alpha^{V_i,A}$*.*

Proof. We shall omit the index α in the proof as we will be talking only about parameters that belong to α. We shall omit the index i as well as we will only be concerned with U_i,V_i and Θ_i.

1. Suppose $u(n)$ remains the same after stage t. We will use what we know from Section 1.4.2, more precisely Proposition 1.4.4, about the approximation to the set U, namely that it is *Good* and Σ_2^0. As α is the strategy responsible for the approximations of the set all the rest of the stages that appear in the proof of (1) can be considered α-true. Let G denote the set of all good stages, then there will be a stage $t_1 > t$ such that:

   Good: $(\forall s > t_1)[s \in G \Rightarrow U \restriction u(n) = U[s] \restriction u(n)]$.

   $\mathbf{\Sigma_2^0}$: $(\forall s > t_1)[U \restriction u(n) \subseteq U[s]]$.

   As $\{U[s] \oplus V[s]\}_{s<\omega}$ is also a good Σ_2^0 approximation to $U \oplus V$, if $n \in \Theta^{U,V}$, there is a stage s such that at all $s' > s$ we have $n \in \Theta^{U,V}[s']$ and if $n \notin \Theta^{U,V}$ then at good stages $s' \in G$ we have that $n \notin \Theta^{U,V}[s']$. It follows that as $E = \Theta^{U,V}$ for any number n there will be a stage t_n such that at all good stages $s > t_n$ we have that $l(\Theta^{U,V}, E)[s] > n$.

   So there will be a good stage $t_2 > \max(t_1, t_n)$ at which $n < l(\Theta^{U,V}, E)[t_2]$. At this stage we will examine the current axiom for n in Γ, say $\langle n, U_n, \{m\}\rangle$. If it is valid then $U_n \subseteq U[t_2] = U \upharpoonright u(n)$. And hence at all stages $s > t_2$ $(U_n \subseteq U[s])$. If it isn't valid then we will enumerate a new axiom $\langle n, U[t_2] \upharpoonright u(n), \{\gamma(n)\}\rangle$ and for this axiom we will have that at all stages $s > t_2(U[t_2] \upharpoonright u(n) \subseteq U[s])$. In both cases the marker $\gamma(n)$ will not be moved at any later stage and the axiom remains valid forever, hence $n \in \Gamma^{U,A}$.

2. Here the strategy α is not responsible for the approximations of the sets. Instead there is a $(\mathcal{P}_i, \Gamma_i)$-strategy $\beta \subset \alpha$ that approximates the sets. All stages considered for the rest of the proof of (2) are β-true. Suppose $v(n)$ remains constant after stage t. As in part (1) we can find a stage $t_1 > t$ such that:

   Good: $(\forall s > t_1)[s \in G \Rightarrow V \upharpoonright v(n) = V[s] \upharpoonright v(n)]$.

   $\mathbf{\Sigma^0_2}$: $(\forall s > t_1)[V \upharpoonright v(n) \subseteq V[s]]$.

   There will be a good stage $t_2 > t_1$ at which $n < l(\Theta^{U,V}, E)[t_2]$. At the next α-true stage $t_3 \geq t_2$ we will examine the current axiom for n in Λ, say $\langle n, V_n, \{m\}\rangle$. If the current axiom is valid, i.e. it was valid at all stages since the last α-true stage t_3^- then $V_n \subseteq V[t_2] \upharpoonright v(n) = V \upharpoonright v(n)$. And hence at all stages $s > t_3(V_n \subseteq V[s])$. If it isn't valid then we will enumerate a new axiom $\langle n, V'_n, \{\lambda(n)\}\rangle$. We choose this V'_n as $V[t] \upharpoonright v(n)$ for some $t : t_3^- < t \leq t_3$ so that it is of least age. Obviously $V[t_2] \upharpoonright v(n)$ would be among these choices. Hence $V'_n \subseteq V[t_2]$. In both cases the marker $\lambda(n)$ will not be moved at any later stage and the axiom remains valid forever, hence $n \in \Lambda^{V,A}$.

□

Propostion 3.5.3. *1. Let α be the biggest $(\mathcal{P}_i, \Gamma_i)$-strategy along the true path h. Suppose that the current A_α-marker for some element n grows unboundedly. If $\Theta_i^{U_i,V_i} = E$ then there is an outcome g_i along the true path.*

2. Let $\alpha \subset h$ be the biggest $\mathcal{P}_i$-strategy. Suppose it builds an operator M_i. Suppose that the current A_α-marker for some element n grows unboundedly. Then $\Theta_i^{U_i,V_i} \neq E$.

Proof. 1. Assume for a contradiction that $\Theta_i^{U_i,V_i} = E$ and there is no g_i-outcome along the true path. Let n be the least element, whose current A_α-marker moves off to infinity. If $n \notin \overline{K}$ then there will be a stage at which n exits $\overline{K}$. After that stage no more axioms for n are enumerated in Γ_α, hence the marker $\gamma_\alpha(n)$ will remain constant. Hence $n \in \overline{K}$.

At every stage s there are finitely many $\mathcal{N}$-strategies that can move n's markers, namely the ones with threshold $d_i[s] \leq n$. Every time a new $\mathcal{N}$-strategy is activated it chooses its threshold $d_i > l(\Theta_i^{U_i,V_i}, E)$. Hence once the length of agreement $l(\Theta_i^{U_i,V_i}, E)$ is above n, no newly activated $\mathcal{N}$-strategy or no strategy whose threshold d_i is cancelled and then re-chosen will have influence on n . So out of the finitely many $\mathcal{N}$-strategies, which have $d_i \leq n$ at any stage, only the ones that are active infinitely often and do not get initialized after they have chosen this threshold can have a permanent effect on n, i.e. only the strategies along the true path. The ones to the right will be initialized and will re-choose their thresholds to be bigger than n, the ones to the left will not be accessible after a certain stage.

We assumed $\Theta_i^{U_i,V_i} = E$, hence there will not be an outcome h_i along the true path. Indeed if $\beta \supset \alpha$ has active P_i-strategy α and true outcome h_i then there is a permanent witness x_β so that $Axioms_{i,\beta}$ changes its entries infinitely often. $Axioms_i$ has finitely many entries, one for each $y \leq x, y \in E$. Hence the entry for at least one element $y \in E$ changes infinitely often, thus $y \notin \Theta_i^{U_i,V_i}$.

Let $h \upharpoonright m$ be the biggest $\mathcal{N}$-strategy which has an active $\mathcal{P}_i$-strategy α and a permanent threshold $d_i \leq n$ after stage t_0. Let $t_2 > \max(s_l(m+1), s_i(m+1), t_0)$ be a stage such that all other $\mathcal{N}$-strategies along the true path and to the right of it have already changed the value of their threshold d_i to a value greater than n.

We claim that after stage t_2 no $\mathcal{N}$-strategy β will change the current i-markers of n. So suppose β is visited at stage $t > t_2$ and has outcome o. Suppose $\beta \subset h$ and has a permanent threshold $d_i[t] < n$. In all other cases it follows from the choice of stage t_2 that β will not change the i-markers of n. Note that according to the choice of $t_2 > s_i(m+1)$ the outcome o is equal to or to the right of the true outcome o_β of β. We shall examine the different possibilities for o_β. Outcome $o_\beta = g_j$ for $j > i$ would cancel d_i at every $\beta\hat{\ }o_\beta$-true stage contradicting the assumption that d_i is permanent. If $o_\beta = g_k$ or $o_\beta = h_k$, for $k < j$ then there will be a new $(\mathcal{P}_i, \Gamma_i)$-strategy along the true path, contradicting the assumption that α is the biggest one. If $o_\beta = f_x$ then it follows from Lemma 3.5.1 and the choice of $t_2 \geq s_i(m+1)$ that $o = f_{x'}$, where $x' \leq x$. If $o_\beta = w$ or $o_\beta = h_j$, for $j > i$ then $o = w$ or $o = h_k$ for $k > j$. In all three cases β will not move any i-markers at stage t.

Proposition 3.5.2 proves that under these circumstances the strategy α will not move the markers either. Hence our assumption is wrong and there is an outcome g_i along the true path.

2. Assume for a contradiction that $\Theta_i^{U_i,V_i} = E$. Let n be the least element whose A_α-marker moves off to infinity. If $M_i = \Gamma_i$ then according to the previous case there will be an $\mathcal{N}$-strategy along the true path with true outcome g_i, followed by another $\mathcal{P}_i$-strategy, namely working with Λ_i. Hence $M_i = \Lambda_i$.

   We will prove that after a certain stage t the current marker of n is not moved by any $\mathcal{N}$-strategy β. The ones that are not in α's subtree do not have access to the markers defined by α. There are only finitely many strategies with permanent threshold $d_i \leq n$. They are all on the true path. Let s be a stage bigger than $s_i(m+1)$, where $h \restriction m$ is the greatest such $\mathcal{N}$-strategy and such that all nonpermanent thresholds are already bigger than n, $n < l(\Theta_i^{U_i,V_i}, E)[s]$ and all strategies to the right of $h \upharpoonright m$ are initialized. Note that after this stage, whenever we visit $\beta \supset \alpha$ such that $\beta \subseteq h \upharpoonright m$ then β can only have an outcome equal to or to the right of the true path.

Let $\beta \supset \alpha$ be an $\mathcal{N}$-strategy along the true path with true outcome o_β. Outcomes $o_\beta = g_j$ for $j > i$ would mean that $d_i > n$ and β does not influence n's marker after stage t. There is no outcome g_i. Outcomes $o_\beta = g_k$ and $o_\beta = h_k$, $k < i$ would activate a bigger $\mathcal{P}_i$-strategy. As in (1) the only possible true outcomes turn out to be $o_\beta = h_j$, $j > i$, outcomes $o_\beta = w$ and $o_\beta = f_x$. But we have seen that in this case the β does not move any i-markers after stage t.

If $n \notin \overline{K}$ then there will be a stage s at which n exits $\overline{K}$ and after which the $\lambda_\alpha(n)$ remains the same. Hence $n \notin K$ and Proposition 3.5.2 proves that in this case α will also stop moving the current marker.

We have reached a contradiction, hence $\Theta_i^{U_i,V_i} \neq E$.

□

Corollary 3.5.1. *The $\mathcal{P}_i$-requirements are satisfied.*

Proof. If $\Theta^{U_i,V_i} \neq E$ then $\mathcal{P}_i$ is trivially satisfied. Assume $\Theta^{U_i,V_i} = E$. Consider the biggest $\mathcal{P}_i$-node α on the true path. It follows from Proposition 3.5.3 that for all its elements all its current markers eventually settle down. Hence by Proposition 3.5.2 for any $n \in \overline{K}$ we have that $n \in \Gamma_i^{U_i,A}$ if α is constructing Γ_i and $n \in \Lambda_i^{V_i,A}$ if α is constructing Λ_i.

If $n \notin \overline{K}$ then $n \notin \overline{K}[t]$ for all $t > s_0$. If $T(\alpha) = (\mathcal{P}_i, \Gamma_i)$ then $n \notin \Gamma_i^{U_i,A}[t]$ at all α-true expansionary stages $t > s_0$, thus $n \notin \Gamma_i^{U_i,A}$. If $T(\alpha) = (\mathcal{P}_i, \Lambda_i)$ then for each axiom $\langle n, V_n, m\rangle \in \Lambda_i$ there are infinitely many stages $t > s_0$ at which this axiom is not valid. Namely for each α-true stage $t > s_0$ with previous α-true stage t^- either there is a stage t_n, $t^- < t_n \leq t$ at which $V_n \nsubseteq V_i[t_n]$ or else at stage t we extract m from A. Thus $n \notin \Lambda_i^{V_i,A}$. □

3.5.3 Satisfaction of the $\mathcal{N}$-requirements

We start by stating two properties of the construction, whose proof is exactly the same as in section 2.5.3 and will be omitted here.

Propostion 3.5.4. *For every node α along the true path $\lim_s R_\alpha[s] = \infty$.*

Proof. See Proposition 2.5.4. □

Propostion 3.5.5. *Let $\alpha \subset h$ be an $\mathcal{N}$-strategy with $ins(\alpha) = \beta$. Suppose $\alpha \supset \beta\hat{\ }g_j$ and α attacks with a witness $\widehat{x}$ at stage t together with an attack of β with x. Then $Attack(\widehat{x})[k] = Attack(x)[k]$ for all $k \leq j$.*

Proof. See Proposition 2.5.5. □

Next we state a technical property whose proof is essentially the same as the proof of Preposition 2.5.6. The definition of an applicable axiom is however different in this construction, so we will give a short proof for completeness.

Propostion 3.5.6. *Suppose an $\mathcal{N}_i$-strategy $\alpha \subset h$ with active $\mathcal{P}_j$-strategy β_j starts an attack with witness x at stage s. Let $y \leq x$, $y \in E$ be a number with entry $\langle y, U_{y,j}, \overline{W}_{y,j}\rangle$ in $Axioms_j[s]$. Let $n \geq d_j$ be a number with an α-potentially applicable axiom $\langle n, Z_{n,j}, \{m\}\rangle$ at stage s at β_j. Then $Z_{n,j} \supseteq U_{y,j}$ if β_j is a $(\mathcal{P}, \Gamma)$-strategy and $Z_{n,j} \supseteq \overline{W}_{y,j}$ if β_j is a $(\mathcal{P}, \Lambda)$-strategy.*

Proof. We consider the case when β_j is a $(\mathcal{P}, \Gamma)$-strategy. During $\alpha's$ cycle with x, the strategy completes $Initialization$ at stage s_0, say. Let $s_1 \geq s_0$ be the last stage at which α performs $Honestification(k)$ for $k \leq j$. At stage s_1 the strategy extracts the markers of all potentially applicable axioms for n, enumerates them in $O_\alpha[s_1]$ and cancels its marker $u_j(n)$. Thus at stages $t > s_1$ the marker $u_j(n)[t]$ is defined to be bigger than $u(d_j)(n)[s_1] \geq \theta(x)[s_1] \geq \max(U_{y,j})$. The axiom $\langle n, Z_{n,j}, \{m\}\rangle$ is potentially applicable at α at stage s, so $m \notin O_\alpha[s] \supseteq O_\alpha[s_1]$. Thus the axiom is enumerated in Γ_j at a stage

$s_2 > s_1$ and $Z_{n,j} = U[s_2] \upharpoonright u_j(n)[s_2]$. As α attacks at stage s, the operator Γ_j is honest at stages t with $s_1 < t \leq s$, in particular it is honest at stage s_2, i.e. $U_{y,j} \subseteq U[s_2]$, and hence $Z_{n,j} \supseteq U_{y,j}$. □

The main aim now is to prove that if an $\mathcal{N}_i$-strategy α on the true path has outcome f_x for some x then the requirement $\mathcal{N}_i$ is satisfied as $x \in \Psi_i^A$. To ensure this we will need to establish that the set $In(x)$ that α is trying to restrain in A ends up indeed in A. Various strategies around α might try to prevent this from being true by extracting elements from A. We will first prove that a $\mathcal{P}$-strategy that is not active at α cannot extract any elements that α is trying to restrain in A. Then we shall prove that neither can any of the other $\mathcal{N}$-strategies. Finally we will establish this for the active $\mathcal{P}$-strategies at α.

Propostion 3.5.7. *Suppose we have an $\mathcal{N}_i$-strategy $\alpha = h \upharpoonright n$ along the true path with active $\mathcal{P}_j$-strategies $\beta_j \subset \alpha$ for $j < i$ and true outcome $h(n+1) = g_j$ or h_j, where $j < i$. Suppose $h \upharpoonright (n+1)$ is visited at stage $s > s_i(n+1)$ with right boundary $R[s]$. Then if $m < R[s]$ is an A_{β_k}-marker, where $k \geq j$, and m is extracted at stage $t > s$ by the active $\mathcal{P}_k$ strategy β_k then m is extracted from A at all $h \upharpoonright (n+1)$-true stages $t \geq s$.*

Proof. After stage $s_i(n+1)$ defined in Lemma 3.5.1 α has permanent thresholds d_k and permanent sets O_{d_k} for $k \geq j$ and $Out_\alpha[t] \supseteq \bigcup_{k \leq j} O_{d_j} \cup O_\alpha[s]$ at all $t \geq s$.

Suppose m is extracted by β_k, where $k \geq j$, at stage $t > s$. Then m is an A_{β_k}-marker of an axiom for an element $e \notin \overline{K}$ such that $e > d_k$ as otherwise a new element would enter O_{d_k} contradicting our choice of stage s. If the marker m was defined after stage s then it is bigger than $R[s]$. If the marker is defined before stage s then so is the axiom Ax_m that it belongs to.

If $h(n+1) = h_j$ then $Out_\alpha[s] \subseteq Out_\alpha[t]$ for all $t > s$. At stage s the axiom Ax_m is examined by α and if m is not already in Out_α then m is enumerated in Out_α at stage s. Hence $m \in Out_\alpha[t]$ for all $t \geq s$.

If $h(n+1) = g_j$ then $Wit_j[s] \subseteq Wit_j[t]$ for all $t > s$ and the current witness $x[s]$ is in the set Wit_j. If $m \notin O_\alpha[s]$ at stage s then it is enumerated in $O_{x[s],own}$ and $O_\alpha[s] \cup O_{x,own}[s] \subseteq Out_\alpha[t]$ at all $\alpha\hat{\ }g_j$-true stages t. □

Propostion 3.5.8. *Suppose $\beta \subset h$ is visited at stages $s_1 > s_i(\beta)$ and $s_2 > s_1$. Suppose at stage s_1 β attacks and then restrains an element m in A until stage s_2. If the active $\mathcal{P}$-strategies at β do not extract m at stages t $s_1 < t \leq s_2$ then neither do the other strategies.*

Proof. It follows that β is an $\mathcal{N}$-strategy that has outcome f_w at all β-true stages t, $s_1 < t < s_2$. The stage of the attack with w is $t_w \geq s_i(\beta)$. The set that β restrains in A is $In(w) \subseteq A[t_w] \upharpoonright R_\beta[t_w]$ and $m < R_\beta[t_w]$. Suppose $\alpha \neq \beta$ extracts m at a stage t, $s_1 < t \leq s_2$. And let that be the least stage and α be the least strategy. We will prove that it is an active $\mathcal{P}$-strategy at β by examining the different possible cases for α.

- $\alpha <_L \beta$ is not possible, as α would not be accessible at stage t.
- $\alpha >_R \beta$, then at stage s_1 α is initialized. If α is a $\mathcal{P}$-strategy then all its markers would be defined after stage s_1 and would be greater than $R_\beta[s_1] > R_\beta[t_w] \geq m$. If α is an $\mathcal{N}$-strategy then it chooses its thresholds after stage s_1 as fresh numbers whose markers are not yet defined. The only markers $m' < R_\beta[s_1]$ that can enter $Out_\alpha[t]$ are the ones that enter $\alpha's$ parameter O_{d_i} and have to be already extracted from A after stage s_1 by a smaller strategy, an active $\mathcal{P}$-strategy at α.
- $\alpha \supset \beta$, then α extracts markers only on active stages, hence if it is visited after stage s_1 then $\alpha \supseteq \beta\hat{\ }f_w$. Then α was initialized on the stage s_1. Similarly to the previous case it cannot be a $\mathcal{P}$-strategy and if $m \in Out_\alpha[t]$ then it must have been first extracted by an active $\mathcal{P}$ at α after s_1 which is smaller than α.
- $\alpha \subset \beta$. If α is a $\mathcal{P}_j$-strategy different from the active one at α then there is an $\mathcal{N}$-strategy $\sigma\hat{\ }o \subset \beta$ with $o \in \{h_j, g_j\}$ that destroys α. Proposition 3.5.7 proves that

α does not extract m at stage t as otherwise $m < R_\beta[t_w] \leq R_\sigma[t_w]$ is extracted at all $\sigma\hat{}o$-true stages after and including t_w contradicting $m \in A[t_w]$.

If α is an $\mathcal{N}_i$-strategy then we have the following cases:

1. $\beta \supseteq \alpha\hat{}w$ or $\beta \supseteq \alpha\hat{}f_x$. Then after stage $t_w \geq s_i(\beta)$ the strategy α has this outcome at all true stages, the set Out_α is constant. No new elements enter $O_{\alpha,d_j}, j < i$, otherwise we initialize β. The sets $Wit_j^\alpha, j < i$ are permanent as is the current witness. The strategy α does not enumerate more elements in O_α as it needs to have some h_j to do so.

2. $\beta \supseteq \alpha\hat{}h_j$, then the elements that enter Out_α at stages $t > t_w$ are markers m_k $k \geq j$ for axioms from the operators of the active $\mathcal{P}$-strategies at α that are potentially applicable at stage t for elements bigger than d_k, hence markers defined after stage t_w. Indeed all markers defined before stage t_w that ever get extracted by α would already be in $Out_\alpha[t_w]$ but $m \in A[t_w]$.

3. $\beta \supseteq \alpha\hat{}g_j$. Then α had an active outcome g_j at the last active stage t_w^- before the attack with w at stage t_w. The marker m was not extracted by α at stage t_w^- and after stage t_w^- α does not enumerate elements $m' < R_\beta[t_w] = R_\beta[t_w^-]$ in O_α or in $O_{x,own}$ for witnesses x defined after stage t_w^-.

   At stage s_1 the strategy α attacks again. If α extracts an element $m < R_\beta[t_w]$ at a stage $t > s_1$ there are two possibilities. The first one is that $m < d_k$ and m enters O_{d_k}. If $k < j$ then it is first extracted by the active $\mathcal{P}_k$-strategy at α after stage s_1 which is smaller. If $k \geq j$ then this would initialize β.

   The second possibility is that $m \in O_{x,else}$ at stage t for some witness x of α defined after stage s_1 and after stage s_1' at which α had an active g-outcome after the attack at s_1. Then m was extracted from $A[s_1']$ and $s_1 < s_1' < t$, contradicting our choice of stage t.

□

Lemma 3.5.2. *Let $\alpha = h \upharpoonright n$ be the last $\mathcal{N}_i$-node along the true path. Then α is successful.*

Proof. Suppose $T(\alpha) = (\mathcal{N}_i, S_0, \dots, S_{i-1})$ and let β_j, $j < i$, be the active $\mathcal{P}_j$-nodes at α, where $\mathcal{P}_j$ is undefined if $S_j = FM_j$. We know that no other node can interfere with α and injure its restraint except for α itself and β_j. As α is the last $\mathcal{N}_i$-node on the true path, it must have outcome w or outcome f_x for some x. Every other outcome is followed by another copy of an $\mathcal{N}_i$-strategy.

If the outcome is w then at all α-true stages $t > s_i(n)$ defined in Lemma 3.5.1, α has a permanent witness x and $x \notin \Psi^A[t]$ with $use(\Psi, A, x)[t] < R_\alpha[t]$. By Proposition 3.5.4 the right boundary R_α is unbounded, hence $x \notin \Psi^A$ and as α never attacks with x, we have $x \in E$. Thus the requirement is satisfied.

Suppose the outcome is f_x and let $Attack(x) = \langle \bar{x}, U_{\bar{x},0} V_{\bar{x},0}, \dots, U_{\bar{x},i-1}, V_{\bar{x},i-1} \rangle$. Then once we visit $\alpha\hat{\ } f_x$ after stage $s_i(n+1)$ the strategy α will permanently restraint $In(x)$ in A. We will prove that the active $\mathcal{P}$-strategies at α do not extract markers from $In(x)$ after stage $s_i(\alpha\hat{\ } f_x)$, the last stage of the attack, and by Proposition 3.5.8 no other strategy will, hence $x \in \Psi^A$.

First we will establish that markers extracted by the active $\mathcal{P}$-strategies after $s_i(\alpha\hat{\ } f_x)$ cannot belong to elements $n < d_j[t_x]$ for all $j < i$. Here t_x is the stage of the attack with witness x. Let q be the greatest index such that $S_q = \Gamma_q$. Then $x \in Wit_q$. After stage $s_i(\alpha)$ the thresholds $d_{i-1}, \dots, d_q$ are not cancelled. If an element enters O_{d_j} or the value of d_j is shifted then we initialize $\alpha\hat{\ } f_x$. Hence this does not happen after stage $s_i(\alpha\hat{\ } f_x)$. Now lets look at $j < q$. Every time we visit α we start from $Result(q)$, examine all witnesses in Wit_q and reach x. Note that once we've reached x, then for all $w < x$ we have established one of the two properties that make us move to the next witness automatically until an active g-outcome is visited, so in this case forever. And our assumption tells us that we will never establish either of the two properties for x. For all Γ_k, $k \leq q$ there is no $V_{\bar{x},k}$ change and for all S_k, $k < q$

there is no $m \in O_{d_k}$ such that $m < L(x)$ and $m \in A[t_x]$. Otherwise we would move to the left of f_x. Hence the only markers restrained in A that might be extracted by the active $\mathcal{P}$-strategies after $s_i(\alpha\hat{}f_x)$ need to belong to elements greater than $d_j[t_x]$ for all j.

If there are no $\Gamma_k = S_k$ for any $k < i$ then no thresholds are ever cancelled, if they are shifted or an elements enters O_{d_j} for $j < i$ then f_x is initialized. So this does not happen after stage $s_i(\alpha\hat{}f_x)$.

Thus, suppose β_j extracts $m < L(x)$ such that $m \in In(x) \subseteq A[t_x]$ at stage $t > s_i(\alpha\hat{}f_x)$. Then m is a marker of an axiom $\langle n, Z_n, \{m\}\rangle$ for some $n \geq d_j[t_x]$ which is valid at stage t and was defined at stage $t_0 < t_x$. The marker m was in $A[t_x]$ hence the axiom was potentially applicable at stage t_x.

If $S_j = \Gamma_j$ $(T(\beta_j) = (\mathcal{P}_j, \Gamma_j))$ then $j \leq q$ and by Proposition 3.5.6 we have $Z_n \supseteq U_{\bar{x},j}$. Hence $V_{\bar{x},j} \nsubseteq V_j[t]$. But then at the next α-true stage one of the conditions for the unsuccessfulness of x would be valid and α would have outcome to the left of f_x contradicting our assumptions.

The only case left to consider is $S_j = \Lambda_j$. We shall deal with all Λ-strategies at once. Suppose that the Λ-strategies at α are $S_{j_0}, S_{j_1}, \ldots, S_{j_r}$, with $j_0 < j_1 < \cdots < j_r$. Then there are strategies $\alpha_0, \ldots, \alpha_r$ such that $\alpha_0\hat{}g_{j_0} = \beta_{j_0} \subset \cdots \subset \alpha_r\hat{}g_{j_r} = \beta_{j_r} \subset \alpha$. Then $ins(\alpha) = \alpha_r$, $ins(\alpha_r) = \alpha_{r-1}, \ldots, ins(\alpha_1) = \alpha_0$.

When α attacks at stage t_x, it times its attack with all of the listed strategies: α_0 which attacked with x_0, $\ldots$, α_r, which attacked with x_r. By Proposition 3.5.5 $Attack(x)[k] = Attack(x_0)[k]$ for all $k < j_0$, $\ldots$, $Attack(x)[k] = Attack(x_r)[k]$ for all $k < j_r$. At the previous α-active stage t_x^- the strategy α_0 had outcome g_{j_0}, α_1 had outcome g_{j_1}, $\ldots$, α_r had outcome g_{j_r}. And so $Out_\alpha[t_x^-] \subseteq O_{x_r,else} \cdots \subseteq O_{x_0,else}$.

We claim that every time α_p has outcome g_{j_p} after stage $t_{x_p} = t_x$ there is a $V_{\bar{x},j_p}$-change for all $p \leq r$. So when we take $j = j_p$, we have a $V_{\bar{x},j}$-change at all β_j-true stages after t_x. Now we have that the axiom $\langle n, Z_n, \{m\}\rangle$, potentially applicable at stage t_x

has by Proposition 3.5.6 the property that $V_{\bar{x},j} \subseteq Z_n$ and so β_j will not extract m at any stage after t_x.

Suppose the claim is true for $k < p$ and α_p has outcome g_{j_p} at stage $t > t_{x_p}$. One reason for this outcome would be the desired $V_{\bar{x},j_p}$-change. The other possible reasons for α_p to have this outcome are for some $k < j_p$:

- $S_k = \Gamma_k$ and there was a change in $V_{\bar{x}_p,k} = V_{\bar{x},k}$ since this witness was last examined, i.e. there is a stage t' such that t' is bigger than the stage of the last attack such that $V_{\bar{x},k} \nsubseteq V_k[t']$. But then when we visit α at the next α-true stage after t it would have an outcome to the left of f_x, so this reason is not possible.

- A marker $m_k < L(x_p)$ of an element $n < d_{k,\alpha_p}[t_{x_p}]$ such that $m_k \in A[t_{x_p}]$ was enumerated in O_{d_k} of α_p.

  Recall that the active $\mathcal{P}_k$-strategy at α_p and α is the same as $k < j_p$. We already established that $n > d_{k,\alpha}[t_x = t_{x_p}]$. Also the marker m_k was defined before stage t_x and even t_x^- as otherwise it would be greater than $L(x_p)$. The marker was not extracted by α on stage t_x^- or else it would be in $O_{x_p,else}$ and not in $A[t_{x_p}]$. So at stage t_x the corresponding axiom $\langle n, Z_n, \{m_k\}\rangle$ was potentially applicable at α and $Z_{x,k} \subseteq Z_n$. The marker m was extracted by the active $\mathcal{P}_k$-strategy at a stage t' after the attack, so $Z_{x,k}$ was a subset of $Z_k[t']$ at an expansionary stage t'. Now if $S_k = \Gamma_k$ this would result in a $V_{\bar{x},k}$-change at stage t' and α would once again have an outcome to the left of f_x at the next true stage, contradicting our assumptions. If $S_k = \Lambda_k$ this would result in no $V_{\bar{x},k} = V_{\bar{x}_p,k}$-change at a β_k-true stage t' contradicting the induction hypothesis.

This concludes the proof of the claim, this lemma and the theorem. □

Chapter 4

Genericity and Non-bounding in the Σ^0_2 Enumeration Degrees

In this chapter we will investigate a second algebraic property of the Σ^0_2 enumeration degrees, namely the existence of degrees that bound and that do not bound *minimal pairs* (see Definition 1.5.2).

Cooper, Li, Sorbi and Yang show in [CLSY05] that every nonzero Δ^0_2 enumeration degree bounds a minimal pair and construct a nonzero Σ^0_2 enumeration degree that does not bound a minimal pair. This non-bounding property can be viewed as the dual of the non-splitting property of the properly Σ^0_2 enumeration degrees proved in Chapter 3. It provides further insight into the structure of the properly Σ^0_2 enumeration degrees within $\mathcal{D}_e(\leq 0'_e)$, allowing us to define a different infinite set of enumeration degrees $\mathcal{I}$ consisting entirely of properly Σ^0_2 enumeration degrees in $\mathcal{D}_e$:

$$\mathcal{I} = \{\ a|\ a > 0_e \wedge (\forall x, y \leq a)[0_e < x \wedge 0_e < y \Rightarrow (\exists d)[d \leq x \wedge d \leq y \wedge d \neq 0_e]]\}.$$

The set $\mathcal{I}$ is downwards closed in the properly Σ^0_2 enumeration degrees, thus $\mathcal{I} \cup \{0_e\}$ is an ideal in $\mathcal{D}_e(\leq 0'_e)$.

In the article [CLSY05] the authors state that their construction can be used to build a 1-generic enumeration degree that does not bound a minimal pair.

We will use lower case Greek letters (especially ρ, τ) for finite binary strings and let $\tau \subseteq \rho$ indicate that τ is an initial segment of ρ. When A is a set we let $\tau \subset A$ denote that τ is an initial segment of $A's$ characteristic function χ_A considered as an infinite binary sequence.

Definition 4.0.1. *A set A is* 1-generic *if for every c.e. set X of finite binary strings*

$$(\exists \tau \subset A)[\tau \in X \vee (\forall \rho \supseteq \tau)[\rho \notin X]].$$

An enumeration degree is 1*-generic if it contains a 1-generic set.*

Copestake [Cop88] had already investigated the properties of the n-generic enumeration degrees for every $n < \omega$. She proved that every 2-generic enumeration degree bounds a minimal pair and announced that there exists a 1-generic enumeration degree that does not bound a minimal pair. Her proof has not appeared in the academic press.

In this chapter we present a complete proof of this longstanding conjecture following the basic ideas presented in [CLSY05]. Our proof however introduces some new features, not present in the construction by Cooper, et al. The enumeration degree that is constructed is also properly Σ^0_2 and generalizes the result from [CLSY05].

Theorem 4.0.1. *There exists a 1-generic Σ^0_2 enumeration degree* **a** *that does not bound a minimal pair in the semi-lattice of the enumeration degrees.*

Let us further note that our result is in contrast with the properties of the 1-generic Turing degrees. Although there are c.e. Turing degrees that do not bound any minimal pair in $\mathcal{D}_T(\leq 0')$ as proved by Lachlan [Lac79], every 1-generic Turing degrees bounds a minimal pair. For a proof of this property see [Odi99].

The work presented in this chapter is published in [Sos07], see Appendix A.1.

4.1 Requirements

Once again we will use the priority method and follow the basic steps, outlined in Section 1.3. We will construct a set A whose e-degree **a** will have the intended properties.

We will have the following list of requirements:

1. The set A should be 1-generic. Let $\{W_e\}_{e<\omega}$ be a computable enumeration of all c.e. sets. For every e we shall have a requirement:

$$\mathcal{G}_e : (\exists \tau \subseteq A)[\tau \in W_e \vee (\forall \mu \supseteq \tau)[\mu \notin W_e]],$$

   where τ and μ are finite binary strings.

2. A does not bound a minimal pair. Let $\{(\Theta, \Psi)_i\}_{i<\omega}$ be a computable enumeration of all pairs of enumeration operators. For every i we will have a requirement:

$$\mathcal{R}_i : \Theta_i^A \text{ is c.e.} \vee \Psi_i^A \text{ is c.e.} \vee$$
$$\vee (\exists D_i)[D_i \leq_e \Theta_i^A \wedge D_i \leq_e \Psi_i^A \wedge D_i \text{ is not c.e.}].$$

   Fix an $\mathcal{R}$-requirement $\mathcal{R}_i$. Let $X_i = \Theta_i^A$ and $Y_i = \Psi_i^A$. This requirement is too complicated to be satisfied at once and we will break it up into subrequirements. Let $\{W_j\}_{j<\omega}$ be a computable enumeration of all c.e. sets:

$$\mathcal{R}_i : (\exists \Gamma_i)(\exists \Lambda_i)(\forall j)[\mathcal{S}_{i,j}],$$

   where $\mathcal{S}_{i,j}$ is the subrequirement:

$$\mathcal{S}_{i,j} : X_i \text{ is c.e.} \vee Y_i \text{ is c.e.} \vee [\Gamma_i^{X_i} = \Lambda_i^{Y_i} = D_i \wedge W_j \neq D_i].$$

We order all requirements linearly. Every requirement $\mathcal{S}_{i,j}$ has lower priority than $\mathcal{R}_i$. One possible way to order the requirements is:

$$\mathcal{G}_0 < \mathcal{R}_0 < \mathcal{G}_1 < \mathcal{S}_{0,0} < \mathcal{G}_2 < \mathcal{R}_1 < \mathcal{G}_3 < \mathcal{S}_{0,1} < \mathcal{G}_4 < \mathcal{S}_{1,0} < \dots .$$

4.2 Strategies and outcomes

In this section we shall describe the basic strategies for satisfying each requirement. The construction will again be carried out on a tree of strategies T, which we shall keep it in mind during our discussions.

The set A will be constructed as Σ^0_2 in a similar way to the one used in Chapter 3. At every stage the set shall initially be approximated by the set of natural numbers

$\mathbb{N}$. Each node (say of length n) visited at stage s will build its own approximation to the set A, denoted by $A^n[s]$, enumerating and extracting elements from $A^{n-1}[s]$. The approximation to the set A at stage s will be $A[s] = A^s[s]$, the resulting set after all nodes have completed their actions. The nodes will obey restrictions on A and $\overline{A}$ that are set by higher priority strategies. Ultimately A will be the set of all natural numbers a such that:

$$(\exists s)(\forall t > s)[a \in A[t]].$$

We will proceed to describe what general actions the different types of strategies, corresponding to the different types of requirements, will make. We have three types of strategies corresponding to the three types of requirements $\mathcal{G}$, $\mathcal{R}$ and $\mathcal{S}$. Every $\mathcal{S}$-strategy β is a substrategy of one particular $\mathcal{R}$-strategy $\alpha \subset \beta$, which we call its *superstrategy.*

Let γ be a $\mathcal{G}$-strategy working with the set W. This strategy is a bit more complicated than our usual FM-strategy, while still maintaining some resemblance to it. In order to be successful it needs to find a witness τ for the genericity of A regarding the c.e. set W. A witness τ for γ will be a finite binary string such that $\tau \subseteq A$ and either $\tau \in W$ or $\forall \rho \supseteq \tau$ we have that $\rho \notin W$. The strategy γ chooses a finite string τ according to rules that ensure compatibility with strategies of higher priority. Then it searches for a string ρ such that $\tau\hat{}\rho \in W$. If it never finds such a string, it will be successful as it will have satisfied the second condition for genericity. While it is searching for such a string it will have outcome w. If it does find such a string eventually then γ remembers the shortest one, μ, and has outcome f. The witness of γ is extended to $\tau\hat{}\mu$. We have the usual order between the two outcomes, $f <_L w$. To ensure that the witness τ in the first case and the extended witness $\tau\hat{}\mu$ in the second case are initial segments of the characteristic function of the set A, the strategy γ will restrain some elements out of A and in A.

Let α be an $\mathcal{R}$-strategy, working with $X = \Theta^A$ and $Y = \Psi^A$. It acts as a mother strategy to all its substrategies ensuring that they work correctly. We assume that on this level the two enumeration operators Γ and Λ are built. They are common to all substrategies of α. This strategy has only one outcome: e.

Let β be an $\mathcal{S}$-strategy with superstrategy $\alpha \subset \beta$. The strategy β is automatically assigned some of the parameters it works with from the superstrategy α. These are the sets $X = \Theta^A$, $Y = \Psi^A$ and the two operators Γ and Λ constructed at α. Furthermore it has its own parameter W, the c.e. set it is working with.

The strategy β has three options to satisfy its requirement: it can prove that X is c.e., it can prove that Y is c.e. or it can prove that $\Gamma^X = \Lambda^Y = D \neq W$. The first two options are considerably easier than the third, so the strategy tries to prove them first. Only when both attempts fail, will β switch to the third option. Each failure of an attempt provides β with some control over the sets X and Y that it will use.

The strategy first tries to prove that the set X is c.e. by building a c.e. set U which approximates the set X. On each stage it adds elements to U that seem currently in the set X and then looks if any errors have occurred in the set. While there are no errors the outcome is ∞_X.

If an error occurs then some element, that was assumed to be in the set $X = \Theta^A$, has been extracted from X, i.e an axiom in the current approximation to Θ for some element has been invalidated by an extraction from the set A. The strategy cannot fix the error by extracting the corresponding element from U because we want U to remain c.e. In this case β gives up on its desire to make X c.e. It finds the least error $k \in U \setminus X$ and forms a set E_k which is called an *agitator set* for k. The agitator contains an element a for every axiom for k in the current approximation of Θ, say $\langle k, D_k \rangle$, such that $a \in D_k$. So extracting the agitator set from A will ensure that each axiom for k in Θ will not be valid for $\Theta^A = X$, that is it will ensure that $k \notin X$. With some additional actions we will ensure that if the agitator is a subset of A then $k \in X$. And so the agitator will have the following property which we will refer to as *the control property*:

$$k \in X \Leftrightarrow E_k \subseteq A.$$

The strategy now turns its attention to Y. It tries to prove that it is c.e. by constructing a set V_k, aiming to make it equal to Y. The set V_k is built in a similar way. Simultaneously the strategy always checks if the agitator E_k for k preserves its control property. Note that the agitator will lose this property if a new axiom for k is enumerated in Θ and if so the error in U is corrected, so the strategy can return to its initial strategy to prove that X is c.e. While there are no errors in V_k and E_k has its control property, the outcome is $\langle \infty_Y, k\rangle$.

If an error is found in V_k, the strategy chooses the least $l \in V_k \setminus Y$ and forms an agitator F^k_l for l in a similar way. F^k_l also has a control property:

$$l \in Y \Leftrightarrow F^k_l \subseteq A.$$

The strategy β finally has some control over the sets X and Y, namely using the agitators it can determine whether or not $k \in X$ and $l \in Y$. It adds axioms $\langle d, \{k\}\rangle \in \Gamma$ and $\langle d, \{l\}\rangle \in \Lambda$ for some witness d, constructing a difference between D and W. If $d \in W$ the outcome is $\langle l, k\rangle$ and the agitators are kept out of A. If $d \notin W$ then the agitators are enumerated in A, so $d \in D$ and the outcome is the symbol d_0.

The possible outcomes of an $\mathcal{S}$-strategy are:

$$\infty_X <_L T_0 <_L T_1 <_L \cdots <_L T_k <_L \cdots <_L d_0,$$

where T_k is the following group of outcomes:

$$\langle \infty_Y, k\rangle <_L \langle 0, k\rangle <_L \langle 1, k\rangle <_L \cdots <_L \langle l, k\rangle \ <_L \ldots$$

4.3 The tree of strategies

The tree of strategies is defined in the usual way, as described in Section 1.3. It is a computable function T with $D(T) \subseteq \{\ w, d, e, \infty_X, \langle \infty_Y, k\rangle, \langle l, k\rangle, d_0 |\ k, l \in \mathbb{N}\}^{<\omega}$ and $R(T)$ the set of all requirements. The nodes on every level n of the tree are assigned to the n-th requirement in our priority listing. Thus nodes on even levels will always be $\mathcal{G}$-strategies.

As we already discussed for every subrequirement $\mathcal{S}_{i,j}$ assigned to a node β on the tree, there is a unique node $\alpha \subset \beta$ such that $T(\alpha) = \mathcal{R}_i$. The node α is the superstrategy of β and β is a substrategy of α. If the strategy β succeeds to prove that the set $X_\alpha = X_\beta$ is c.e. or that the set $Y_\alpha = Y_\beta$ is c.e. then the requirement $\mathcal{R}_i$ will be globally satisfied and no further subrequirements will need a strategy on the tree to satisfy them. For this reason below outcome ∞_X and $\langle \infty_Y, k\rangle$ of an $\mathcal{S}_{i,j}$-strategy there are no further $\mathcal{S}_{i,j'}$-substrategies of α, where $j < j'$. In this case we will rearrange the requirements so that we still have a $\mathcal{G}$-strategy at every node of even length. Let $\{\mathcal{P}_e\}_{e<\omega}$ be some computable listing of all $\mathcal{R}$- and $\mathcal{S}$-requirements such that $\mathcal{R}_i$ has higher priority than $\mathcal{S}_{i,j}$. We shall define the tree inductively. We will make use of an also inductively defined set of $\mathcal{P}$-requirements at every node α, denoted by $\mathbb{P}_\alpha$, consisting of the $\mathcal{P}$-requirements that *need attention*. At the root of the tree this set consists of all $\mathcal{P}$-requirements.

1. If α is of even length $2e$ then $T(\alpha) = \mathcal{G}_e$. We define $\mathbb{P}_{\alpha\hat{\ }o} = \mathbb{P}_\alpha$, where $o \in \{d, w\}$.

2. If α is of odd length then it is assigned the least $\mathcal{P}$-requirement in $\mathbb{P}_\alpha$.

3. If α is an $\mathcal{R}_i$-strategy then $\mathbb{P}_{\alpha\hat{\ }e} = \mathbb{P}_\alpha \setminus \{\mathcal{R}_i\}$.

4. If α is an $\mathcal{S}_{i,j}$-strategy then for $o' \in \{\ \infty_X, \langle \infty_Y, k\rangle |\ k < \omega\}$, we define $\mathbb{P}_{\alpha\hat{\ }o'} = \mathbb{P}_\alpha \setminus \{\ \mathcal{S}_{i,j'} |\ j' \geq j\}$. For $o'' \in \{\ d_0, \langle l, k\rangle |\ l, k < \omega\}$, we define $\mathbb{P}_{\alpha\hat{\ }o''} = \mathbb{P}_\alpha \setminus \{\mathcal{S}_{i,j}\}$.

4.4 Interactions between strategies

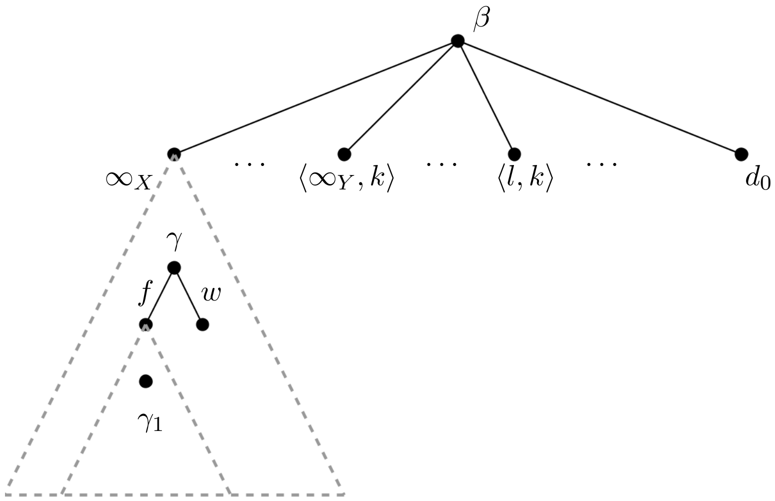
Before we give the formal construction we shall consider some possible interactions between different strategies, that will reveal to us some difficulties and the need for some extra parameters. In order to have any organization whatsoever we make use of a global parameter, a *counter* **b**, whose value will be an upper bound to the numbers that have appeared in the construction up to the current moment.

4.4.1 A $\mathcal{G}$-strategy below an $\mathcal{S}$-strategy

Our first concern will be the interactions between a higher priority $\mathcal{S}$-strategy β and a lower priority $\mathcal{G}$-strategy γ. The interesting cases are when $\gamma \supseteq \beta\hat{\ }\infty_X$ and similarly when $\gamma \supseteq \beta\hat{\ }\langle\infty_Y, k\rangle$. We shall concentrate on the case $\gamma \supseteq \beta\hat{\ }\infty_X$, describing a possible situation that if repeated infinitely often shall lead to the incorrect work of the strategy β. We shall then modify the design of the strategies to avoid the described situation.

Suppose β is of length n and is visited at stage s at which it adds an element k to the set U. Every element that enters the set U is currently in the set $X = \Theta^A$. So there is an axiom $\langle k, E'\rangle \in \Theta$ which is currently valid, i.e. $E' \subseteq A^n[s]$, where $A^n[s]$ is the approximation to A at substage n of stage s, the substage at which we visit β. The strategy β will keep a list $\mathbb{U}$ of the axioms from Θ that it assumes to be valid when enumerating new elements in U. At stage s this list shall contain the axioms in Θ for every element in U of least age, see Definition 1.2.1, and will be updated at β-true stages. Thus if $k \in X$ then the entry in $\mathbb{U}$ for k will eventually stop changing.

Now suppose that after the entry of k in U (possibly even at the same stage) γ chooses a string μ_γ and extracts a member of E' from A. If there aren't any other axioms for k in the corresponding approximation of Θ and γ is on the true path, then we shall have an error in U. Fortunately when the strategy β checks for errors in its set U, it examines all stages and substages since it was last visited. At the next β-true stage, s_1 say, β will therefore find this error, choose an agitator for k and move on to the right with outcome $\langle\infty_Y, k\rangle$. It is possible that later a new axiom for k is enumerated in the corresponding approximation to Θ and thus the error in U is corrected. At the next β-true stage s_2, β returns to its initial aim to prove that X is c.e. But then another $\mathcal{G}$-strategy $\gamma_1 \supseteq \gamma\hat{\ }f$ chooses a string μ_{γ_1} and again extracts k from X by extracting an element that invalidates the new axiom for k. If this situation repeats infinitely often, ultimately we will claim to have $X = U$ but k will be extracted from X at infinitely many stages and thus our claim would be wrong. The corresponding $\mathcal{S}$-subrequirement will not be satisfied.



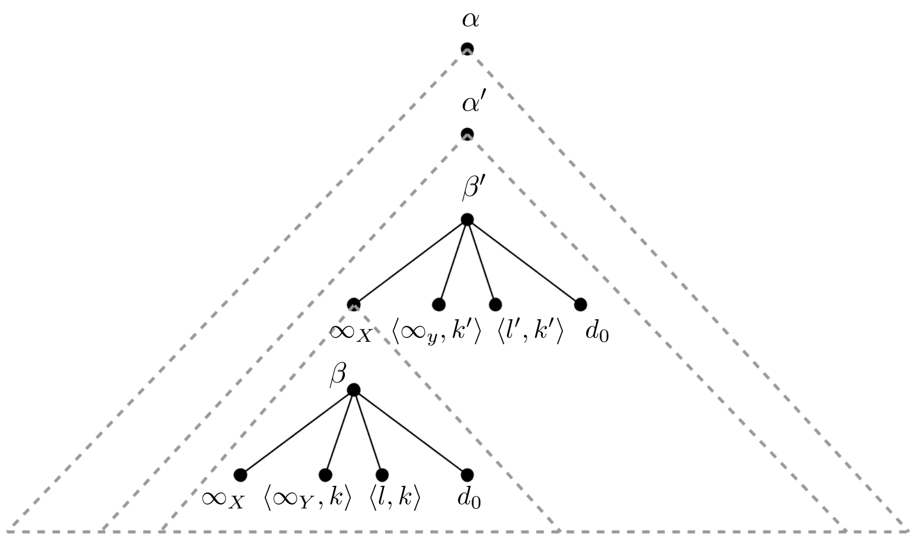
To avoid this we will have to ensure some sort of stability for the elements that we put in U, more precisely for the corresponding axioms in $\mathbb{U}$ that we assume to be valid. This is how the idea for *applying an axiom* arises. We apply an axiom $\langle k, E'\rangle$ by changing the value of the global parameter $\mathbf{b}$ so that it is larger than the elements of the axiom and then by initializing those strategies that might invalidate the axiom.

The first thing that comes to mind is to initialize *all* strategies $\delta \supseteq \beta\hat{\ }\infty_X$. This way we would avoid errors at all. If the set X is infinite though, we would never give a chance to strategies $\delta \supseteq \beta\hat{\ }\infty_X$ to satisfy their requirements as we will initialize them infinitely often. This problem is solved with the notion of local priority. Every $\mathcal{G}$-strategy $\gamma \supseteq \beta\hat{\ }\infty_X$ will have a fixed local priority regarding β. This priority is given by a computable bijection $\sigma_\beta : \mathbb{G}_\beta \to \mathbb{N}$, where $\mathbb{G}_\beta$ is the set of all $\mathcal{G}$-strategies in the subtree of $\beta\hat{\ }\infty_X$. If $\gamma \subset \gamma_1$ then $\sigma_\beta(\gamma) < \sigma_\beta(\gamma_1)$. A strategy $\gamma \supseteq \beta\hat{\ }\infty_X$ has local priority $\sigma_\beta(\gamma)$ in relation to β. When we apply the axiom $\langle k, E'\rangle$ only strategies γ with $\sigma_\beta(\gamma)$ greater than k will be initialized. Then as the stages grow so do the elements that enter U and with them grows the number of $\mathcal{G}$-strategies that are preserved. Ultimately each strategy will get a chance to satisfy its requirement.

We will extend the definition of local priority to all strategies $\delta \supseteq \beta\hat{\ }\infty_X$. This will be a useful way to refer to them in the description of the construction. A strategy $\delta \supset \beta$ which is not a $\mathcal{G}$-strategy is an immediate successor of a $\mathcal{G}$-strategy γ. It will be give the local priority of γ, i.e. $\sigma_\beta(\delta) = \sigma_\beta(\gamma)$. Naturally all strategies extending $\beta\hat{\ }\langle\infty_Y, k\rangle$ for every $k < \omega$ shall also be given local priority relative to β, defined in a similar way.

4.4.2 An $\mathcal{S}$-strategy below an $\mathcal{S}$-strategy

Our second concern is the interaction between two $\mathcal{S}$-strategies β and β', with corresponding superstrategies α and α'. As every $\mathcal{R}$-strategy has infinitely many substrategies it is possible that β is of lower priority than β' on the tree, while the corresponding superstrategies are arranged in the opposite way, namely α is of higher priority than α'. We shall examine precisely this case, assuming further that $\beta \supseteq \beta'\hat{\ }\infty_X$.



Suppose that at stage s the strategy β chooses its agitators E_k and F_l^k and extracts them from A having outcome $\langle l, k\rangle$. This means that β has already modified the operators Γ_α and Λ_α by enumerating the corresponding axioms $\langle d, \{k\}\rangle$ and $\langle d, \{l\}\rangle$ in them. From this stage on until (if ever) the witness d is cancelled it is important for the correctness of α to keep either $k \in X$ and $l \in Y$ or to keep both $k \notin X$ and $l \notin Y$.

Otherwise we risk $\Gamma_\alpha^X(d) \neq \Lambda_\alpha^Y(d)$. To achieve this the two agitators E_k and F_l^k need to be either both entirely extracted from $A[t]$ or subsets of $A[t]$ at every stage $t > s$. (If the witness is cancelled we enumerate the axiom $\langle d, \emptyset \rangle$ in both operators, preserving the equality at d forever.)

Suppose now that at the next β'-true stage $s' > s$ the strategy β' decides to build its own agitators for elements k' and l'. These new agitators include as a subset β's agitator E_k and do not contain any elements from β's other agitator F_l^k, i.e. $E_k \subset E_{k'} \cup F_{l'}^{k'}$ and $F_l^k \cap (E_{k'} \cup F_{l'}^{k'}) = \emptyset$. Then β' has outcome d_0 and enumerates its agitators in $A[s']$, causing the anticipated difference between the sets Γ_α^X and Λ_α^Y.

To avoid this β' will choose its agitators carefully: along with the elements needed to form the agitator with the requested control property it will add also all elements of all agitators that were chosen and out of A at the previous β'-true stage s. This action will ensure that β's agitators will not be separated as $E_k \cup F_l^k \subset E_{k'} \cup F_{l'}^{k'}$ and whatever β decides to do, it will not cause errors in α's operators.

Unfortunately this will not solve the problem completely. It is possible that at a later stage a new axiom is enumerated in Θ_α for k or a new axiom is enumerated in Ψ_α for l, causing one of the agitators E_k or F_l^k to lose its control property and creating a difference between the sets Γ_α^X and Λ_α^Y at the element d again. If β is visited again then it will fix this mistake by cancelling the false witness d. If not, the error will remain unfixed and the $\mathcal{R}$-strategy α might not satisfy its requirement. To prevent this from happening we will attach a new parameter to α: a list $Watched_\alpha$ through which α will keep track of all its $\mathcal{S}$-substrategies. The list will have an entry for every substrategy, which will contain information about its agitators. If α sees that one of the agitators has lost its control property then it will go ahead with the actions on discarding the false witness and correcting the mistake in the operators Γ_α and Λ_α in advance. This action will not interfere with β's work. In fact if β is ever visited again it will cancel the witness and give up the agitator that has lost its control property anyway. In this sense α is just preempting the actions of β.

4.5 Construction

We will begin the description of the construction by listing again all parameters that are associated with each strategy. Their purpose was explained intuitively in the previous two sections. While describing the parameters we will suppress the subscripts that indicate the strategy to which they belong. These subscripts will appear only when more than one strategies are involved in a discussion and we need to distinguish between their parameters.

We have one global parameter **b**, common to all strategies, which is an upper bound to all elements that have appeared so far in the construction. Its initial value is 0.

In addition every strategy δ visited at stage s will have two more parameters $E[s]$ and $F[s]$. The set $E[s]$ contains all elements restrained out of A at this stage s by strategies $\delta' \subset \delta$. The set $F[s]$ contains all elements that are restrained in A by strategies of higher priority $\delta'' < \delta$. Note that these elements may have been restrained at a previous stage.

Each $\mathcal{G}_e$-strategy γ working with the set W_e will have two parameters: finite binary strings τ and μ, with initial value the empty string $\emptyset$.

Each $\mathcal{R}_i$-strategy α working with the enumeration operators Θ_i and Ψ_i is equipped with a list $Watched$ with entries of the form $\langle \beta : \langle E, E_k, F^k_l \rangle, d \rangle$, where β is a substrategy of α, E_k and F^k_l are β's current agitators, the set E contains information needed to assess if the agitators still have the control property and d is the witness that must be *cancelled* in case one of the agitators loses its control property. The initial value of the list is $\emptyset$. Also α has parameters Γ and Λ, the enumeration operators that α and all its substrategies construct together. Their initial value is $\emptyset$ as well.

Each $\mathcal{S}_{i,j}$-strategy β inherits the two parameters Γ and Λ from its superstrategy. In addition it constructs a c.e. approximation to a set U and c.e. approximations to sets

V_k for all k, all initially the empty set. Corresponding to these c.e. approximations the strategy has lists $\mathbb{U}$ and $\mathbb{V}_k$, with initial values the empty list. During the construction β might form agitators E_k for all k and F_l^k for all k and l or choose a witness d, but initially the agitators are empty and the witness is undefined.

At stage $s = 0$ all nodes of the tree are initialized, $\mathbf{b}[0] = 0$, $\delta[0] = \emptyset$ and $A[0] = \mathbb{N}$.

At each stage $s > 0$ we will have $A^0[s] = \mathbb{N}$, $\delta^0[s] = \emptyset$ and $\mathbf{b}^0[s] = \mathbf{b}^{s-1}[s-1]$.

Assume that we have already built $\delta^n[s]$, $A^n[s]$ and $\mathbf{b}^n[s]$. If $n = s$ then we end this stage, initializing all nodes to the right of $\delta[s]$ and move on to the next stage $s + 1$. Otherwise $n < s$ the strategy $\delta^n[s]$ will be activated and will choose an outcome o. Then $\delta^{n+1}[s] = \delta^n[s]\hat{\ }o$. Let s^- be the previous stage at which $\delta^n[s]$ was visited if it has not been initialized since this stage and $s^- = s$ if δ is in initial state. We have three cases depending on the type of the strategy $\delta^n[s]$:

(I.) $\delta^n[s] = \gamma$ is a $\mathcal{G}$-strategy. The actions that γ makes are as follows:

1. If $\tau = \emptyset$ then define τ to be the binary string of length $\mathbf{b}^n[s] + 1$ such that:
$$\tau(a) \simeq 0 \text{ iff } a \in E[s].$$
Increase the value of the counter to $\mathbf{b}^{n+1}[s] = \mathbf{b}^n[s] + 1$ and go to step 2.

2. If $\mu = \emptyset$ then search for a string μ such that $\tau\hat{\ }\mu \in W[s]$. If there is no such string then let $A^{n+1}[s] = A^n[s]$. All elements for which $\tau(a) = 1$ are restrained by γ in A and the outcome is $o = w$. If there is such a string then define μ to be the least binary string such that $\tau\hat{\ }\mu \in W[s]$ and increase the value of the counter to $\mathbf{b}^{n+1}[s] = \max(\mathbf{b}^{n+1}[s], |\tau\hat{\ }\mu| + 1)$, where $|\tau\hat{\ }\mu|$ denotes the length of the string $\tau\hat{\ }\mu$. Go to step 3.

3. Restrain in A all numbers a such that $\tau\hat{\ }\mu(a) = 1$. Restrain out of A all numbers a such that $a \geq |\tau|$ and $\tau\hat{\ }\mu(a) = 0$. Let
$A^{n+1}[s] = A^n[s] \setminus \{\, a |\ a$ is restrained out of A by $\gamma\}$ and let the outcome be $o = f$.

(II.) $\delta^n[s] = \alpha$ is an $\mathcal{R}$-strategy. The strategy α scans all entries in the list $Watched[s]$. For each entry $\langle \beta : \langle E, E_k, F_l^k \rangle, d \rangle \in Watched[s]$ it checks if there is an axiom $\langle k, E' \rangle \in \Theta$ such that $E' \cap (E \cup E_k) = \emptyset$ or $\langle l, F' \rangle \in \Psi$ such that $F' \cap (E \cup E_k \cup F_l^k) = \emptyset$. If there is such an axiom then α *cancels* d by enumerating in both sets Γ and Λ the axiom $\langle d, \emptyset \rangle$. After all entries have been checked, α sets $A^{n+1}[s] = A^n[s]$ and has outcome $o = e$.

(III.) $\delta^n[s] = \beta$ is an $\mathcal{S}$-strategy, a substrategy of α. If β is watched by α then it deletes the corresponding entry from $Watched_\alpha$. Unless otherwise specified $\mathbf{b}^{n+1}[s] = \mathbf{b}^n[s]$. The actions that β makes depend on the outcome o^- that the strategy had at the previous β-true stage s^-. If this is the first β-true stage in the construction or if β is initialized after the last β-true stage then $o^- = \infty_X$.

- The outcome o^- is ∞_X.

  1. Choose the least $k \in X \setminus U$. Here $X = \Theta^{A^n}[s]$. If there is such an element then there is an axiom $\langle k, E' \rangle \in \Theta[s]$ with $E' \subseteq A^n[s]$. Enumerate k in the set U. Let $\langle k, E' \rangle$ be the axiom of least age $a(\Theta^A[s], E', s)$ for k in $\Theta[s]$. Enumerate this axiom in the list $\mathbb{U}$ and *apply* this axiom by initializing all strategies $\delta \supseteq \beta\hat{\ }\infty_X$ of local β-priority with value greater than k and by setting $\mathbf{b}^{n+1}[s] = \max(\mathbf{b}^n[s], E')$.

  2. Proceed through the elements of U until an element for which there is no *applicable axiom* is found or until all elements are scanned. Here we have a very specific definition of an applicable axiom. For every k let $\mathbb{R}_{\infty_X}(k)[s]$ be the set of all elements restrained out of A by strategies $\delta \subseteq \delta[s^-]$ that extend $\beta\hat{\ }\infty_X$ and have local priority $\sigma_\beta(\delta) < k$.

     Definition 4.5.1. *An axiom $\langle k, E' \rangle \in \Theta[s]$ is* applicable *at stage s if:*

     i. $E' \cap E_\beta[s] = \emptyset$ and

     ii. $E' \cap \mathbb{R}_{\infty_X}(k)[s] = \emptyset$.

The intuition behind this definition is that it is plausible that the axiom will end up valid. The set $\mathbb{R}_{\infty_X}(k)[s]$ contains all elements that are restrained by strategies with local priority less than k along what seems to be the true path.

If there is an applicable axiom for k then let $\langle k, E'\rangle$ be the applicable axiom for k of least age $a(\Theta^A[s], E', s)$. If the entry for k in $\mathbb{U}$ is different, replace it with $\langle k, E'\rangle$. If the axiom $\langle k, E'\rangle$ is not yet applied, then apply it.

If all elements $k \in U$ are scanned and an applicable axiom is found for each then let $A^{n+1}[s] = A^n[s]$ and $o = \infty_X$.

If there is no applicable axiom for k then proceed as follows:

- Initialize all strategies $\delta \supseteq \beta\hat{\ }\infty_X$ of local $\beta\hat{\ }\infty_X$-priority with value greater than k.
- Examine all $\mathcal{S}$-strategies β' in the subtree with root $\beta\hat{\ }\infty_X$. If β' was visited at stage s^-, had outcome $\langle l', k'\rangle$ with witness d', corresponding agitators $E_{k'}$, $F^{k'}_{l'}$ and was not initialized after stage s^- then add to the list $Watched_{\alpha'}$, where α' is the superstrategy of β', an element of the following structure:

$$< \beta' : \langle E_{\beta'}[s^-], E_{k'}, F^{k'}_{l'}\rangle, d' > .$$

- Finally define the agitator for k as $E_k = \mathbb{R}_{\infty_X}(k)[s] \setminus E_\beta[s]$. All elements $a \in E_k$ are restrained out of A by β. Let $A^{n+1}[s] = A^n[s] \setminus E_k$ and $o = \langle \infty_Y, k\rangle$.

- The outcome o^- is $\langle \infty_Y, k\rangle$.

1. Check if there is an axiom $\langle k, E'\rangle \in \Theta$ such that $E' \cap (E_\beta[s] \cup E_k) = \emptyset$. If so then act as in step 1. of the case $o^- = \langle l, k\rangle$, described below.
2. Choose the least element $l \in Y[s] \setminus V_k$. If there is such an element then let $\langle l, F'\rangle \in \Psi[s]$ be the axiom with $F' \subseteq A^n[s] \setminus E_k$ of least age $a(\Psi^A[s], F', s)$.

Enumerate the element l in V_k and the axiom $\langle l, F' \rangle$ in $\mathbb{V}_k$. *Apply* this axiom by initializing all strategies $\delta \supseteq \beta\hat{}\langle \infty_Y, k \rangle$ of local β-priority with value greater than l and by setting $\mathbf{b}^{n+1}[s] = \max(\mathbf{b}^n[s], F')$.

3. Proceed through the elements of V_k, searching for an element for which there is no applicable axiom in $\Psi[s]$. For every l let $\mathbb{R}_{\langle \infty_Y, k \rangle}(l)[s]$ be the set of all elements that are restrained out of A by strategies $\delta \subset \delta[s^-]$ that extend $\beta\hat{}\langle \infty_Y, k \rangle$ and have local priority less than l.

Definition 4.5.2. *An axiom* $\langle l, F' \rangle \in \Psi$ *is* applicable *if:*

i. $F' \cap E_\beta[s] = \emptyset$*;*

ii. $F' \cap E_k = \emptyset$*;*

iii. $F' \cap \mathbb{R}_{\langle \infty_Y, k \rangle}(l)[s] = \emptyset$.

If there is an applicable axiom for l then let $\langle l, F' \rangle$ be the one with least age $a(\Psi^A[s], F', s)$. If the entry for l in $\mathbb{V}_k$ is different, replace it with $\langle l, F' \rangle$. If the axiom $\langle l, F' \rangle$ is not yet applied, apply it.

If all elements $l \in V_k$ are scanned and an applicable axiom is found for each then let $A^{n+1}[s] = A^n[s] \setminus E_k$ and $o = \langle \infty_Y, k \rangle$.

If there is no applicable axiom for l then proceed as follows:

- Initialize all strategies $\delta \supseteq \beta\hat{}\langle \infty_Y, k \rangle$ of local β-priority with value greater than l.
- Examine all $\mathcal{S}$-strategies extending $\beta\hat{}\langle \infty_Y, k \rangle$. If β' was visited at stage s^-, had outcome $\langle l', k' \rangle$ with witness d', corresponding agitator $E_{k'}$, $F^{k'}_{l'}$ and was not initialized after stage s^- then add to the list $Watched_{\alpha'}$, where α' is the superstrategy of β', an element of the following structure:
$$< \beta' : \langle E_{\beta'}[s^-], E_{k'}, F^{k'}_{l'} \rangle, d >.$$
- The agitator for l is $F^k_l = \mathbb{R}_{\langle \infty_Y, k \rangle}(l)[s] \setminus (E_\beta[s] \cup E_k)$. All elements $a \in (E_k \cup F^k_l)$ are restrained in A by β.

  - Finally find the least element d that has not been used in the definition of Γ yet. This will be a witness for β. Enumerate the axiom $\langle d, \{k\}\rangle$ in Γ and the axiom $\langle d, \{l\}\rangle$ in Λ. Let $A^{n+1}[s] = A^n[s]$ and $o = d_0$.

- The outcome o^- is d_0.

  1. If $d \notin W[s]$ then let $A^{n+1}[s] = A^n[s]$ and $o = d_0$.
  2. If $d \in W[s]$ then β restrains all elements $a \in (E_k \cup F_l^k)$ out of A. Let $A^{n+1}[s] = A^n[s] \setminus (E_k \cup F_l^k)$ and $o = \langle l, k\rangle$.

- The outcome o^- is $\langle l, k\rangle$. Then the agitators E_k and F_l^k and the witness d are defined.

  1. If there is an axiom $\langle k, E'\rangle \in \Theta[s]$ such that $E' \cap (E_\beta[s] \cup E_k) = \emptyset$, i.e. E_k has lost its control property, then *cancel* d and let $V_k = \mathbb{V}_k = E_k = F_l^k = \emptyset$. Update the entry for k in $\mathbb{U}$ with $\langle k, E'\rangle$. *Apply* the axiom $\langle k, E'\rangle$. The strategy β stops restraining elements $a \in E_k \cup F_l^k$. Let $A^{n+1}[s] = A^n[s]$ and $o = \infty_X$.
  2. If there is an axiom $\langle l, F'\rangle \in \Psi$ such that $F' \cap (E_\beta[s] \cup E_k \cup F_l^k) = \emptyset$ then *cancel* d and let $F_l^k = \emptyset$. Update the entry for l in $\mathbb{V}_k$ with $\langle l, F'\rangle$. *Apply* the axiom $\langle l, F'\rangle$. The strategy β stops restraining elements $a \in F_l^k$. Let $A^{n+1}[s] = A^n[s] \setminus E_k$ and $o = \langle \infty_Y, k\rangle$.
  3. If neither of the above two conditions hold, and hence both agitators still have their control property, then let $A^{n+1}[s] = A^n[s] \setminus (E_k \cup F_l^k)$ and $o = \langle l, k\rangle$.

4.6 Proof

The proof of the theorem is divided into four groups of lemmas. The first group concerns the relationships between the various restrictions that strategies impose at different stages. The second group of lemmas concentrates on the properties of the agitator sets. Then follows the group dedicated to the existence of the true path. Each of these groups provide us with essential properties of the construction. These are then collectively used to prove that all requirements are indeed satisfied in the last group of lemmas.

4.6.1 Restriction lemmas

The construction involves various restrictions of elements in and out of the set A. We start our analysis of the construction by establishing some fundamental rules about these restriction. This will help us later to determine properties of the characteristic function of A. We start off with a simple property of the agitator sets that will be helpful for the rest of the restriction lemmas.

Propostion 4.6.1. *Let β be an $\mathcal{S}$-strategy that is visited and chooses an agitator Ag at stage s. Let $s' < s$ be the greatest stage at which β is initialized. The elements of the agitator Ag are restrained out of A by some $\mathcal{G}$-strategy $\gamma \supset \beta$ at a stage s_0 such that $s' < s_0 < s$.*

Proof. The proof is by induction on s. Suppose the statement is true for all strategies visited at stages $t < s$ and let β be visited at stage s. Assume β chooses its agitator E_k and let $a \in E_k$ (the case when β chooses F_l^k is proved similarly). Then $a \in \mathbb{R}_{\infty_X}(k)[s]$ and hence is restrained out of A at stage s^- by some strategy extending $\beta\hat{\ }\infty_X$. Obviously $s^- > s'$, otherwise $\mathbb{R}_{\infty_X}(k)[s] = \emptyset$ because all strategies that extend β would also be initialized and would not restrain any elements out of A.

If a is restrained out of A by a $\mathcal{G}$-strategy then we have established the statement for this element a. Suppose a is restrained by an $\mathcal{S}$-strategy, say $\beta' \supseteq \beta\hat{\ }\infty_X$. Then a is in an agitator $Ag_{\beta'}$ of β'. This agitator was defined by β' on a previous β'-true stage, say t, such that $s' < t < s$. Applying the induction hypothesis for t we obtain the required statement for a in this case as well. □

Strategies δ on the tree work under the assumption that the set E_δ approximates correctly the elements that their predecessors extract from A. The next lemma will prove in a sense that this assumption is correct.

Lemma 4.6.1. *Let s_1 and s_2 be two consecutive δ-true stages. If δ is not initialized at any intermediate stage t such that $s_1 < t \leq s_2$ then $E_\delta[s_1] = E_\delta[s_2]$.*

Proof. We will prove the lemma by induction on the length of δ. If $|\delta| = 0$ then $\delta = \emptyset$ and $E_\delta[s_1] = E_\delta[s_2] = \emptyset$. So let us assume that the statement is true for strategies of length n and let δ be a strategy with $|\delta| = n$. We will prove that the statement holds for $\delta\hat{}o$, an arbitrary immediate successor of δ.

Suppose $\delta' = \delta\hat{}o$ is visited at stages s_1 and s_2 and not initialized at stages t such that $s_1 < t \leq s_2$. Then δ is also visited at stages s_1 and s_2 and is not initialized at any stage t such that $s_1 < t \leq s_2$. The induction hypothesis gives us $E_\delta[s_1] = E_\delta[s_2]$. We only need to prove that the elements that δ restrains at stages s_1 and s_2 are the same. We will examine the different cases depending on the type of δ and the outcome o:

1. If δ is an $\mathcal{R}$-strategy, a $\mathcal{G}$-strategy with $o = w$ or an $\mathcal{S}$-strategy with $o = \infty_X$ or $o = d_0$ then δ does not restrain any elements at stages s_1 and s_2.

2. Suppose δ is a $\mathcal{G}$-strategy with outcome $o = f$. Then the value of δ's parameters τ and μ are the same at stages s_1 and s_2, as they can change only after initialization. Therefore the elements that δ restrains at both stages s_1 and s_2 are the same as well, namely the elements $a > |\tau|$ such that $\tau\hat{}\mu(a) = 0$.

3. Suppose δ is an $\mathcal{S}$-strategy with outcome $o = \langle \infty_Y, k \rangle$. Then the elements that δ restrains out of A at stages s_1 and s_2 are the ones in its agitators $E_k[s_1]$ and $E_k[s_2]$ respectively. If we assume that $E_k[s_1] \neq E_k[s_2]$ then at some stage t such that $s_1 < t \leq s_2$, the strategy δ would have had outcome $o' = \infty_X$. Indeed δ can only choose a new value for its agitator E_k at a stage t' if it had outcome ∞_X

at the previous true stage t. But $\infty_X <_L \langle \infty_Y, k\rangle$ and δ' would be initialized at stage t contrary to our assumption.

4. Suppose δ is an $\mathcal{S}$-strategy with outcome $o = \langle l, k\rangle$. Then the elements that δ restrains at stages s_1 and s_2 are the ones in $E_k[s_1] \cup F_l^k[s_1]$ and $E_k[s_2] \cup F_l^k[s_2]$ respectively. If we assume that $E_k[s_1] \neq E_k[s_2]$ or $F_l^k[s_1] \neq F_l^k[s_2]$ then at some stage t such that $s_1 < t \leq s_2$ the strategy δ would have had an outcome $o' = \infty_X$ or $o' = \langle \infty_Y, k\rangle$ to the left of o and δ' would again be initialized at stage t contrary to our assumption.

□

Propostion 4.6.2. *If s is a δ-true stage and $a \in E_\delta[s]$ then δ cannot restrain a (in or out of A) at stage s.*

Proof. Let $s_0 \leq s$ be the least δ-true stage such that δ is not initialized at stages t with $s_0 < t \leq s$. According to Lemma 4.6.1, $E_\delta[s_0] = E_\delta[s]$ and therefore $a \in E_\delta[s_0]$. Only $\mathcal{G}$- and $\mathcal{S}$-strategies restrain elements in or out of A. We treat the two cases separately:

1. Let δ be a $\mathcal{G}$-strategy. The value of δ's parameter τ is chosen at stage s_0 and remains the same at all stages t such that $s_0 < t \leq s$. Then $a < |\tau|$ and $\tau(a) = 0$, hence δ does not restrain a at stage s.

2. Let δ be an $\mathcal{S}$-strategy. Then δ restrains only elements in its agitators. Any agitator that δ chooses at stages $t \geq s_0$ does not intersect $E_\delta[t] = E_\delta[s_0]$. Therefore δ does not restrain a.

□

We next consider the properties of the parameters F_δ for strategies δ. We will show that for every strategy δ the elements that it restrains in A will remain in A unless the strategy is initialized. First we prove that strategies of lower priority than δ obey these restrictions. Then we prove that higher priority strategies always initialize δ, should they decide to injure its restrictions. These two properties clarify which elements will end up in the constructed set A.

Lemma 4.6.2. *If s is a δ-true stage and $a \in F_\delta[s]$ then δ does not restrain a out of A at stage s.*

Proof. Assume that a is restrained in A by $\delta_1 < \delta$ at stage $s_1 \leq s$. Note that $a \in F_\delta[s]$ until δ_1 is initialized or is visited and stops restraining a in A. Hence δ_1 is not initialized at stages t such that $s_1 < t \leq s$. Let $s_2 \geq s_1$ be the first δ-true stage after the imposition of the restraint at stage s_2 by δ_1. We will prove that at stage s_2 the strategy δ is in initial state. We have the following cases:

1. $\delta_1 <_L \delta$. Then δ is initialized at stage s_1.

2. $\delta_1 \subset \delta$.

    a. δ_1 is a $\mathcal{G}$-strategy. Then at stage s_1 the strategy δ_1 picks a new value for one of its parameters τ or μ after its last initialization at a stage $t < s_1$. At stage t the strategy δ was also initialized. If δ_1 chooses τ at stage s_1 then at stage s_1 the strategy δ_1 is in initial state, i.e. s_1 is the first δ_1-true stage after stage t. As any δ-true stage is a δ_1-true stage, it follows that s_2 is the first δ-true stage after δ's initialization at stage t. If on the other hand δ_1 chooses a new value for the parameter μ at stage s_1 then it has outcome f at this stage for the first time after its initialization at stage t. The strategy δ_1 will have outcome f at all stages until its next initialization, in particular at stage s. We can conclude that $\delta \supseteq \delta_1\hat{\ }f$ and thus is not accessible at stages t' with $t \leq t' < s_1$.

    b. δ_1 is an $\mathcal{S}$-strategy. Then at stage s_1 it has outcome d_0. This is the only case when an $\mathcal{S}$-strategy restrains elements in A. Furthermore δ_1 had outcome

$\langle \infty_Y, k\rangle$ at its previous visit at stage s_1^- and has outcome d_0 at each visit after s_1 while it is restraining the element in A. In particular it has outcome d_0 at stage s. Hence $\delta \supseteq \delta_1\hat{\ }d_0$ and δ was initialized at stage s_1^-, when δ_1 had outcome $\langle \infty_Y, k\rangle$.

So, if $\gamma \supseteq \delta$ is a $\mathcal{G}$-strategy then for any τ_γ that γ chooses at stages after stage s_1 we have $a < |\tau_\gamma|$ and γ does not restrain a out of A.

If δ is an $\mathcal{S}$-strategy and we assume that δ restrains a out of A then a is included in some agitator Ag. By Proposition 4.6.1 any element that enters the agitator has been restrained out of A by some $\mathcal{G}$-strategy $\gamma \supset \delta$ after $\delta' s$ last initialization. But we just established in the paragraph above that no such γ restrains a out of A. Hence $a \notin Ag$. □

Lemma 4.6.3. *Suppose that at stage s we visit δ_1. Suppose that δ_1 restrains out of A an element a that is currently restrained in A by a lower priority strategy $\delta_2 \supset \delta_1$. Then δ_2 is initialized at stage s.*

Proof. The proof is by induction on the distance $d(\delta_1, \delta_2) = |\delta_2| - |\delta_1|$. Assume that the statement is true for all pairs of strategies with distance $d < n$.

Let $d(\delta_1, \delta_2) = n$. Suppose that δ_2 restrains an element a in A at stage $s_0 < s$. This element remains restrained until stage s, thus both strategies δ_1 and δ_2 are not initialized at stages t with $s_0 < t < s_1$. By proposition 4.6.2 the element a is not restrained out of A by δ_1 at stage s_0. So at stage s the elements that δ_1 restrains out of A are different from the ones it restrained at stage s_0.

If δ_1 is a $\mathcal{G}$-strategy, this could only happen if it had outcome w at stage s_0 and outcome f at stage s, restraining new elements included in the definition of its parameter μ. As s_0 is a δ_2-true stage it follows that $\delta_2 \supseteq \delta_1\hat{\ }w$ and is initialized at stage s.

If δ_1 is an $\mathcal{S}$-strategy then a is included in some agitator Ag. This agitator is chosen at a stage $t \leq s$. It is extracted from A at stage s, but was not extracted from A at stage s_0.

It follows from the construction that once δ_1 chooses its agitator Ag, the agitator is extracted from A at all stages at which this agitator is valid unless δ_1 has outcome d_0. Thus if $t \leq s_0$ then the only possibility is $\delta_2 \supseteq \delta_1\hat{\ }d_0$. Then at stage s, δ_1 has outcome $\langle l, k\rangle$ and initializes δ_2.

The other possibility is that $t > s_0$. If the agitator $Ag = E_k$ for some k, then at stage $t^- \geq s_0$ the strategy δ_1 has outcome ∞_X. Then δ_2 would be initialized at stage t^- unless $\delta_2 \subseteq \delta_1\hat{\ }\infty_X$.

By the construction the element a was restrained out of A by some $\sigma \supset \delta_1$ at stage $t^- \geq s_0$. By Lemma 4.6.2 we have that $\sigma < \delta_2$ and by the assumption that δ_2 is not initialized at stages $t' > s_0$ it follows that $\sigma \not<_L \delta_2$. Thus $\sigma \subset \delta_2$. Furthermore by Proposition 4.6.2 the stage t^- cannot be equal to s_0 as otherwise δ_2 would not be able to restrain a at stage s_0. Now by applying the induction hypothesis for the strategies σ and δ_2 we obtain again a contradiction with our assumptions, namely that δ_2 is initialized at stage $t^- < s$.

The case when $Ag = F_l^k$ is treated similarly. □

4.6.2 Lemmas about the agitators

Using the established properties of the restrictions we can now obtain a simpler definition of the agitators. Suppose β is an $\mathcal{S}$-strategy that chooses an agitator for the element m at stage s. We have two similar cases depending on β's previous outcome o^- . If $o^- = \infty_X$ then $Ag = \mathbb{R}_{\infty_X}(m)[s] \setminus E_\beta[s]$. By Lemma 4.6.1 we have that $E_\beta[s] = E_\beta[s^-]$. By Proposition 4.6.2 these elements cannot be restrained by strategies extending β, thus $\mathbb{R}_{\infty_X}(m)[s] \cap E_\beta[s] = \emptyset$. Thus the agitator has a simpler definition $Ag = \mathbb{R}_{\infty_X}(m)[s]$. If $o^- = \langle \infty_Y, k\rangle$ then we have a similar situation. In addition $\mathbb{R}_{\langle \infty_Y, k\rangle}(m)[s] \cap E_k = \emptyset$ and we get $Ag = \mathbb{R}_{\langle \infty_Y, k\rangle}(m)[s]$.

Furthermore suppose $\beta' \supset \beta$ is an $\mathcal{S}$-strategy visited at stage s^- with outcome

$\langle l', k' \rangle$. Let $E_{\beta'-\beta} = E_{\beta'}[s^-] \setminus E_\beta[s^-]$, the elements that are restrained out of A by strategies below β, but above β'. If β' is not initialized at stage s then $E_{\beta'-\beta} \cup E_{k'} \cup F^{k'}_{l'} \subset Ag$.

Similarly if $\beta' \supset \beta$ is an $\mathcal{S}$-strategy and at stage s^- it was visited and had outcome $\langle \infty_y, k' \rangle$. If β' is not initialized at stage s then $E_{\beta'-\beta} \cup E_{k'} \subset Ag$.

In this section we shall prove two more properties of the agitators. The first lemma establishes that the agitators have the control property discussed in Section 4.2. The second lemma guarantees that the choice of agitators does indeed prevent the undesirable situation described in Section 4.4.2.

Lemma 4.6.4. *Let β be an $\mathcal{S}_{i,j}$-strategy visited at stage t_0. Denote by X the set Θ_i^A and by Y the set Ψ_i^A.*

1. *Suppose β chooses an agitator E_k for k at stage t_0. If the node $\beta\hat{}\infty_X$ is not initialized or visited at any stage $t > t_0$ and $E_k \subseteq A$ then $k \in X$.*
2. *Suppose β chooses an agitator F_l^k for l at stage t_0. If the node $\beta\hat{}\langle \infty_Y, k \rangle$ is not initialized or visited at any stage $t > t_0$ and $F_l^k \subseteq A$ then $l \in Y$.*

Proof. We will concentrate on the first part of the lemma; the second part is proved similarly. To prove that $k \in X = \Theta_i^A$ we need to find an axiom $\langle k, E' \rangle \in \Theta_i$ with $E' \subset A$.

Consider the axiom $\langle k, E' \rangle$ for k listed in $\mathbb{U}[t_0]$. We will prove that it has this property. Assume for a contradiction that an element $a \in E'$ is extracted from A at infinitely many stages. This axiom was applied not later than at stage t_0. Furthermore it was valid when it entered $\mathbb{U}$ hence $E' \cap E_\beta[t_0] = \emptyset$ according to Lemma 4.6.1.

The strategy β chooses an agitator for k at stage t_0. We initialize all strategies δ such that $\beta <_L \delta$. Furthermore $o[t_0^-] = \infty_X$, hence at stage t_0^- we have initialized all strategies δ' such that $\beta\hat{}\infty_X <_L \delta'$. These are not visited again before stage t_0.

Therefore all nodes δ such that $\beta\hat{}\infty_X <_L \delta$ do not restrain a out of A at any stage $t \geq t_0$. The only strategy that can extract a out of A at stage t_0 is β. This follows from Proposition 4.6.1, the fact that $\mathcal{G}$-strategies restrain out of A only elements larger than the length of their current parameter τ and that if a $\mathcal{G}$-strategy is in initial state at stage $t \geq t_0$ it will choose a new value for its parameter τ of length greater than a.

As $\beta\hat{}\infty_X$ is not visited at stages $t > t_0$ and $E_k \subset A$ by the same Proposition 4.6.1, β does not extract the element a at infinitely many stages. Elements that enter F_l^k for any l must be first extracted by a $\mathcal{G}$-strategy extending $\beta\hat{}\langle\infty_Y, k\rangle$.

We only need to further consider strategies $\delta \subset \beta$. Let $t_1 > t_0$ be the first stage at which $a \notin A[t_1]$ and δ be the strategy that extracts it. We treat $\mathcal{G}$- and $\mathcal{S}$-strategies separately.

If δ is a $\mathcal{G}$-strategy that extracts a at stage $t_1 > t_0$ then it has outcome f at stage t_1. As β is not initialized at stage t_1, $\delta\hat{}f \subseteq \beta$ and $\delta\hat{}f$ is not initialized at stages t such that $t_0 < t \leq t_1$. Thus by Lemma 4.6.1 we have $a \in E_\beta[t_0]$ contradicting $a \in E'$.

If δ is an $\mathcal{S}$-strategy then a is included in some agitator Ag which is taken out of A at stage t_1. As in the proof of Lemma 4.6.3 if Ag is chosen before or at stage s_0 then $\beta \supseteq \delta\hat{}d_0$ and is initialized at stage t_1. Thus the agitator is chosen at stage $t > t_0$, and as usual a was extracted from A at the previous δ-true stage t^- by one of the strategies extending δ. Our choice of t_1 as the first stage after t_0 at which a is extracted from A guarantees that $t^- = t_0$. But we know that the only strategy that can extract a at stage t_0 is β, hence $a \in E_k \subset A$. □

Lemma 4.6.5. *Let $\beta\hat{}\langle l, k\rangle$ be visited at stage t_0. If β is not initialized or visited at stages $t > t_0$ and $(E_k \cup F_l^k) \not\subset A$ then $(E_k \cup F_l^k \cup E_\beta[t_0]) \cap A = \emptyset$.*

Proof. Let $(E_k \cup F_l^k) \not\subset A$. First we will prove that $(E_k \cup F_l^k) \cap A = \emptyset$. Let $a \in E_k \cup F_l^k$. Then a is restrained out of A by some $\mathcal{G}$-strategy $\gamma \supset \beta$ at some stage $t' < t_0$ after $\beta's$ last initialization as we established in Proposition 4.6.1. As β is not initialized or visited anymore, no other $\mathcal{G}$-strategy can restrain the element a out of A. Indeed $\mathcal{G}$-strategies of higher priority than β would initialize β if they restrained a new element. The ones to the right of β are initialized at stage t' and choose their parameter τ to be of length greater than a. So if $a \notin A[t]$ then a is restrained out of A by some $\mathcal{S}$-strategy $\delta \subset \beta$. We can even say that $\delta\hat{}\infty_X \subseteq \beta$, if a is included in some agitator $E_{k'}$, and $\delta\hat{}\langle\infty_Y, k'\rangle \subseteq \beta$, if a is included in some agitator $F_{l'}^{k'}$, again using the result from Proposition 4.6.1. Moreover the agitator is chosen at stage $t_1 > t_0$, as after the strategy δ chooses its agitator it has outcomes to the right of β until the agitator is cancelled.

Suppose a is extracted from A at stage $t > t_0$ by $\beta_1 \subset \beta$. Then a is included in the agitator Ag_1 of β_1, chosen at stage $t_1 > t_0$. So $a \notin A[t_1^-]$ and $t_1^- \geq t_0$. If $t_1^- = t_0$ then $E_k \cup F_l^k \subseteq Ag_1$. If $t_1^- > t_0$ then there is another strategy β_2 such that $\beta_1 \subset \beta_2 \subset \beta$ and a is included in one of its agitators Ag_2. With a similar argument we get a monotone decreasing sequence of stages $t_1 > t_2 > \dots$ bounded by t_0, hence finite.

Therefore always when $a \notin A[t]$, we have a finite sequence of $\mathcal{S}$-strategies:

$$\beta_1 \subset \beta_2 \subset \cdots \subset \beta$$

and a corresponding monotone sequence of their agitators:

$$Ag_1 \supset Ag_2 \supset \cdots \supset (E_k \cup F_l^k)$$

such that Ag_1 is restrained out of A at stage t. If $a \notin A[t]$ and $t > t_0$ then $(E_k \cup F_l^k) \cap A[t] = \emptyset$ and ultimately $(E_k \cup F_l^k) \cap A = \emptyset$.

Let us assume now that $b \in E_\beta[t_0] \cap A \neq \emptyset$. Then there is a stage t_b such that $b \in A[t]$ for all $t > t_b$. Let t' be a stage at which $(E_k \cup F_l^k) \cap A[t] = \emptyset$ and $t' > t_b$. By the argument above there is a series of $\mathcal{S}$ strategies $\beta_1 \subset \beta_2 \subset \cdots \subset \beta_n \subset \beta$ and a corresponding series of their agitators $Ag_1 \supset Ag_2 \supseteq \cdots \supseteq (E_k \cup F_l^k)$ and Ag_1 is restrained out of A at stage t'. These agitators are chosen at stages $t_1 > t_2 \cdots > t_n > t_0$ respectively. We can express $E_\beta[t_0]$ in the following way:

$$E_\beta[t_0] = E_{\beta_1}[t_0] \cup E_{\beta_2 - \beta_1}[t_0] \cup \cdots \cup E_{\beta - \beta_n}[t_0].$$

Now using Lemma 4.6.1 we can modify this expression to:

$$E_\beta[t_0] = E_{\beta_1}[t_1] \cup E_{\beta_2-\beta_1}[t_2] \cup \cdots \cup E_{\beta-\beta_n}[t_0].$$

If $b \in E_{\beta_2-\beta_1}[t_2] \cup \cdots \cup E_{\beta-\beta_n}[t_0]$ then $b \in Ag_1$. If $b \in E_{\beta_1}[t_1]$ then by Lemma 4.6.1 we have $b \in E_{\beta_1}[t']$. Thus in both cases $b \notin A[t']$ contradicting the choice of $t' > t_b$. Therefore $E_\beta[t_0] \cap A = \emptyset$. □

4.6.3 The true path

This section proves the existence of the true path. It will be defined as usual as the leftmost path of nodes visited at infinitely many stages. As in Chapter 3, the tree of strategies is infinitely branching and we must start with a formal proof that the so defined true path is of infinite length. A much harder task will be to prove that the strategies along this path are initialized only finitely often. The difficulty comes from the introduced local priority and frequent initialization performed by $\mathcal{S}$-strategies during the application of axioms. We will prove separately that $\mathcal{S}$-strategies along the true path initialize every lower priority strategy along the true path only finitely often. Using these two results we can finally establish the existence of a true path with all required properties as defined in Section 1.3.

Lemma 4.6.6. *There exists an infinite path h in the tree of strategies T with the following properties:*

1. $(\forall n)(\exists^\infty s)[\ h \upharpoonright n \subseteq \delta[s]\ ]$.
2. $(\forall n)(\exists s_l(n))(\forall s > s_l(n))[\delta[s] \not<_L h \upharpoonright n]$.

Proof. We will define h inductively and simultaneously prove that it has the desired properties. First $h \upharpoonright 0 = \emptyset$ obviously has both properties. It is visited at every stage and $s_l(0) = 0$. Now let's assume we have defined $h \upharpoonright n$ with the desired properties. We will define $h \upharpoonright (n+1) = (h \upharpoonright n)\hat{\ }o$, where o is the true outcome of the strategy $h \upharpoonright n$.

If $h \upharpoonright n$ is an $\mathcal{R}$-strategy then $o = e$. We always visit $h \upharpoonright (n+1)$ when we visit $h \upharpoonright n$, hence infinitely often and $s_l(n+1) = s_l(n)$.

If $h \restriction n$ is a $\mathcal{G}$-strategy then let $o \in \{ f, w\}$ be the leftmost outcome visited at infinitely many $h \restriction n$-true stages. If $o = f$ then $s_l(n+1) = s_l(n)$. Otherwise there is a stage s_1 such that $(\forall t > s_1)[h \restriction n \subseteq \delta[t] \Rightarrow (h \restriction n)\hat{\ }w \subseteq \delta[t]]$. Then $s_l(n+1) = \max(s_l(n), s_1)$.

If $h \restriction n$ is an $\mathcal{S}$-strategy then we have several cases to consider:

1. If the outcome ∞_X is visited at infinitely many stages then $o = \infty_X$ and $s_l(n+1) = s_l(n)$. Otherwise there is a least $h \restriction n$-true stage s_1 such that $(h \restriction n)\hat{\ }\infty_X \not\subseteq \delta[t]$ at all $h \restriction n$-true stages $t \geq s_1$. At stage s_1 the strategy $h \restriction n$ chooses an agitator E_k for some fixed number k and has outcome $\langle \infty_Y, k\rangle$. At all stages greater than s_1 the possible outcomes for $h \restriction n$ are $\langle \infty_Y, k\rangle$, $\{ \langle l, k\rangle \mid l \in \mathbb{N}\}$ and d_0.

2. If the outcome $\langle \infty_Y, k\rangle$ is visited at infinitely many stages then $o = \langle \infty_Y, k\rangle$ and $s_l(n+1) = \max(s_l(n), s_1)$. Otherwise there is a least $h \restriction n$-true stage $s_2 > s_1$ such that $(h \restriction n)\hat{\ }\langle \infty_Y, k\rangle \not\subseteq \delta[t]$ at all $h \restriction n$-true stages $t \geq s_2$. At stage s_2 the strategy $h \restriction n$ chooses a second agitator F_l^k for some fixed l and has outcome d_0. At all stages $t > s_2$ the possible outcomes are d_0 and $\langle l, k\rangle$.

3. If at some stage $s_3 > s_2$ the strategy has outcome $\langle l, k\rangle$ then at all stages $t \geq s_3$ the strategy has this outcome, as in order to return from outcome $\langle l, k\rangle$ back to d_0, the strategy needs to have outcomes $\langle \infty_Y, k\rangle$ or ∞_X at an intermediate stage. Thus in this case $o = \langle l, k\rangle$ and $s_l(n+1) = \max(s_l(n), s_3)$.
   Otherwise at all stages $t > s_2$ the strategy has outcome d_0 and $o = d_0$, $s_l(n+1) = \max(s_l(n), s_2)$.

□

Lemma 4.6.7. *For every $\mathcal{S}$-strategy β working with the operators Θ and Ψ and with parameters U and V_k, for $k < \omega$ the following statement is true:*

1. *If $\beta\hat{\ }\infty_X \subseteq h$ then for every $k \in U$ there exists an axiom $\langle k, E'\rangle \in \Theta$ and a stage*

s_k such that if $t > s_k$ and β is visited at stage t with $o^- = \infty_X$ then $\langle k, E' \rangle$ is applicable for k. Furthermore $E' \subseteq A$.

2. If $\beta\hat{\ }\langle \infty_Y, k \rangle \subseteq h$ then for every $l \in V_k$ there exists an axiom $\langle l, F' \rangle \in \Psi$ and a stage s_l such that if $t > s_l$ and β is visited at t with $o^- = \langle \infty_Y, k \rangle$ then $\langle l, F' \rangle$ is applicable for l. Furthermore $F' \subseteq A$.

Proof. Assume that this is not the case and choose $\beta \subseteq h$ as the least strategy for which the proposition is false. Suppose $\beta\hat{\ }\infty_X \subseteq h$. The case $\beta\hat{\ }\langle \infty_Y, k \rangle \subseteq h$ is similar. Let $k \in U$ be the least number for which there is no applicable axiom at infinitely many β-true stages.

Let $\mathbb{G} = \{ \gamma \supseteq \beta\hat{\ }\infty_X | \ \gamma$ is a $\mathcal{G}$-strategy with local priority less than k $\}$.

We choose a stage s so big that:

a. $\mathcal{S}$-strategies $\beta' \subset \beta$ do not initialize any strategy $\gamma \in \mathbb{G}$ at stages $t \geq s$. Our choice of β as the least strategy for which the proposition is not true guarantees that this choice of t is satisfiable.

b. For all elements $m \in U$ such that $m \leq k$ we have $m \in U[s]$.

c. For each element $m < k$, $m \in U$ there is an axiom in $\Theta[s]$ that is applicable at every stage $t \geq s$ which is furthermore valid at all stages $t \geq s$.

d. Let $M = \max\{ \ |\gamma| \ | \ \gamma \in \mathbb{G}\} + 2$ and let $s > s_l(M)$, where $s_l(M)$ is defined in Lemma 4.6.6.

Our choice of s, precisely conditions a, b and d, guarantees that for all $t > s$, β does not get initialized at stage t. Then Lemma 4.6.1 gives us that $E_\beta[t]$ is the same at all β-true stages $t > s$. We can therefore omit the index t in further discussions and refer to this set as E_β.

Let $s_1 > s$ be a stage at which $h \upharpoonright M$ is visited. At the next β-true stage s_1^+ the outcome $o^- = o[s_1]$ is ∞_X. We scan the elements of U and change their corresponding entries in $\mathbb{U}$ if needed. The elements $m < k$ will not require any actions but it is still possible that k does.

If there is an applicable axiom for k in $\Theta[s_1^+]$ then it will be recorded in $\mathbb{U}[s_1^+]$, applied and will have the following properties:

1. $E' \cap E_\beta = \emptyset$,
2. $E' \cap \mathbb{R}_{\infty_X}(k)[s_1^+] = \emptyset$.

If there is no applicable axiom then we define an agitator $E_k = \mathbb{R}_{\infty_X}(k)[s_1^+]$ and move to the right of the true path. Let s_2 be the next stage at which $\beta\hat{}\infty_X$ is visited. At this stage we must have found an axiom $\langle k, E''\rangle$, applied it and enumerated in $\mathbb{U}$ for which again:

1. $E'' \cap E_\beta = \emptyset$,
2. $E'' \cap \mathbb{R}_{\infty_X}(k)[s_1^+] = \emptyset$.

In both cases we have an axiom $\langle k, E^0\rangle$ for which the two conditions hold. Let $s_3 > s_1$ be a $h \upharpoonright M$-true stage by which this axiom is applied, i.e. $s_3 \leq s_1^+$ in the first case and $s_3 \leq s_2$ in the second case. We will prove that no strategy extracts elements from E^0 at stages $t > s_3$. Hence this axiom will be the one we are searching for.

First note that at stages after the axiom is applied, once a strategy is initialized, it will not restrain elements from E^0 out of A at any further stage. The actions that we make when applying the axiom include the initialization of strategies extending $h \upharpoonright M$. Strategies to the right of $h \upharpoonright M$ are initialized at stage s_1. In the first case the axiom $\langle k, E^0\rangle$ is applied at stage s_1^+. Strategies to the left of β are initialized again and strategies extending β are not accessible before this event. In the second case the axiom is applied at stage s_2. Strategies to the left $\beta\hat{}\infty_X$ are initialized again at stage s_2 and strategies extending $\beta\hat{}\infty_X$ are not accessible after their initialization at stage s_1 until the event of the application of the axiom. Strategies to the left of $h \upharpoonright M$ are not accessible after stage $s > s_l(M)$, thus will also not extract any elements from A at stages $t > s$.

The only danger is that a strategy δ along $h \upharpoonright M$ restrains an element $a \in E^0$ out of A at a stage $t > s_3$. We will prove that this also does not happen.

First of all if δ is a $\mathcal{G}$-strategy then by stage s_1, which is an $h \upharpoonright M$-true stage, its outcome is final and so are all elements that it restrains out of A. These elements are in E_β if $\delta \subset \beta$ or in $\mathbb{R}_{\infty_X}(k)[s_1^+]$ if $\delta \supset \beta$.

If δ is an $\mathcal{S}$-strategy the elements it restrains out of A are the ones in its agitators. At stages s_1 and s_3 the strategy has its true outcome o. If this is $o = d_0$ then δ does not restrain any elements at any stage $t \geq s_1$. If $o = \langle l, k \rangle$ for some l and k then again the elements that δ restrains out of A are the same at all further stages and are in E_β if $\delta \subset \beta$ or in $\mathbb{R}_{\infty_X}(k)[s_1^+]$ if $\delta \supset \beta$ hence do not contain elements from E^0.

The other two possibilities are $o = \infty_X$, in which case there is no defined agitator at stage s_3 and every agitator that δ chooses is eventually cancelled, or $o = \langle \infty_Y, k \rangle$ for some fixed k. In the latter case δ has a permanent agitator E_k already extracted at stage s_1, it has no defined agitator F_l^k at stage s_3 and can only choose values for the agitators F_l^k, for elements $l < \omega$, all of which are eventually cancelled.

We will prove that agitators formed after stage s_3 cannot contain elements from E^0. It is convenient to consider each $\mathcal{S}$-strategy $\delta \subset h \upharpoonright M$ in order of its length, starting from the longest. The reason is that strategies of lower priority determine the elements that enter agitators of higher priority strategies.

Let δ be the longest $\mathcal{S}$-strategy along $h \upharpoonright M$. Suppose δ chooses an agitator Ag at stage $t' > s_3$. All of Ag's elements were restrained by strategies extending δ at the previous δ-true stage $t^- \leq s_3$. These are either strategies that were initialized when the axiom $\langle k, E^0 \rangle$ was applied and hence cannot restrain elements from E^0, or $\mathcal{G}$-strategies $\gamma \subset h \upharpoonright M$ which as we already proved do not restrain elements from E^0.

By induction we can prove the same for the shorter $\mathcal{S}$-strategies. □

Corollary 4.6.1. *For every n there is an $h \upharpoonright n$-true stage $s_i(n)$ such that $h \upharpoonright n$ does not get initialized after stage $s_i(n)$.*

Proof. The proof is by induction on n. The case $n = 0$ is trivial because $h \upharpoonright 0 = \emptyset$ is never initialized and is visited at every stage, so $s_i(0) = 0$.

Assume that the statement is true for $h \upharpoonright n$. If $h \upharpoonright (n+1)$ is not a $\mathcal{G}$-strategy then $h \upharpoonright n$ is a $\mathcal{G}$-strategy and it does not initialize any strategies that extend it. So we can simply define $s_i(n+1)$ to be the least $h \upharpoonright (n+1)$-true stage after $\max(s_i(n), s_l(n+1))$, where $s_l(n+1)$ is defined in Lemma4.6.6.

If $h \upharpoonright (n+1)$ is a $\mathcal{G}$-strategy then we choose $s_i(n+1)$ to be the least $h \upharpoonright (n+1)$-true stage with the following properties:

1. $s_i(n+1) \geq s_i(n)$.

2. $s_i(n+1) > s_l(n+1)$.

3. Let β be an $\mathcal{S}$-strategy with $\beta\hat{}\infty_X \subseteq h \upharpoonright (n+1)$ and $k \in U_\beta$ be an number less than the local β-priority of $h \upharpoonright (n+1)$. By Lemma 4.6.7 there is a stage s_k and an axiom $\langle k, E^0 \rangle \in \Theta_\beta$ such that $E^0 \subseteq A$ and the axiom is applicable at every stage $t > s_k$. This axiom has limit age $a(A, E^0)$. There are only finitely many axioms in Θ_β whose age is of lesser value at any stage $t \geq s$, as any axiom enumerated after the age of this axiom has reached this limit value will have greater age. We choose $s_i(n+1)$ to be a stage by which all these finitely many axioms have been applied if they ever get applied.

4. Similarly if β is an $\mathcal{S}$-strategy with $\beta\hat{}\langle\infty_Y, k\rangle \subseteq h \upharpoonright (n+1)$ and $l \in V_k^\beta$ is a number less than the local β-priority of $h \upharpoonright (n+1)$, using the result from Lemma 4.6.7 we can choose $s_i(n+1)$ to be a stage by which no more axioms for l in Ψ_β ever get applied.

□

4.6.4 Satisfaction of the requirements

Lemma 4.6.8. *Every $\mathcal{R}$-requirement is satisfied.*

Proof. Fix an $\mathcal{R}_i$-requirement. Let α be the corresponding $\mathcal{R}$-strategy on the true path. We will prove that $\Theta_i^A = X$ and $\Psi_i^A = Y$ do not form a minimal pair. The proof is divided into three cases depending on the true outcomes of the $\mathcal{S}$-substrategies of α along the true path:

1. For all $\mathcal{S}$ strategies $\beta \subset h$, substrategies of α,

$$(\exists k)(\exists l)[\beta\hat{\ }\langle l, k\rangle \subset h] \vee \beta\hat{\ }d_0 \subset h.$$

First we will prove that $\Gamma^X = \Lambda^Y$. Now the properties of the agitators proved in Section 4.6.2 will play an important role as the operators Γ and Λ are constructed by *all* of α's substrategies, not only the ones along the true path. So we have to prove that $\Gamma^X(d_\beta) = \Lambda^Y(d_\beta)$, for every witness d_β that any substrategy β has ever used.

We automatically have this equality for any witness d_β that is cancelled. Cancelling the witness includes enumerating the axiom $\langle d_\beta, \emptyset\rangle$ in both operators. So $\Gamma^X(d_\beta) = \Lambda^Y(d_\beta) = 1$.

This means that substrategies to the right of the true path will not cause problems. Substrategies to the left of and on the true path may have witnesses that are never cancelled. So let β be a substrategy of α and d be a witness chosen at stage s_0 that is never cancelled. Then β has outcome d_0 at stage s_0. After stage s_0 the strategy β is not initialized and does not have outcomes ∞_X or $\langle \infty_Y, k\rangle$, as in those cases we would cancel β's witness d. Let the corresponding agitators for d be E_k and F_l^k, so we have axioms $\langle d, \{k\}\rangle \in \Gamma$ and $\langle d, \{l\}\rangle \in \Lambda$. We have the following three possibilities:

   (a) $\beta <_L h$. Then let $s \geq s_0$ be the last stage at which β is visited. If $\beta\hat{\ }\langle l, k\rangle \subseteq \delta[s]$ then the conditions of Lemma 4.6.5 are true. Therefore if $(E_k \cup F_l^k) \not\subseteq A$

then $(E_k \cup F_l^k \cup E_\beta[t]) \cap A = \emptyset$. If $(E_k \cup F_l^k) \subseteq A$ then according to Lemma 4.6.4 we have $k \in X$ and $l \in Y$ and therefore $\Gamma^X(d) = \Lambda^Y(d) = 1$.

If $(E_k \cup F_l^k \cup E_\beta[t]) \cap A = \emptyset$ then from the proof of Lemma 4.6.5 it follows that $E_k \cup F_l^k$ are included in an agitator of a higher priority strategy β' visited at stage $s_1 > s$. By construction that there is an entry $\langle \beta : \langle E[s], E_k, F_l^k \rangle, d \rangle \in Watched_\alpha$. In this case we claim $\Gamma^X(d) = \Lambda^Y(d) = 0$. Suppose for a contradiction that this is not true, say $\Gamma^X(d) = 1$. Then the only axiom in Γ for d is true, so $k \in X = \Theta_i^A$. Therefore there is an axiom $\langle k, E' \rangle \in \Theta_i$ such that $E' \subseteq A$ and hence $E' \cap (E_k \cup E_\beta[s]) = \emptyset$. It appears in $\Theta_i[t]$ at some stage t. The strategy $\alpha \subset h$ will be visited after stage t, as it is visited infinitely often. It shall then spot this axiom while examining the entry for β in $Watched$ and cancel d. Similarly we may prove that $\Lambda^Y(d) = 0$.

If $\beta\hat{\ } d_0 \subseteq \delta[s]$, as β is not initialized at stages $t > s$, we have that $E_k \cup F_l^k$ is restrained in A by β. From Lemmas 4.6.2 and 4.6.3 it follows that $E_k \cup F_l^k$ is a subset of A. Lemma 4.6.4 gives us $k \in X$ and $l \in Y$. Hence $\Gamma^X(d) = \Lambda^Y(d) = 1$.

(b) Suppose $\beta\hat{\ } d_0 \subseteq h$. Then $s_0 = s_i(|\beta|)$. Here as well by Lemmas 4.6.2 and 4.6.3 we have $E_k \cup F_l^k \subset A$. Lemma 4.6.4 gives us $k \in X$ and $l \in Y$. Hence $\Gamma^X(d) = \Lambda^Y(d) = 1$.

(c) If $\beta\hat{\ }\langle l, k \rangle \subseteq h$ then by Lemma 4.6.6 there is a stage $s_1 > s_0$ such that at β-true stages $t > s_1$ the strategy β always has this outcome and $E_k \cup F_l^k$ is extracted from $A[t]$. Also by Lemma 4.6.1 $E_\beta[t] = E_\beta[s_1]$ for all β-true stages $t > s_1$ and we will refer to this set as E_β. As β is visited at infinitely many stages $(E_k \cup F_l^k \cup E_\beta) \cap A = \emptyset$. In this case $\Gamma^X(d) = \Lambda^Y(d) = 0$. If we assume otherwise then there would be an axiom in Θ_i for k or in Ψ_i for l, which does not intersect $E_k \cup F_l^k \cup E_\beta$ and by the actions in the construction β would spot this axiom at one of its true stages and cancel the witness d.

This gives us a set $D = \Gamma^X = \Lambda^Y$. Now it follows quite easily from the construction that D is not c.e. Let W_j be any c.e. set and consider the $\mathcal{S}_{i,j}$-substrategy β along the true path. Let $n = |\beta|$. After stage $s_l(n+1)$ from Lemma 4.6.6 β always has its true outcome whenever it is visited and a permanent witness d which is never cancelled. If $\beta\hat{\ }\langle l, k\rangle \subset h$ then $W_j(d) = 1$ and as we just proved $D(d) = 0$. If $\beta\hat{\ }d_0 \subset h$ then $W_j(d) = 0$ and again as we just saw $D(d) = 1$.

2. There is a strategy β, substrategy of α, with $\beta\hat{\ }\infty_X \subseteq h$. Let $n = |\beta|$. Let $U_\beta = \bigcup_{t \geq s_i(n+1)} U_\beta[t]$. We will prove that $U_\beta = X$ and so X is c.e. Assume for a contradiction that this is not true.

   If there is an element $k \in X \setminus U_\beta$ then choose the least one. As $k \in X = \Theta_i^A$, there is an axiom $\langle k, E'\rangle \in \Theta_i$ such that $E' \subset A$. By Lemma 4.6.1 we have that $E_\beta[s] = E_\beta[s_i(n+1)]$ at all β-true stages $t > s_i(n+1)$. It follows that $E_\beta[s] \subset \overline{A}$ for every stage s and hence $E' \cap E_\beta[s] = \emptyset$. Let $s > s_i(n+1)$ be a stage at which the axiom $\langle k, E'\rangle$ is already enumerated in Θ_i, the set E' has reached its limit age $a(A, E')$ and all numbers less than k that ever enter U_β are already in U_β. Then k will enter U_β at the next β-true stage at which $o^- = \infty_X$, if not before.

   By Lemma 4.6.7 for every $k \in U_\beta$ there is an axiom $\langle k, E'\rangle \in \Theta_i$ for which $E' \subseteq A$, therefore $k \in X$ and $U_\beta \subseteq X$. Ultimately we get $X = U_\beta$.

3. There is an $\mathcal{S}$-strategy β which is a substrategy of α with $\beta\hat{\ }\langle \infty_Y, k\rangle \subset h$ for some k. Let $V_{k,\beta} = \bigcup_{t \geq s_i(n+1)} V_{k,\beta}[t]$. We show in this case that $V_{k,\beta} = Y$ and therefore Y is c.e. The proof is similar to part 2.

□

Lemma 4.6.9. *Every $\mathcal{G}$ requirement is satisfied.*

Proof. Fix a c.e. set W_e and consider the $\mathcal{G}_e$-strategy $\gamma \subset h$. Let $n = |\gamma|$. Let τ and μ denote the values of γ's parameters at stage $s_i(n+1)$ defined in Corollary

4.6.1. It follows from the construction that these values remain the same at further stages. Indeed τ changes value only after initialization and μ changes value only when γ switches to outcome f. By construction we have that $\tau\hat{}\mu \in W_e$ if the true outcome of γ is f. If the true outcome is w then for every extension $\rho \supseteq \tau\hat{}\mu$ we have $\rho \notin W_e$. Thus we only need to further prove that $\tau\hat{}\mu \subset A$.

By Lemma 4.6.1 the value of the set $E_\gamma[t]$ does not change at γ-true stages $t > s_i(n)$ and we will refer to it as E_γ. Finally γ always has its true outcome at true stages $t > s_i(n+1)$.

If $\tau\hat{}\mu(a) = 1$ then a is restrained in A by γ and by Lemmas 4.6.2 and 4.6.3 $a \in A$. If $\tau\hat{}\mu(a) = 0$ and $a < |\tau|$ then $a \in E_\gamma \subset \overline{A}$ so $A(a) = 0$. If $\tau\hat{}\mu(a) = 0$ and $a \geq |\tau|$ then a is extracted at every γ-true stage $t \geq s_i(n+1)$ and $A(a) = 0$. Therefore $\tau\hat{}\mu \subset A$.

This concludes the proof of the lemma and the theorem. $\square$

Chapter 5

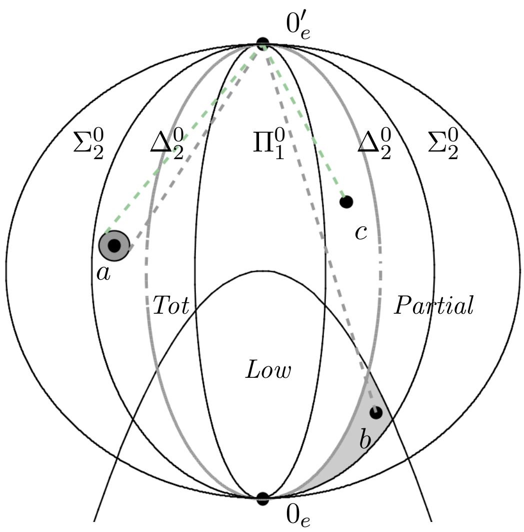
Cupping and Non-cupping in the Δ_2^0 Enumeration Degrees

In the previous two chapters we investigated the structure of the properly Σ_2^0 enumeration degrees. Both splitting/non-splitting and bounding/non-bounding turned out to be properties that distinguish the Δ_2^0 enumeration degrees from the properly Σ_2^0 enumeration degrees. One more property of this sort is cupping/non-cupping, see Definition 1.5.1. Cooper, Sorbi and Yi [CSY96] prove that every nonzero Δ_2^0 enumeration degree can be cupped by a total incomplete Δ_2^0 enumeration degree and that there exists a nonzero Σ_2^0 enumeration degree that cannot be cupped by any incomplete Σ_2^0 enumeration degree.

In this chapter we shall complement the first result by proving that every nonzero Δ_2^0 enumeration degree can be cupped by a partial Δ_2^0 enumeration degree. We shall once again use genericity to establish this property, see Definition 4.0.1. In Section 1.1 we already mentioned the existence of partial Δ_2^0 enumeration degrees, obtained as quasi-minimal degrees, see Definition 1.1.3. Copestake [Cop88] proves that there is a strong connection between these two notions as every 1-generic enumeration degree is quasi-minimal. Furthermore Copestake [Cop90] proves that a 1-generic enumeration degree is low, see Definition 1.1.5, if and only if it is Δ_2^0. Thus by constructing a Δ_2^0 1-generic *cupping partner* for every nonzero Δ_2^0 enumeration degree we obtain the following result.

Theorem 5.0.1. *Every nonzero Δ_2^0 enumeration degree can be cupped by a partial low Δ_2^0 enumeration degree.*

This result, together with the original result by Cooper, Sorbi and Yui, shows that we have some flexibility when searching for cupping partners for Δ_2^0 enumeration degrees. On the other hand it shows that we can limit our search for a cupping partner to a small subclass of the Δ_2^0 enumeration degrees.



It would be natural to ask whether or not we can narrow this class even further, perhaps there is a finite set that contains a cupping partner for every nonzero Δ_2^0 enumeration degree. Lewis [Lew04] proves that this is not true for the Δ_2^0 Turing degrees in $\mathcal{D}_T(\leq 0')$. Our second result in this chapter shows that the Δ_2^0 enumeration degrees are not any different in this respect. In fact we will prove that one cannot even *computably enumerate* a sequence of Δ_2^0 enumeration degrees, which contains within its members a cupping partner for every nonzero Δ_2^0 enumeration degree. The notion of a Δ_2^0-computably enumerable sequence of enumeration degrees is defined in 1.2.2.

Theorem 5.0.2. *Let* $\{a_i\}_{i<\omega}$ *be a* Δ_2^0*-computably enumerable sequence of enumeration degrees. There exists a nonzero* Δ_2^0 *enumeration degree* b *such that for every* $i < \omega$ *if* a_i *is incomplete then* $a_i \vee b \neq 0_e'$.

Theorem 5.0.1 is joint work with Guohua Wu, published in [SW07], see Appendix A.2. Both theorems are proved by modifying the constructions suggested in [CSY96].

5.1 Cupping by a partial enumeration degree

In this section we shall give a proof of Theorem 5.0.1. Let A be a nonzero Δ_2^0 set with Δ_2^0 approximating sequence $\{A[s]\}_{s<\omega}$. We shall construct a Δ_2^0 1-generic set B whose enumeration degree cups the degree of A. As usual we start by formalizing the requirements:

1. We have a global requirement which guarantees that the degree of B cups the degree of A. We shall construct an enumeration operator Γ so that:
$$\mathcal{S} : \Gamma^{A,B} = \overline{K}.$$
Here $\overline{K}$ denotes as usual any Π_1^0 representative of the degree $0_e'$.

2. The set B must be 1-generic. Let $\{W_i\}_{i<\omega}$ be a computable enumeration of all c.e. sets. For every $i < \omega$ we have a requirement:
$$\mathcal{G}_i : (\exists \tau \subset B)[\tau \in W_i \vee (\forall \mu \supseteq \tau)[\mu \notin W_i]],$$
where τ and μ denote finite binary strings.

The degree $0_e'$ is total, hence constructing B as a 1-generic set ensures that the degree of B is not complete.

5.1.1 Basic strategies

To satisfy the global requirement $\mathcal{S}$ we will construct an enumeration operator Γ, following the basic ideas for the design of the $(\mathcal{P}, \Gamma)$-strategy from Chapter 2 and

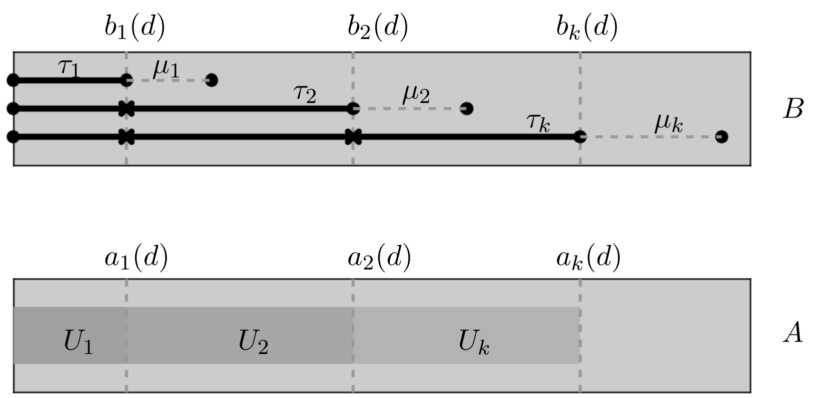
Chapter 3. In this case though we will have a much simpler strategy. At every stage s we shall examine all elements $n < s$, ensuring that $\Gamma^{A,B}[s](n) = \overline{K}[s](n)$. For elements $n \in \overline{K}[s]$ we shall have a current axiom of the form $\langle n, A[s] \upharpoonright a(n)+1, B[s] \upharpoonright b(n)+1 \rangle$, where $a(n)$ and $b(n)$ are markers defined by us. If $n \notin \overline{K}$ then we can rectify Γ by extracting the marker $b(n)$ from B.

The basic strategy for satisfying a $\mathcal{G}_i$ requirement is the same as the one used in Chapter 4. We select a witness τ following some basic rules imposed by higher priority strategies that ensure $\tau \subset B$. At every stage we check if there is an extension $\mu \supseteq \tau$ so that $\mu \in W_i$. If there is such an extension, we select the least one μ and make sure that it is an initial segment of the characteristic function of B by extracting or enumerating elements $n < |\mu| + 1$, where $|\mu|$ denotes the length of the binary string μ. Then we restrain $B \upharpoonright |\mu| + 1$.

When looking at all strategies collectively we notice a conflict between the global $\mathcal{S}$-strategy and each $\mathcal{G}$-strategy, similar to the conflict we observed in Chapter 2 between a $\mathcal{P}$-strategy and an $\mathcal{N}$-strategy below it, see Section 2.1.1. The need for Γ-rectification might injure the restraint imposed by a $\mathcal{G}$-strategy on the set B. To avoid this we shall modify the $\mathcal{G}$-strategy. It will again select a threshold d, whose B-marker will determine the length of the witness τ. Before the imposition of the restraint the $\mathcal{G}$-strategy shall try to ensure that all B-markers for elements $n \geq d$ are above this restraint. An extraction from the set A below the A-marker of the threshold $a(d)$ will facilitate this. To force such an extraction the $\mathcal{G}$-strategy shall construct a c.e. set U, approximating the given set A and threatening to prove that A is c.e. The $\mathcal{G}$-strategy shall then run in cycles, preforming many attempts to satisfy its requirement. Each new cycle k shall have a new witness τ_k and shall search for an extension $\mu_k \supseteq \tau_k$ in the set W_i. If this extension is found the strategy shall end the k-th cycle by approximating a larger initial segment of the set A up to $a_k(d)$, where $a_k(d)$ is the current marker of the threshold during the k-th cycle. Then it shall perform capricious destruction on the operator Γ

by extracting the B-marker for the threshold which is current during the k-th cycle, $b_k(d)$, thereby moving the action of the $\mathcal{S}$-strategy and the next cycle of this strategy to elements larger than the required restraint on the set B, $|\mu_k|$. Thus the only number that will conflict the restraint on B for this cycle will be the marker $b_k(d)$.

The following pictures is meant to show the progress of a $\mathcal{G}$-strategy during its work on the k-th cycle.



As the set A is not c.e. the approximation of A shall be unsuccessful and we shall eventually be able to locate a permanent extraction from the set A, an extraction useful to the last cycle k. Using this extraction we can restore the set B by enumerating the marker $b_k(d)$ back in the set B, making μ_k an initial segment of the characteristic function of B and preserve the restraint $|\mu_k|$ on B at further stages.

5.1.2 Construction

The construction will be carried out in stages. We will not use a tree of strategies, instead we order the requirements linearly:

$$\mathcal{S} < \mathcal{G}_0 < \mathcal{G}_1 \ldots$$

and assign a strategy to every requirement. At the beginning of each stage we shall activate the $\mathcal{S}$-strategy. Then we shall activate the least $\mathcal{G}$-strategy that *requires atten-*

tion, defined below. We will still use small Greek letters α, β to denote $\mathcal{G}$-strategies. At every stage only one $\mathcal{G}$-strategy shall be activated and all $\mathcal{G}$-strategies of lower priority will be initialized.

We shall construct a Δ^0_2 approximation to the set B. Initially it will be the empty set: $B[0] = \emptyset$. The approximation to the set B at stage s shall be obtained from $B[s-1]$ by allowing the two active strategies at stage s to enumerate or extract numbers from it.

The $\mathcal{S}$-strategy

The global $\mathcal{S}$-strategy shall have a parameter Γ, the enumeration operator that it will be constructing. To every element n the strategy shall assign current A- and B-markers, $a(n)$ and $b(n)$, and a current axiom of the form $\langle n, A \upharpoonright a(n)+1, B \upharpoonright b(n)+1\rangle$. Initially $\Gamma = \emptyset$ and all markers and axioms are undefined. At stage s the $\mathcal{S}$-strategy operates as follows:

For very element $n < s$ perform the following actions:

- If $n \notin \overline{K}[s]$ then find all valid axioms in Γ for n, $\langle n, A_n, B_n\rangle$, and extract the greatest element of B_n from $B[s]$.
- If $n \in \overline{K}[s]$ and the current axiom for n is valid then skip to the next element. If the current axiom for n is not defined or is not valid then:

  1. If $a(n)[s] \uparrow$, define $a(n)[s] = a(n-1)[s] + 1$. (if n= 0, define $a(n) = 1$).
  2. If $b(n)[s] \downarrow$ then extract it from B and cancel all markers $b(n')[s]$ for $n' > n$.
  3. Define $b(n)[s]$ as a fresh number greater than any number mentioned in the construction so far. Enumerate $b(n)[s]$ in $B[s]$.
  4. Define the current axiom for n at stage s to be
     $\langle n, A[s] \upharpoonright a(n)+1, B[s] \upharpoonright b(n)+1\rangle$ and enumerate it in $\Gamma[s]$.

Activating the $\mathcal{G}$-strategy α.

Denote by W_α the c.e. set that α is working with. The strategy α is equipped with a threshold d and a current witness τ, initially undefined. The strategy has furthermore a parameter, which we shall call the *current guess*, denoted by G and it shall have the following structure: $\langle U, \mu, b, t\rangle$, where U is α's current approximation to the set A, μ is the binary string that α would like to make an initial segment of the set B, b is a marker whose enumeration in the set B will facilitate this and finally t is the stage at which this guess was made. This parameter has initial value $\langle \emptyset, \emptyset, \uparrow, \uparrow\rangle$.

The strategy α at stage s has threshold d, witness τ and guess $G = \langle U, \mu, b, t\rangle$ all possibly undefined. We list the cases in which it requires attention and the actions it makes. Every time we choose the first case which applies at stage s.

1. The threshold d is not defined.

   Action: Define the threshold $d \in \overline{K}[s]$ as a fresh number.

2. The threshold d is defined but $d \notin \overline{K}[s]$.

   Action: Shift the value of the threshold to the next element in $\overline{K}[s]$. Cancel the current witness τ. If the marker b of the current guess $G[s]$ is defined then extract it from B and cancel the current guess.

3. A B-marker of an element $n < d$ has been extracted from B at stage s.

   Action: Cancel the current witness τ. If the marker b of the current guess $G[s]$ is defined then extract it from B and cancel the current guess.

4. $U \subseteq A[s]$ and $b \downarrow \in B[s]$.

   Action: Extract b from the set $B[s]$. For every element n such that $b < n < |\mu|+1$ set $B[s](n) = \mu(n)$. Cancel the current B-marker for every $n \geq d$ and extract $b(d)$ from $B[s]$.

5. $U \nsubseteq A[s]$ and $b \downarrow \notin B[s]$.

   Action: Enumerate b in the set $B[s]$.

6. The witness τ is not defined or is not an initial segment of $B[s]$.

   Action: If the current marker $b(d)$ is defined then set $\tau = \chi_B[s] \upharpoonright b(d) + 1$.

7. $b \uparrow$ or $b \downarrow \notin B[s]$, and $U \subseteq A[s]$ and there is an extension $\mu \supseteq \tau$ such that $\mu \in W_\alpha[s]$.

   Action: Define a new value for the current guess G to be $\langle A_d, \mu, b(d), s\rangle$, where $\langle d, A_d, B_d\rangle$ is the current axiom for d in $\Gamma[s]$. Extract $b(d)$ from $B[s]$. For every element n such that $b(d) < n < |\mu| + 1$ set $B(n)[s] = \mu(n)$. Cancel all A- and B-markers for elements $n \geq d$ and cancel the witness τ. Define a fresh value of the marker $a(d)$.

5.1.3 Proof

We will prove that this is a finite injury construction, i.e. that every $\mathcal{G}$-strategy eventually stops requiring attention and satisfies its requirement. Before we can do this we will prove two properties of the construction. The first one concerns the axioms used in the construction of the operator Γ.

Propostion 5.1.1. *1. At every stage s if $n < m$, $n, m \in \overline{K}[s]$ and the current axioms for n and m at stage s are $\langle n, A_n, B_n\rangle$ and $\langle m, A_m, B_m\rangle$ then $A_n \subseteq A_m$ and $B_n \subset B_m$.*

2. If α is a $\mathcal{G}$-strategy not initialized at stage s then there is at most one valid axiom in $\Gamma[s]$ for its threshold d different from the current one. This axiom is associated with α's current guess $G[s]$.

Proof. 1. This fact follows directly from the construction of Γ. At every stage if $n < m$ then $a(n) < a(m)$ and $b(n) < b(m)$ as whenever we cancel the value for one of n's markers we also cancel the value for m's corresponding marker. By induction suppose the statement is true for stages $t < s$. If the current axiom for n is not valid at stage s then it is redefined to $A[s] \upharpoonright a(n)+1$ and the B-marker for m is cancelled. Thus at the same stage a new current axiom is defined for m, with the required properties. If the current axiom for n is valid but the one for m is not, then it will be redefined at stage s and will have again the required properties. Finally if both current axioms are valid at stage s then they were defined at a previous stage $t < s$ for which the induction hypothesis is true.

2. As d is α's threshold at stage s, $d \in \overline{K}[s]$. If α is in initial state at stage s then there is no axiom for d in $\Gamma[s]$.

Otherwise any axiom for d that the $\mathcal{S}$-strategy has cancelled at a previous stage is invalid. The $\mathcal{S}$-strategy always extracts the current B-marker before it enumerates a new axiom in Γ. This marker can never be reenumerated in the set B.

Any axiom for d that was used for a previous guess $G[t]$ at a stage $t < s$ is not valid. Whenever α changes the value of the guess it executes the actions under step 7 and the marker recorded in its previous value is not in the approximation to B. If α cancels G at stage t during steps 2 or 3 then the marker recorded in the guess is extracted from B, so the axiom associated with the old value of this guess remains invalid forever. Whenever α cancels the current B-marker of the threshold at step 4 it extracts it from B invalidating the axiom associated with it.

Thus the only axioms for d that can be valid at stage s are the current one and the one used in the current guess $G[s]$. □

Our next concern is to establish that the current guess of a $\mathcal{G}$-strategy approximates a c.e. set.

Propostion 5.1.2. *Let α be a $\mathcal{G}$-strategy. Suppose the value of α's current guess G is not cancelled at stages $t > s$. Denote by $U[t]$ the value of the first component of $G[t]$. Then $\{U[t]\}_{t>s}$ is a c.e. approximation to the set $\mathbb{U} = \bigcup_{t>s} U[t]$.*

Proof. The strategy α is not initialized at stages $t > s$ as otherwise its guess would be cancelled. Whenever it requires attention it receives it. It is enough to prove that $U[t] \subseteq U[t+1]$ for all $t > s$. If $U[t] \neq U[t+1]$ then α receives attention at stage $t+1$ and executes step 7. The current guess at the beginning of stage $t+1$ is $G = \langle U[t], \mu, b, t_0 \rangle$ and $U[t] \subseteq A[t]$, $b \notin B[t]$. The current axiom at stage $t+1$ for the threshold d is defined after stage t_0 as at t_0 the strategy α executes step 7 and cancels the current markers of its threshold. The A-marker of the threshold d is has a fresh value and thus is greater then $\max U[t]$. Suppose that the current axiom for d at stage $t+1$ is defined at stage t_1, $t_0 < t_1 \leq t+1$. It is enough to prove that $U[t] \subseteq A[t_1]$.

Suppose for a contradiction that $U[t] \nsubseteq A[t_1]$. If $b \in B[t_1]$ then it will remain in $B[t_1]$ at stage t_1. If $b \notin B[t_1]$ at the beginning of stage t_1 then α requires attention under step 5 at stage t_1 and enumerates the marker b in $B[t_1]$. Thus at the end of stage t_1 we will have $b \in B[t_1]$. As $b \notin B[t+1]$, there must be a stage t_2 such that $t_1 < t_2 < t+1$ at which α extracts b again from the set $B[t_2]$. This can only happen if step 4 is executed, but then α cancels the current marker for the threshold d, contradicting the fact the axiom discussed is current at stage $t+1$. □

Lemma 5.1.1. *1. Every strategy requires attention only at finitely many stages.*
2. Every $\mathcal{G}$-requirement is satisfied.

Proof. We prove both parts of this lemma with simultaneous induction: Suppose that the statement is true for strategies of higher priority than α and let s_1 be the least stage such that at stages $t \geq s_1$ strategies of higher priority than α do not require attention.

First let us note once and for all that every time α receives attention, all lower priority strategies are initialized, their thresholds are cancelled and later redefined as fresh numbers. As every time $B[t] \restriction b(d)[t]+1$ changes the strategy α requires attention under step 6, it follows that at stages $t > s_0$ only $\mathcal{S}$ and α can modify the approximation to $B[t] \restriction b(d)[t] + 1$.

At stages $t \geq s_1$ the strategy α is not initialized. Hence stage s_1 is the last stage at which α requires attention under step 1. It selects a threshold d. Every time it requires attention under step 2, the threshold d is shifted to the next available element in the approximation of $\overline{K}$. The set $\overline{K}$ is infinite hence this shifting process will eventually lead α to an element $d \in \overline{K}$. There is a least stage $s_2 \geq s_1$ after which α does not execute step 2. The A-markers for elements $n < d$ do not change after stage s_2. As the set A is Δ_2^0, the approximation to $A \restriction \max_{n<d} a(n) + 1$ will stabilize and hence there is a least stage $s_3 \geq s_2$ such that $A[t] \restriction \max_{n<d} a(n) + 1 = A \restriction \max_{n<d} a(n) + 1$ and the $\mathcal{S}$-strategy will not extract from the set B any markers of elements $n < d$ at stages $t > s_3$. So s_3 is the last stage at which α executes step 3. After stage s_3 the value of α's guess will not be cancelled. By Proposition 5.1.2 the set $\mathbb{U} = \bigcup_{t>s_3} U[t]$ is a c.e. set. As A is not, $A \neq \mathbb{U}$.

Suppose that there is an element $n \in \mathbb{U} \setminus A$, and let s be the least stage such that $n \in U[t]$ and $n \notin A[t]$ for all $t \geq s$. Then at stages $t \geq s$ we always have $U[t] \nsubseteq A[t]$ and steps 4 and 7 will never again be executed. Thus after stage s the current markers of the threshold will not be modified by α. Let $s_7 > s_3$ be the last stage at which step 7 is executed. Then at stage s_7 the final value of the guess $G = \langle U, \mu, b, s_7 \rangle$ is recorded and $\mu \in W_\alpha[s_7]$. Let $\widehat{\mu}$ be the string μ with position b set to 0. At all stages $t \geq s_7$ either $\widehat{\mu}$ is an initial segment of $B[t]$ or μ is an initial segment of $B[t]$. As $s_7 > s_3$ and $d \in \overline{K}$ the $\mathcal{S}$-strategy does not modify $B \restriction b+1$. If it modifies B below $|\mu|+1$ at stage $t > s_7$ then it extracts an old marker of an element $n > d$ for a valid axiom in $\Gamma[t]$. By Proposition 5.1.1 there is an axiom for the threshold d which is also valid at stage t and this is the one associated with the guess G. Thus $U \subseteq A[t]$, $b \in B[t]$ and α executes step 4, restoring $\widehat{\mu}$ as an initial segment of $B[t]$. As this does not happen after stage s, the $\mathcal{S}$-strategy does not modify B below $|\mu|+1$ at stages $t \geq s$. At stage s the strategy α enumerates the marker b back in the set $B[s]$ executing step 5 for the last time and ensuring that $\mu \subseteq \chi_B[s]$. Thus after stage s the strategy α can only require attention under step 6. Furthermore $B[t] \upharpoonright |\mu|+1$ is not modified at all $t > s$ and thus the requirement associated with α is satisfied by $\mu \subseteq \chi_B$ and $\mu \in W_\alpha$. As the current markers of the threshold d are not modified by α after stage s, there will be a least stage $s_6 > s$ at which the $\mathcal{S}$-strategy will modify the current axiom for d and the approximation to $B \upharpoonright b(d)+1$ for the last time, after $A \upharpoonright a(d)+1$ has stabilized. Then at stage s_6 the strategy α will execute step 6 for the last time and will never require attention at further stages.

Suppose that $\mathbb{U} \subseteq A$. Then let n be a number such that $n \in A \setminus \mathbb{U}$. If we assume that α requires attention under step 7 at infinitely many stages then the marker $a(d)$ will grow unboundedly. Then let s' be a stage such that $n \in A[t]$ and $a(d)[t] > n$ at all $t > s'$. The number n would be enumerated in every current axiom for d and will eventually enter the set $\mathbb{U}$. Let $s_7 > s_3$ be the least stage after which α does not execute step 7. The final value of the guess is $G = \langle U, \mu, b, s_7\rangle$ or $G = \langle \emptyset, \emptyset, \uparrow, \uparrow\rangle$ if the strategy does not execute the actions at step 7 after stage s_3. Let $s > s_7$ be the least stage such that $U \subseteq A[t]$ at all stages $t \geq s$. Then α can execute step 4 for the last time at stage s. After this $b \notin B[t]$ or $b\uparrow$ and α can only require attention under step 6. The current markers of the threshold d are not changed after stage s by α. As in the previous case there is a least stage $s_6 > s$ at which α executes step 6 for the last time and at stages $t \geq s_6$ the witness has a permanent value such that $\tau \subseteq \chi_B$. As α does not execute step 7 after stage $s_6 > s_7$ there is no extension of τ in the set $W_\alpha[t]$ at all $t > s_6$. Thus the requirement is satisfied by $\tau \subseteq \chi_B$ and $(\forall \mu \supseteq \tau)[\mu \notin W_\alpha]$. The strategy does not require attention at stages $t > s_6$. □

Corollary 5.1.1. *The $\mathcal{S}$-requirement is satisfied.*

Proof. Let m be any element. Let α be a $\mathcal{G}$-strategy with permanent threshold $d > m$. By Lemma 5.1.1 every $\mathcal{G}$-strategy has a permanent threshold and by construction every $\mathcal{G}$-strategy chooses a threshold of value greater than any chosen before, thus this choice of α is satisfiable. There is a stage s such that α does not require attention at stages

$t \geq s$. At stage s the $\mathcal{S}$-strategy ensures that $\Gamma^{A,B}[t](m) = \overline{K}(m)$. If at a later stage $t > s$ this equality is injured then $\mathcal{S}$ would extract a marker $b(m)$ from the set B. If $m \in \overline{K}[t]$ then the current marker for m would be extracted, otherwise a marker of the valid axiom for m in Γ at stage t would be extracted. In both cases α would require attention under step 3 contradicting our choice of s. Thus the equality is preserved at all stages $t > s$. If $m \notin \overline{K}$ then there is no valid axiom for m in Γ at any stage $t > s$. If $m \in \overline{K}$ then the current axiom for m at stage s is valid at all stages $t > s$. □

Corollary 5.1.2. *The set B is Δ_2^0.*

Proof. We use the same trick as in the proof of the previous lemma. Let m be any element and let α be a $\mathcal{G}$-strategy along the true path with permanent threshold $d > m$. If m is extracted from $B[t]$ then m is a marker of an element $n < d$ and is extracted by the $\mathcal{S}$-strategy or by a strategy of higher priority than α. Then α requires attention at stage t under step 3 in the first case or is initialized in the second case. By Lemma 5.1.1 there is a stage s after which α is not initialized and does not require attention and hence m is extracted at finitely many stages from B. □

5.2 No computably enumerable cupping sequence

In this section we prove Theorem 5.0.2. Given a Δ_2^0-computable sequence of enumeration degrees $\{a_i\}_{i<\omega}$, we shall construct a Δ_2^0 set B whose enumeration degree is nonzero and is not cupped by any incomplete member of the sequence. As usual we start by formalizing the requirements.

For every i let A_i be a representative of the given Δ_2^0 enumeration degree a_i. Let $\{A_i[s]\}_{s<\omega}$ be a good Δ_2^0 approximation to A_i, obtained from the given one using the method described in Section 1.4.1. The requirements that the constructed set B needs to satisfy are:

1. The set B is not c.e. Let $\{W_e\}_{e<\omega}$ be a computable enumeration of all c.e. sets. For every natural number e we have a requirement:
$$\mathcal{N}_e : W_e \neq B.$$

2. The degree of the set B is not cupped by any incomplete member of the sequence. Let $\{\Theta_j\}_{j<\omega}$ be a computable enumeration of all enumeration operators. For every i and every j we will have a requirement:
$$\mathcal{P}_{i,j} : \Theta_j^{A_i,B} = \overline{K} \Rightarrow (\exists \Gamma_{i,j})[\Gamma_{i,j}^{A_i} = \overline{K}].$$

5.2.1 Basic strategies

We shall describe the basic strategies with the context of the tree in mind.

A $\mathcal{P}$-strategy

Consider a $\mathcal{P}$-strategy α working on the requirement $\mathcal{P}_{i,j}$. We shall denote Θ_j by Θ_α, A_i by A_α and $\Gamma_{i,j}$ by Γ_α. The basic goal of α is to construct the operator Γ_α so that $\Gamma_\alpha^{A_\alpha} = \overline{K}$. It shall have two outcomes $i <_L w$.

The strategy will perform cycles k of increasing length, examining each element $n < k$ on each cycle. The cycles do not necessarily correspond to the stages at which α is active. In fact α can take any number of stages to complete one of its cycles. When examining a particular element n, the strategy α shall try to rectify the operator Γ_α at this element n, using information from the current approximation of the set $\Theta_\alpha^{A_\alpha,B}$. The strategy will act differently depending on whether or not the element is in the current approximation of the set $\overline{K}$. If $n \in \overline{K}$ then the strategy will try to find an axiom to enumerate in Γ_α which is valid at almost all stages s. Candidates for such an axiom come from the axioms currently enumerated in Θ_α. The strategy α shall select the axiom $\langle n, A_n, B_n\rangle$ that has been valid the longest (i.e. of least age) including at all stages since the strategy last examined the element n during the previous cycle. If there is such an axiom then α will record it as its current guess in a special parameter $Ax_\alpha(n)$

and enumerate a corresponding axiom $\langle n, A_n \rangle$ in the operator Γ_α. Then during the same stage it will move on to examine the next element in the cycle. If the current guess recorded in $Ax_\alpha(n)$ has not been valid at some stage since the last time n was examined or if there is no appropriate axiom valid long enough in the current approximation of the operator Θ_α then the strategy shall indicate that it has been unsuccessful to rectify Γ_α at n via the outcome i, ending its actions for this particular stage. When α is active again it will move on to examine the next element in the cycle. Thus if the outcome i is visited infinitely often in relation to a particular element n, this yields that Θ_α is unable to supply α with an axiom for n that is valid at all but finitely many stages and hence $n \notin \Theta_\alpha^{A_\alpha, B}$.

If $n \notin \overline{K}$, then to rectify Γ_α the strategy should ensure that all previously enumerated axioms for n in Γ_α are invalid. It is enough to ensure that there are infinitely many stages s at which $n \notin \Gamma_\alpha^{A_\alpha}$. Thus the strategy first searches for such a stage since the last time that n was examined. If such a stage is found, α assumes that the operator will be rectified eventually and moves on to the next element, without any further actions related to n. If α is not able to spot a stage at which $n \notin \Gamma_\alpha^{A_\alpha}$, then it shall enumerate the element n back in the set $\Theta_\alpha^{A_\alpha, B}$ by enumerating back in B the B-part, B_n, of each axiom that is used as an axiom for n in Γ_α and we shall say that α is restraining these elements in B. The strategy will indicate that it has been unsuccessful to rectify Γ_α at n via the outcome w. It shall next concentrate its attention on this element at further stages not moving on to the next element in the cycle, until it observes a stage at which the operator is rectified. Thus if α has outcome w at all but finitely many stages then the strategy is never able rectify Γ_α at some element n. From the properties of a good approximation we can deduce that in this case $n \in \Theta^{A_\alpha, B} \setminus \overline{K}$.

To sum up we have three possibilities for the outcomes of a $\mathcal{P}$-strategy α:

1. The strategy α has outcome w at all but finitely many stages. Then α performs finitely many cycles, reaching an element $n \in \Theta_\alpha^{A_\alpha, B} \setminus \overline{K}$.

2. The strategy α has outcome i related to a particular element n at infinitely many stages. Then α performs infinitely many cycles and for this element n we have that $n \in \overline{K} \setminus \Theta_\alpha^{A_\alpha,B}$.

3. The strategy α has outcome i at infinitely many stages, but for every element n the outcome i is related to n only at finitely many stages. In this case the construction of Γ_α is successful, i.e. $\Gamma_\alpha^{A_\alpha} = \overline{K}$.

An $\mathcal{N}$-strategy

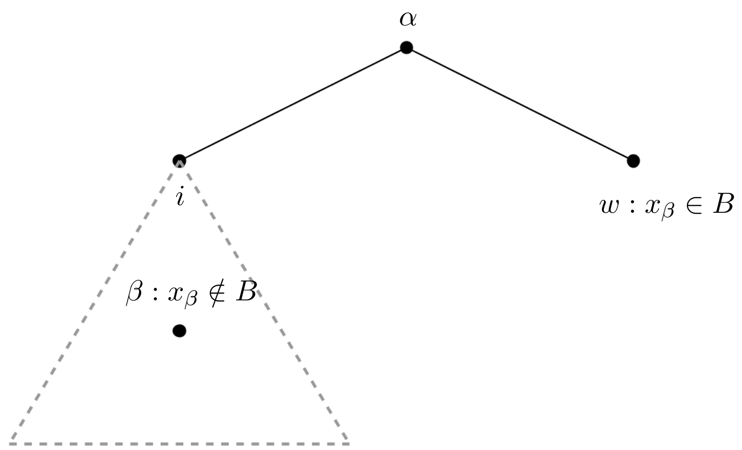
Let β be an $\mathcal{N}$-node working on the requirement $\mathcal{N}_e$. We shall denote W_e by W_β. This strategy attempts to prove that $B \neq W_\beta$. It uses a strategy even simpler than the basic Friedberg-Muĉnik strategy: First it selects a fresh witness x_β, one that has not appeared in the construction so far. While $x_\beta \notin W_\beta$ the strategy will keep x_β in B and indicate this via a rightmost outcome w. If x_β enters W_β then every time the strategy β is visited it will extract x_β from B and indicate this by a leftmost outcome d.

5.2.2 Interactions between strategies

The strategies are designed so that they do not interfere with each other. Every $\mathcal{N}$-strategy β is responsible for its unique witness x_β, which will never be extracted unless β decides to extract it. If it is extracted then β will extract it at every true stage. A $\mathcal{P}$-strategy is most of the time only an observer, it does not modify the approximation to B except in one case, when it restrains elements in B. In this case it has outcome w and we make sure that all strategies extending this outcome are in initial state. Thus lower priority strategies will not injure this restraint. Higher priority strategies initialize α if they injure this restraint.

The only risk we face is that the set B can turn out to be properly Σ_2^0 as an element n might be extracted and enumerated back in B infinitely often. Consider a $\mathcal{P}$-strategy α and an $\mathcal{N}$-strategy $\beta \supseteq \alpha\hat{\ }i$. The strategy β has a witness x_β which is extracted from

B. The strategy α has used in the definition of its operator Γ_α an axiom $\langle n, A_n, B_n\rangle$ such that $x_\beta \in B_n$. At stage s α examines the element n, which is not in $\overline{K}[s]$ and enumerates the marker x_β back in the set B. Before the strategy β is visited again, α moves on to a new element n', enumerates a new axiom for it using $\langle n', A_{n'}, B_{n'}\rangle$ and again $x_\beta \in B_{n'}$. Then β is visited and extracts x_β again. If this situation repeats infinitely often with elements $n', n'', \dots$, the number x_β will be extracted at infinitely many stages from B.



To avoid this risk we shall require that a $\mathcal{P}$-strategy α always restores the set B in its initial state after it has observed a rectifying stage. In this way after α is done with the element n it extracts the witness x_β, preempting β's actions, before it enumerates a new axiom for n'. Thus the new axiom used for n' will not contain the marker x_β in its B-part. This action makes the $\mathcal{P}$-strategies a bit more aggressive as now they will extract numbers from B as well. This turns out not to provoke further conflicts and is dealt with in detail in Section 5.2.5.

5.2.3 Parameters and the tree of strategies

A $\mathcal{P}$-strategy α will have a parameter Γ_α, the enumeration operator that it will construct when visited. At initialization Γ_α is set to the empty set. It will have also parameters k_α denoting the current cycle of the strategy and $n_\alpha \leq k_\alpha$ denoting the current element of the cycle that α is working with. At initialization the values of the parameters are set to $k_\alpha = 0$ and $n_\alpha = 0$. For every n it will have a parameter $Ax_\alpha(n)$ denoting the axiom in Θ_α for n which is currently assumed to be permanently valid. Finally it will keep track in a set Out_α of all witnesses that the strategy has currently re-enumerated back in B. At initialization α will give up any restraints.

An $\mathcal{N}$-strategy β shall have a parameter x_β, which will be undefined if β is initialized. At initialization β will give up any restraints.

The requirements will be ordered linearly as follows:

$$\mathcal{P}_{0,0} < \mathcal{N}_0 < \mathcal{P}_{0,1} < \mathcal{N}_1 \dots$$

The tree of strategies is defined as usual. Its domain is a subset of $\{w, d, i\}^{<\omega}$ and:

1. $T(\varnothing) = \mathcal{P}_{0,0}$.

2. Let α be a $\mathcal{P}$-node in the domain of T. Then $\alpha\hat{\ }o$, where $o \in \{i, w\}$, is also in the domain of T and $T(\alpha\hat{\ }o) = \mathcal{N}_{|\alpha|/2}$, the least $\mathcal{N}$-requirement in the priority listing which is not yet assigned to any node..

3. Let β be an $\mathcal{N}$-node in the domain of T. Then $\beta\hat{\ }o$, where $o \in \{d, w\}$, is in the domain of T and $T(\beta\hat{\ }o) = \mathcal{P}_{i,j}$, where $\mathcal{P}_{i,j}$ is the least $\mathcal{P}$-requirement in the priority listing which is not yet assigned to any node.

5.2.4 Construction

We shall perform the construction in stages. At each stage s we shall construct a string $\delta[s]$ of length s through the domain of T. The set B shall be approximated by a sequence of cofinite sets with $B[0] = \mathbb{N}$ and every set $B[s]$ obtained from $B[s-1]$ by allowing the active strategies at stage s to enumerate or extract numbers from it. A traditional Δ^0_2 approximation $\{\widehat{B}[s]\}_{s<\omega}$ to the set B can be obtained by setting $\widehat{B}[s] = B[s] \restriction s$.

At stage 0 all nodes are initialized. Suppose we have constructed $\delta[t]$ for $t < s$. We construct $\delta[s](n)$ with an inductive definition. We always start at the root of the tree: $\delta[s](0) = \emptyset$. Suppose that we have constructed $\delta[s] \upharpoonright n$. If $n = s$, we end this stage and move on to $s+1$, initializing all nodes $\sigma > \delta[s]$. Otherwise we visit the strategy $\delta[s] \upharpoonright n$ and let it determine its outcome o. We define $\delta[s](n+1) = o$. We have two cases depending on the type of strategy associated with $\delta[s] \upharpoonright n$:

(I.) $\delta[s] \upharpoonright n = \alpha$ is a $\mathcal{P}$-node:

Let s^- be the previous α-true stage, if α has not been initialized since, and $s^- = s$ otherwise. The strategy α will inherit the values of its parameters from stage s^- and during its actions it can change their values several times. Thus we will omit the subscript indicating the stage when we discuss α's parameters.

If the current element n_α does not need further actions we shall move on to the next element. As this is a subroutine which is frequently performed in the construction, we define it here once and for all, and we refer to it with the phrase **reset the parameters**. Denote the current values of n_α by n and of k_α by k. We *reset the parameters* by performing the following actions: Initialize the strategies extending $\alpha\hat{\ }w$. If Out_α is not empty then extract Out_α from B and set $Out_\alpha = \emptyset$. Remove any restraint imposed by α. If $n < k$ then set $n_\alpha := n+1$. Otherwise $n = k$ and we set $k_\alpha := k+1$, $n_\alpha := 0$ and end this sub-stage with outcome i.

1. Let $k = k_\alpha$ and $n = n_\alpha$. Let s_n^- be the previous stage when n was examined, if α has not been initialized since, $s_n^- = s$ otherwise.

2. If $n \in \overline{K}[s]$ and $n \in \Gamma_\alpha^{A_\alpha}[t]$ for all stages t with $s_n^- < t \leq s$ then *reset the parameters* and go to step 1.

3. If $n \in \overline{K}[s]$, but $n \notin \Gamma_\alpha^{A_\alpha}[t]$ at some stage t with $s_n^- < t \leq s$ then:

   a. If $Ax_\alpha(n) \uparrow$ then define it as the axiom that has been valid longest including at all stages $s_n^- < t \leq s$ and move on to step c. If there is no such axiom then let the outcome be i and *reset the parameters.*

b. If $Ax_\alpha(n)$ is defined but was not valid at some stage t with $s_n^- < t \leq s$, then cancel its value (make it undefined) and let the outcome be i, *reset the parameters.*

c. If $Ax_\alpha(n) = \langle n, A_n, B_n \rangle$ is defined and has been valid at all stages t with $s_n^- < t \leq s$ then enumerate in Γ_α the axiom $\langle n, A_n \rangle$. *Reset the parameters* and go back to step 1.

4. If $n \notin \overline{K}[s]$ and $n \notin \Gamma_\alpha^{A_\alpha}[t]$ at some stage t: $s_n^- < t \leq s$ *reset the parameters* and go back to step 1.

5. Suppose $n \notin \overline{K}[s]$ but $n \in \Gamma_\alpha^{A_\alpha}[t]$ at all t such that $s_n^- < t \leq s$. For each axiom $\langle n, A_n \rangle \in \Gamma_\alpha[s]$, consider the corresponding B-part B_n of the axiom $\langle n, A_n, B_n \rangle \in \Theta_\alpha$. If $B_n \not\subseteq B[s]$ then enumerate all elements from B_n that are not in $B[s]$ back in the set B. Out of these elements enumerate in the set Out_α the ones that are currently restrained out of B. Restrain the elements of B_n in B.

   Let the outcome be w. Note that we will not reset the parameters at this point, thus the construction will keep going through this step while there is no change in A_α. If later on there is a change in A_α then the strategy will move on to the next element in the cycle but only after it has restored the set B to its original state by extracting Out_α from B.

(II.) $\delta[s] \upharpoonright n = \beta$ is an $\mathcal{N}$-node:

Let s^- be the previous β-true stage if β has not been initialized since. The strategy β inherits the values of its parameters from stage s^- and goes to the step indicated at stage s^-. Otherwise $s^- = s$ and the strategies starts from step 1.

1. Define x_β as a fresh number - one that has not appeared in the construction so far. Go to the next step.

2. If $x_\beta \notin W_\beta[s]$ then let the outcome be w, return to this step at the next stage. Otherwise go to the next step.

3. If $x_\beta \in B[s]$, then extract x_β from $B[s]$ and restrain it out of B. Let the outcome be d, come back to this step at the next stage.

This completes the construction.

5.2.5 Proof

We define the true path h as usual to be the leftmost infinite path in the tree of strategies of nodes visited at infinitely many stages.

1. $(\forall n)(\exists^\infty s)[h \upharpoonright n \subseteq \delta[s]]$;

2. $(\forall n)(\exists s_l(n))(\forall s > s_l(n))[\delta[s] \not<_L h \upharpoonright n]$.

The true path exists as the tree is finitely branching. We shall prove that the strategies along the true path do not get initialized infinitely often.

Propostion 5.2.1. *For all n there exists a stage $s_i(n)$ such that $h \upharpoonright n$ does not get initialized at stages $t \geq s_i(n)$.*

Proof. We prove this proposition with induction on n. The case $n = 0$ is trivial as $h \upharpoonright 0$ does not get initialized at any stage $t > 0$, thus $s_i(0) = 1$.

Suppose that we have proved the statement for n. Then $h \upharpoonright (n+1)$ does not get initialized at any stage $t \geq \max(s_i(n), s_l(n+1))$ unless it is initialized by $h \upharpoonright n$. The only case when this is possible is when $h \upharpoonright n$ is a $\mathcal{P}$-strategy and $h \upharpoonright (n+1) = (h \upharpoonright n)\hat{\ }w$. It follows from the construction that $h \upharpoonright n$ performs only finitely many cycles, the actions on *resetting the parameters* ensure that every time the strategy starts a new cycle it has outcome i. Thus after a certain stage $s_i(n+1)$ the strategy $h \upharpoonright n$ will not reset its parameters and hence will not initialize strategies below outcome w. □

The next lemma shows that the only elements that are ever extracted from the set B are the witnesses that are extracted by an $\mathcal{N}$-strategy.

Propostion 5.2.2. *1. Let x_β be a witness of an $\mathcal{N}$-strategy β. If β does not extract x_β at any stage, then $x_\beta \in B[s]$ for all s.*

2. If x is not a witness to an $\mathcal{N}$-strategy, then $x \in B[s]$ for all s.

Proof. 1. It follows that the witness x_β will never be restrained out of B. From the choice of a witness in step $II.1$ of the construction it follows that x_β is not a witness to any other $\mathcal{N}$-strategy. On the other hand it cannot be extracted by a $\mathcal{P}$-strategy α as in order to be extracted by α it must first enter the set Out_α and elements in this set are necessarily restrained out of B.

2. Part two is proved by a similar argument as part (1). □

This is all we need to prove that the $\mathcal{N}$-requirements are satisfied.

Lemma 5.2.1. *The set B is not c.e.*

Proof. For every i there is a strategy β along the true path working with $\mathcal{N}_i$. This strategy is visited infinitely often and not initialized at any stage $t \geq s_i(|\beta|)$, where $s_i(|\beta|)$ is defined in Proposition 5.2.1. Let $x = x_\beta[s_i(|\beta|)]$ be $\beta's$ permanent witness at stages $t \geq s_i|\beta|$. If $\beta\hat{\ }w \subset h$ then x is never enumerated in W_i and Proposition 5.2.2 yields $x \in B$, hence $x \in B \setminus W_i$. If $\beta\hat{\ }d \subset h$ then there is a stage s_x such that $x \in W_i[t]$ at all $t \geq s_x$. At every β-true stage $t \geq s_x$ the strategy β ensures $x \notin B[t]$, hence $x \in W_i \setminus B$. □

We shall turn our attention to the $\mathcal{P}$-strategies. Before we can prove that they are successful we will show that the restraints that they impose on B are respected.

Lemma 5.2.2. *Let α be a $\mathcal{P}$-strategy. If α restrains an element n in B at stage s then $n \in B[t]$ at all stages $t > s$ until α removes the restraint.*

Proof. Suppose for a contradiction that a strategy γ extracts n from B at stage $s_1 \geq s$ strictly before α has removed the restraint. And let s_1 be the least such stage and γ be the least such strategy. We have to consider different cases depending on the type and priority of the strategy γ.

1. $\gamma <_L \alpha$. Then γ is visited at stage s_1 and hence α is initialized at stage s_1 and removes its restraints.

2. $\alpha <_L \gamma$. Then γ is initialized at stage s. If γ is an $\mathcal{N}$-strategy then γ chooses its witness after stage s hence bigger than n. If γ is a $\mathcal{P}$-strategy then γ will extract only elements from B that enter Out_γ at a stage t such that $s < t < s_1$. Elements that enter the set $Out_\gamma[t]$ are not in $B[t]$. By our choice of stage s_1 as the least stage greater than s at which $n \notin B$, we have that $n \in B[t]$ and hence does not enter Out_γ.

3. $\alpha \subset \gamma$. The only strategies that are accessible while α is restraining elements in B are the strategies extending outcome w. By the actions in *Resetting the parameters* these strategies are in initial state at stage s. Thus the argument in 2 is valid for these strategies as well.

4. $\gamma \subset \alpha$ and γ is an $\mathcal{N}$-strategy. Then n is the witness of γ. If $\gamma\hat{\ }w \subseteq \alpha$ then at stage s_1 the strategy γ has outcome d and initializes α, forcing it to drop any restraints. If $\gamma\hat{\ }d \subseteq \alpha$ then the element n is extracted by γ at every α-true stage since the last initialization of α. Thus no axiom $\langle m, A_m, B_m \rangle$ with $n \in B_m$ is valid at an α-true stage after the last initialization of α and hence no such axiom will be used by α in the construction of Γ_α. This contradicts the fact that α restrains n at stage s.

5. $\gamma \subset \alpha$ and γ is an $\mathcal{P}$-strategy. Then $n \in Out_\gamma[s_1]$. Suppose n enters the set Out_γ at stage $t < s_1$. Then $n \notin B[t]$ and by the choice of s_1 it must be that $t \leq s$. Then at stage s the element n is in Out_γ and by the construction γ has outcome w at stage s, as whenever it has outcome i the set Out_γ is empty. As α is visited at stage s, $\gamma\hat{\ }w \subseteq \alpha$. By the actions of *Resetting the parameters* when the set Out_γ is extracted from B, γ initializes α at stage s_1.

□

We are ready to prove that every $\mathcal{P}$-requirement is satisfied.

Lemma 5.2.3. *For every i if $A_i \not\equiv_e \overline{K}$ then $A_i \oplus B \not\equiv_e \overline{K}$.*

Proof. Suppose that A_i is incomplete and for each j consider the strategy $\alpha \subset h$ along the true path labelled by the requirement $\mathcal{P}_{i,j}$. Then $\Theta_j = \Theta_\alpha$ and $A_i = A_\alpha$. We will prove that $\Theta_\alpha^{A_\alpha, B} \neq \overline{K}$. By Proposition 5.2.1 after stage $s_i(|\alpha|)$ the strategy α is not initialized. Let $\Gamma_\alpha = \bigcup_{t > s_i(|\alpha|)} \Gamma_\alpha[t]$. Then by assumption $\Gamma_\alpha^{A_\alpha} \neq \overline{K}$.

Suppose there is an $m \in \Gamma^{A_\alpha} \setminus \overline{K}$. Then there is a valid axiom $\langle m, A_m \rangle$ in Γ_α for m. Let $s > s_i(|\alpha|)$ be the stage at which this axiom is enumerated in Γ_α. As $A_m \subseteq A_\alpha$, A_α is a Δ_2^0 set and $\overline{K}$ is a Π_1 set, there is a stage $s_1 > s$ such that $(\forall t \geq s_1)[A_m \subseteq A_\alpha[t] \wedge m \notin \overline{K}[t]]$. If after stage s_1 the strategy α considers m then by $I.5$ of the construction α will never again move on to a different element and have outcome w forever. Thus α will perform finitely many cycles.

If α performs finitely many cycles then let n be the last element it considers and let s_2 be the least stage such that $n_\alpha[t] = n$ for all $t \geq s_2$. Then again by $I.5$ of the construction $n \notin \overline{K}[t]$ and $n \in \Gamma_\alpha^{A_\alpha}[t]$ at all $t \geq s_2$ or else $I.4$ of the construction would be valid at an α-true stage and α would move on to the next element. The good approximation that we have chosen for A_α and Proposition 1.4.1 guarantee that in this case $n \in \Gamma_\alpha^{A_\alpha}$ and hence there is a valid axiom $\langle n, A_n \rangle$ in Γ_α. By the actions that α performs at stage s_2 under $I.5$ each axiom for n including $\langle n, A_n, B_n \rangle$ is restored, i.e.

$B_n \subseteq B[s_2]$ and α restrains B_n in B at all stages $t \geq s_2$. By Lemma 5.2.2 $B_n \subset B$. Thus $n \in \Theta_\alpha^{A_\alpha, B}$.

Suppose now that $\Gamma_\alpha^{A_\alpha} \subseteq \overline{K}$ and that α performs infinitely many cycles. Let n be the least element such that $n \in \overline{K} \setminus \Gamma_\alpha^{A_\alpha}$. We will prove that in this case $n \notin \Theta_\alpha^{A_\alpha, B}$. Suppose not. Then there is a valid axiom in Θ_α. Consider the oldest valid axiom $\langle n, A_n, B_n \rangle$ in Θ_α, i.e. the one with least limit age $a(A_\alpha \oplus B, A_n \oplus B_n)$.

By assumption the strategy will perform infinitely many cycles and hence at infinitely many stages it will examine n. As $n \notin \Gamma_\alpha^{A_\alpha}$ and we have chosen a good approximation to A_α there will be infinitely many stages at which $n \notin \Gamma_\alpha^{A_\alpha}[t]$. Let s_0 be the first stage at which α examines n and at which the axiom has reached its limit age, i.e. $a(A_\alpha \oplus B, A_n \oplus B_n, t) = a(A_\alpha \oplus B, A_n \oplus B_n)$ at all $t \geq s_0$ and all other axioms for n enumerated in $\Theta[s_0]$ have greater age.

Let t_0 be the least stage after s_0 such that $n \notin \Gamma_\alpha^{A_\alpha}[t_0]$. Consider the least stage $s_1 > t_0$ at which n is again considered by α. Then step $I.3$ of the construction will be executed. If $Ax_\alpha(n)$ is currently undefined then α will select $\langle n, A_n, B_n \rangle$ as the new value of $Ax_\alpha(n)$ and enumerate it in Γ_α. If $Ax_\alpha(n)$ does have a value then it will be cancelled as the corresponding axiom is already enumerated in Γ_α and was not valid at stage t_0. Let t_1 be the next stage at which $n \notin \Gamma_\alpha^{A_\alpha}[t_1]$ and s_2 be the next stage at which α considers n. Finally $I.3.a$ and $I.3.c$ will be executed and the axiom $\langle n, A_n \rangle$ will be enumerated in Γ_α. By assumption this axiom is valid at all stages $t > s_0$ hence $n \in \Gamma_\alpha^{A_\alpha}$ and we have finally reached the desired contradiction. □

Finally to complete the proof we need to show that the constructed set B is in fact a Δ^0_2 set. We will do this in two steps.

Lemma 5.2.4. *Suppose α is a strategy visited at stages s_1 and s_2. Suppose x is a witness of a higher priority strategy $\beta < \alpha$. If $B(x)[s_1] = 1$ and $B(x)[s_2] = 0$ then α is initialized at a stage t, with $s_1 < t \leq s_2$.*

Proof. First note that in order for $B(x)[s_2] = 0$ then β must extract the witness x at stage $s_x \leq s_2$ by Proposition 5.2.2. If $s_1 < s_x$ then β is visited at stage s_x and has outcome d. Then if $\beta <_L \alpha$ or $\beta\hat{\ }w \subseteq \alpha$, the strategy α is initialized at stage s_x. If $\beta\hat{\ }d \subseteq \alpha$ then as at stage s_1 the witness $x \in B[s_1]$, β must have a different witness at stage s_1 and must have been initialized together with all its successors including α at a stage t such that $s_1 < t \leq s_x \leq s_2$.

Suppose $s_x < s_1$. As all strategies of lower priority than β are in initial state at stage s_x, the element x must be enumerated back in B before or at stage s_1 by a $\mathcal{P}$-strategy γ of higher priority than β. By construction at this stage γ executes step $I.5$ of the construction with outcome w and restrains x in B. This restraint is still valid at stage s_1 hence $\alpha \geq \gamma\hat{\ }w$. By Lemma 5.2.2 γ gives up its restraint at a stage $t \leq s_2$ as otherwise $x \in B[s_2]$. By construction when γ gives up its restraint, it initializes all strategies of lower priority than $\gamma\hat{\ }w$, hence α is initialized at stage t. □

Lemma 5.2.5. *The set B is Δ^0_2.*

Proof. We will prove that for every number n the value of $B(n)$ changes only finitely often. By Proposition 5.2.2 this is true for numbers that are not witnesses to any $\mathcal{N}$-strategy and for numbers that are witnesses to an $\mathcal{N}$-strategy and are never extracted by it.

Suppose that n is the witness x_β to the $\mathcal{N}$-strategy β extracted for the first time at stage s_x. No other $\mathcal{N}$-strategy will affect $B(x_\beta)$ as the sets of witnesses to each $\mathcal{N}$-strategy are disjoint. If β is initialized at stage $s > s_x$ then β gives up its restraint on x_β and will not extract x_β at any further stage. Furthermore x_β cannot enter the set $Out_\alpha[t]$ for any $t > s$ and any $\mathcal{P}$-strategy α. At stage s the element x_β can belong to finitely many sets Out_α for finitely many strategies α. Each such strategy can extract the element x_β only once, when emptying the set Out_α. Altogether $B(x_\beta)$ changes finitely often.

Suppose that β is not initialized after stage s_x. Then the element x_β has a permanent restraint out of B and no strategy $\alpha <_L \beta$ is visited after stage s_x.

First we note that by Lemma 5.2.4 $\mathcal{P}$-strategies of lower priority than β will not change the value of $B(x_\beta)$ as in order to do this they must be visited at a stage s_1 at which $B(x_\beta)[s_1] = 1$ to include an axiom that uses x_β and then again at a stage s_2 at which $B(x_\beta)[s_1] = 0$ to enumerate x_β back in B, without being initialized in between.

Thus we only need to prove that the finitely many $\mathcal{P}$-strategies $\alpha \subset \beta$ do not change the value of the $B(x_\beta)$ infinitely often. Assume for a contradiction that this is not true and let $\alpha \subset \beta$ be the largest strategy that changes the value of x_β infinitely often. It follows that α is visited infinitely often, not initialized and performs infinitely many cycles. Let $s > s_x$ be a stage after which no lower priority $\mathcal{P}$-strategy ever changes the value of $B(x_\beta)$. Let $s_0 > s$ be the least stage at which $B(x_\beta) = 0$.

At any stage $t > s_0$ if α is visited and chooses a new axiom to enumerate in Γ_α then $B(x_\beta)[t] = 0$. Indeed higher priority strategies $\alpha'\hat{\ }i \subset \alpha$ always have empty $Out_{\alpha'}$ when they have outcome i. Higher priority strategies $\alpha''\hat{\ }w$ do not enumerate any further elements after stage s_0 or else α and hence β are initialized. If α enumerates the element x_β in B at stage t_0 it executes step $I.5$ and does not define new axioms in Γ_α. The element x_β is permanently restrained out of B and hence enters the set $Out_\alpha[t_0]$. If α chooses a new axiom at stage $t > t_0$ its set Out_α is empty and $Out_\alpha[t_0]$ is extracted from B, hence $x_\beta \notin B[t]$.

Thus α will use only finitely many axioms whose B-part contains x_β in the definition of Γ_α. These are axioms for finitely many numbers, only part of which are not elements of the set $\overline{K}$. For each such element $n \notin \overline{K}$ there will be finitely many axioms Ax_n enumerated in Γ_α. Let s_n be a stage after which the approximation of the Δ_2^0 set A_α does not change on the A-parts of the axioms Ax_n. After stage s_n the value of $\Gamma_\alpha^{A_\alpha}(n)$ does not change. If $\Gamma_\alpha^{A_\alpha}(n) = 1$ then when α examines n after stage s_n it will restrain x_β in B forever and never move on to a different element contrary to the assumption that α performs infinitely many cycles. If $\Gamma_{\alpha}^{A_{\alpha}}(n) = 0$ then whenever α examines n at stages after s_n, step $I.4$ of the construction will be valid and α will not enumerate x_{β} back in B. This proves that our assumption is wrong and hence $B(x_{\beta})$ changes its value only at finitely many stages. □

Chapter 6

Cupping and Non-cupping in the ω-c.e. Enumeration Degrees

In this chapter we will continue the theme of cupping/non-cupping, but this time with respect to the subclasses of the Δ^0_2 enumeration degrees that arise from the Ershov hierarchy. In Section 1.1 we define these subclasses and mention some of their properties. For every n, $3 \leq n \leq \omega$ we have a proper subclass consisting of all n-c.e. enumeration degrees, see Definition 1.1.4.

In Chapter 5 we proved that we cannot computably enumerate a sequence of Δ^0_2 enumeration degrees that contains a cupping partner for every nonzero Δ^0_2 enumeration degree and in Section 1.2 we saw that the class of n-c.e. degrees for every $n \leq \omega$ can be computably enumerated. An immediate corollary of these two results is the following:

Corollary 6.0.1. *There exists a nonzero Δ^0_2 enumeration degree that cannot be cupped by any incomplete ω-c.e. degree.*

Our next result shows that this Δ^0_2 enumeration degree degree cannot be ω-c.e.

Theorem 6.0.1. *For every nonzero ω-c.e. enumeration degree* $\mathbf{a}$ *there exists an incomplete 3-c.e. enumeration degree* $\mathbf{b}$ *such that* $\mathbf{a} \vee \mathbf{b} = \mathbf{0}'_e$.

Cooper and Yates, see [Coo73], prove the existence of a non-cuppable nonzero c.e. Turing degree. This result is transferred into the Π_1^0 enumeration degrees via the standard embedding ι of $\mathcal{D}_T$ into $\mathcal{D}_e$. Using the fact that the class of 2-c.e. enumeration degrees coincides with the class of Π_1^0 enumeration degrees, see [Coo90], we can restate this result as follows: There exists a nonzero ω-c.e. enumeration degree which cannot be cupped by any incomplete 2-c.e. enumeration degree. Thus Theorem 6.0.1 claims the strongest possible result in this respect.

We face again the question of how much further we can limit our search for cupping partners when we restrict our attention to the smaller subclass of all ω-c.e. enumeration degrees. In contrast to the Δ_2^0 enumeration degrees, we can computably enumerate a sequence of ω-c.e. degrees which contains a cupping partner for every nonzero ω-c.e. degrees. Can we further limit this to a finite set? Cooper, Seetapun and (independently) Li prove that there exists a single incomplete Δ_2^0 Turing degree that cups every nonzero c.e. degree. When we transfer this statement into the enumeration degrees we obtain a single incomplete Δ_2^0 enumeration degree that cups all nonzero Π_1^0 enumeration degrees. Our next result shows that any other attempt at a result of this kind is doomed to failure as for every incomplete Σ_2^0 enumeration degree **a** there exists a nonzero member of the second class, a nonzero 3-c.e. enumeration degree **b**, such that **b** is not cupped by **a**. This provides a partial answer to the suggested question: If there is a finite set containing cupping partners for every nonzero ω-c.e. enumeration degree then it cannot be of cardinality 1.

Theorem 6.0.2. *Let* $\mathbf{a}$ *be an incomplete* Σ_2^0 *enumeration degree. There exists a nonzero 3-c.e. enumeration degree* $\mathbf{b}$ *such that* $\mathbf{a} \vee \mathbf{b} \neq \mathbf{0}_e'$.

Theorem 6.0.1 is joint work with Guohua Wu, published in [SW07], see Appendix A.2. Theorem 6.0.2 will be published in [Sos08b].

6.1 Cupping by a 3-c.e. enumeration degree

In this section we give a proof of Theorem 6.0.1. Let A be a nonzero ω-c.e. set with ω-c.e. approximation $\{A[s]\}_{s<\omega}$ bounded by the total computable function g. We shall construct an incomplete 3-c.e. set B whose enumeration degree cups the degree of A. The set B shall satisfy the following group of requirements:

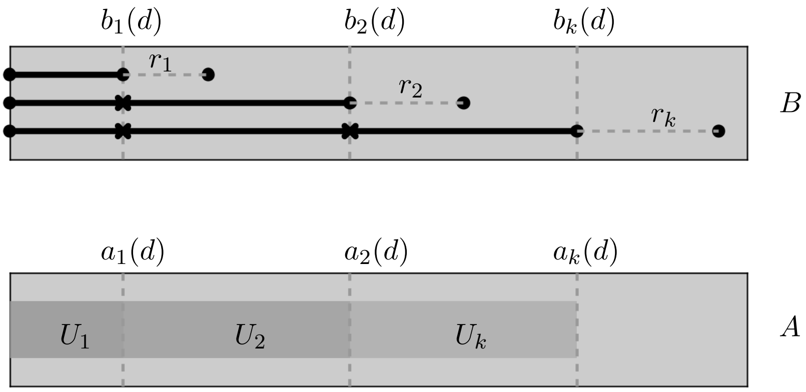
1. We have a global requirement which guarantees that the degree of B cups the degree of A. We shall construct an enumeration operator Γ so that:
$$\mathcal{S} : \Gamma^{A,B} = \overline{K}.$$
Here $\overline{K}$ denotes as usual any Π^0_1 representative of the degree $0'_e$.

2. The set B must be incomplete. We shall construct an auxiliary Π^0_1 set E to witness the incompleteness of B. Let $\{\Phi_i\}_{i<\omega}$ be a computable enumeration of all enumeration operators. For every $i < \omega$ we have a requirement:
$$\mathcal{N}_i : \Phi_i^B \neq E.$$

6.1.1 Basic strategies

This construction repeats almost exactly the construction described in Section 5.1. To satisfy the global $\mathcal{S}$-requirement we will construct an enumeration operator Γ rectifying it at every stage and using axioms of the form $\langle n, A \upharpoonright a(n) + 1, B \upharpoonright b(n) + 1\rangle$, where $a(n)$ and $b(n)$ are markers defined by us.

To satisfy an $\mathcal{N}_i$-requirement we shall use a strategy very similar to the $\mathcal{G}$-strategy, but it will be based on the simple Friedberg-Muĉhnik strategy instead of the strategy for constructing a generic set. It shall select a threshold d and attempt to secure its restraint on the set B by moving the B-markers of elements $n \geq d$ above it. To do this it shall construct a c.e. set U approximating the given set A and threatening to prove that A is c.e. The $\mathcal{N}$-strategy shall then run in cycles, preforming many attempts to satisfy its requirement. Each new cycle k shall have a new witness x_k and shall search for a valid axiom $\langle x_k, D_k\rangle$ in Φ_i, so that $x_k \in \Phi_i^B$. If this axiom is found the strategy shall end the k-th cycle by extracting the witness x_k from the set E and approximating a larger initial segment of the set A up to $a_k(d)$, where $a_k(d)$ is the current marker of the threshold during the k-th cycle. Then it shall perform capricious destruction on the operator Γ by extracting the B-marker for the threshold which is current during the k-th cycle, $b_k(d)$, thereby moving the action of the $\mathcal{S}$-strategy and the next cycle of this strategy to elements larger than the required restraint, $r_k = \max D_k$, on the set B. Thus the only number that will conflict the restraint on B for this cycle will be the marker $b_k(d)$.



As the set A is not c.e. the approximation of A shall be unsuccessful and we shall eventually be able to locate a permanent extraction from the set A, an extraction useful to the last cycle k. Using this extraction we can restore the set B by enumerating the marker $b_k(d)$ back in the set B, making $\langle x_k, D_k\rangle$ a valid axiom that enumerates the witness x_k back in Φ_i^B, and preserve the restraint r_k on B at further stages.

This strategy would work well if we were constructing a Δ_2^0 set. In this case however we are required to construct a 3-c.e. set and thus are only allowed to extract a marker from the set B once. The strategy that we just described might require us to extract the same marker $b_k(d)$ many times. Suppose during cycle $k+1$ we see a useful extraction in the approximation to A. We assume that this is a permanent extraction and reenumerate the marker $b_k(d)$. If our assumption turns out to be wrong, we will need to extract the marker $b_k(d)$ a second time before we carry on with cycle $k+1$. And this can be repeated many times.

On first sight this conflict seems to be of insurmountable difficulty. On closer inspection we notice that the number of times that this situation can arise is bounded and we can compute this bound. Every time we see a useful change in the approximation to A it is below a fixed number $a_k(d)$. As the set A is ω-c.e. the number of times that the approximation to $A \upharpoonright a_k(d)+1$ can change is bounded by $\sum_{x<a_k(d)+1} g(x)$. We will modify the form of the B-markers. For every element n with A-marker $a(n)$ we will define a finite set of B-markers $M(n)$ of size $\sum_{x<a(n)+1} g(x)+1$. The current axioms in the operator Γ will have the form $\langle n, A \upharpoonright a(n)+1, B \upharpoonright M(n)+1\rangle$, where $B \upharpoonright M(n)+1$ just denotes $B \upharpoonright \max M(n)+1$. Every time we are required to extract the B-marker of an element n, we will extract the least element from the set $M(n)$ that has not been extracted from B yet.

6.1.2 Construction

The construction will be designed similarly to the one in Section 5.1. We order the requirements linearly:

$$\mathcal{S} < \mathcal{G}_0 < \mathcal{G}_1 \ldots$$

and assign a strategy to every requirement.

At the beginning of each stage we shall run the $\mathcal{S}$-strategy. Then we shall activate the least $\mathcal{N}$-strategy that *requires attention*, defined below.

We shall construct a 3-c.e. approximation to the set B. Initially it will be the empty set: $B[0] = \emptyset$. The approximation to the set B at stage s shall be obtained from $B[s-1]$ by allowing the two active strategies at stage s to enumerate or extract numbers from it. Every element n will be extracted at most once from the set B and it will thus automatically be 3-c.e. The set E is constructed as a Π^0_1 set: $E[0] = \mathbb{N}$ and $E[s]$ is obtained from $E[s-1]$ after a possible extraction by the active $\mathcal{N}$-strategy at stage s.

The $\mathcal{S}$-strategy

The global $\mathcal{S}$-strategy constructs an enumeration operator Γ. To every element n the strategy shall assign current A- and B-markers, $a(n)$ and $M(n)$, and a current axiom of the form $\langle n, A \upharpoonright a(n)+1, B \upharpoonright M(n)+1\rangle$. Initially $\Gamma = \emptyset$ and all markers and axioms are undefined. At stage s the $\mathcal{S}$-strategy operates as follows:

For very element $n < s$ perform the following actions:

- If $n \notin \overline{K}[s]$ then find all valid axioms in Γ for n, $\langle n, A_n, B_n\rangle$. The finite set B_n ends in a B-marker $M(n)$ defined at a previous stage. Extract the least element of $M(n)$ that has never been extracted before from $B[s]$, if there is one available.

- If $n \in \overline{K}[s]$ and the current axiom for n is valid then skip to the next element. If the current axiom for n is not defined or is not valid then:

  1. If $a(n)[s]\uparrow$, define $a(n)[s] = a(n-1)[s]+1$. (if n= 0, define $a(n) = 1$).
  2. If $M(n)[s]\downarrow$ and there is an element in $M(n)$ that has never been extracted then extract from $B[s]$ the least such element and cancel all markers $M(n')[s]$ for $n' > n$ and go to step 3.
  3. Define $M(n)[s]$ as a fresh set of size $\sum_{x<a(n)+1} g(x)+1$ consisting entirely of numbers greater than any number mentioned in the construction so far. Enumerate $M(n)[s]$ in $B[s]$, go to step 4.
  4. Define the current axiom for n at stage s to be $\langle n, A[s] \upharpoonright a(n)+1, B[s] \upharpoonright M(n)+1\rangle$ and enumerate it in $\Gamma[s]$.

Activating the $\mathcal{N}$-strategy α.

Denote by Φ_α the operator that α is working with. The strategy α is equipped with a threshold d and a current witness x, initially undefined. The strategy has furthermore a parameter, which we shall call the *current guess*, denoted by G and it shall have the following structure: $\langle U, Ax, M, t\rangle$, where U is α's current approximation to the set A, Ax is the axiom in Φ_α that α would like make valid, M is a B-marker whose enumeration in the set B will facilitate this and finally t is the stage at which this guess was made. This parameter has initial value $\langle \emptyset, \uparrow, \uparrow, \uparrow\rangle$.

The strategy α at stage s has threshold d, witness x and guess $G = \langle U, \langle y, D\rangle, M, t\rangle$ all possibly undefined. We list the cases in which it requires attention. Some cases are followed by a *check*-statement which needs to be valid in order for α to make the actions described. Every time we choose the first case which applies at stage s.

1. The threshold d is not defined.

   Action: Define the threshold $d \in \overline{K}[s]$ as a fresh number.

2. The threshold d is defined but $d \notin \overline{K}[s]$.

   Check: If the guess is defined then $M \not\subseteq B[s]$ or there is a member of M that has not yet been extracted.

   Action: Shift the value of the threshold to the next element in $\overline{K}[s]$. Cancel the current witness x. If $M \downarrow\subseteq B[s]$ then extract from B the least member of M that has never been extracted. Cancel the current guess.

3. A member of a B-marker for an element $n < d$ has been extracted from B at stage s.

   Check: If the guess is defined then $M \not\subseteq B[s]$ or there is a member of M that has not yet been extracted.

Action: Cancel the current witness x. If $M \downarrow\subseteq B[s]$ then extract from B the least member of M that has never been extracted. Cancel the current guess.

4. $U \subseteq A[s]$ and $M \downarrow\subseteq B[s]$.

   Check: There are members of M and $M(d)$ that have not yet been extracted.

   Action: Extract from the set $B[s]$ the least member of each M and $M(d)$ that has not yet been extracted. Enumerate $D \setminus M$ in the set $B[s]$. Cancel the current B-marker for every $n \geq d$.

5. $U \not\subseteq A[s]$ and $M \downarrow\not\subseteq B[s]$.

   Action: Enumerate M in the set $B[s]$.

6. The witness x is not defined or the current marker of the threshold $M(d)$ is defined and $B \upharpoonright M(d) + 1$ has been modified at stage s.

   Action: Define the witness x as a fresh number.

7. $M \uparrow$ or $M \downarrow\not\subseteq B[s]$, and $U \subseteq A[s]$ and $x \in \Phi^B_\alpha[s]$.

   Check: There is a member of the current B-marker $M(d)$ that has not yet been extracted.

   Action: Let $\langle x, D_x \rangle$ be an axiom in $\Phi_\alpha[s]$ valid at stage s. Define a new value for the current guess G to be $\langle A_d, \langle x, D_x \rangle, M(d), s \rangle$, where $\langle d, A_d, B_d \rangle$ is the current axiom for d in $\Gamma[s]$. Extract from $B[s]$ the least member of $M(d)$ that has not yet been extracted. Cancel all A- and B-markers for elements $n \geq d$. Extract x from $E[s]$ and then cancel x. Define a fresh value of the marker $a(d)$.

8. α was not allowed to execute its actions at a previous stage due to an invalid check.

   Actions: No actions are executed.

6.1.3 Proof

We will follow the structure of the proof described in Section 5.1.3. In some cases to obtain a proof of a statement we only need to adapt the notation used in a proof from 5.1.3. We will not repeat such proofs but instead refer to the appropriate statement from Section 5.1.3.

One significant difficulty that distinguishes this construction from the one described in Section 5.1 is that in order to prove that every $\mathcal{N}$-strategy requires attention at finitely many stages we need to show that the *check*-clauses are always valid. Before we can do this we will examine the properties of the axioms used in Γ.

Propostion 6.1.1. *1. At every stage s if $n < m$ and $n, m \in \overline{K}[s]$ and the current axioms for n and m at stage s are $\langle n, A_n, B_n \rangle$ and $\langle m, A_m, B_m \rangle$ then $A_n \subseteq A_m$ and $B_n \subset B_m$.*

2. If $M(n)$ is the current marker for n at stage s then no member of $M(n)$ has been extracted from B.

3. If α is an $\mathcal{N}$-strategy not initialized at stage s then there is at most one valid axiom in $\Gamma[s]$ for its threshold d different from the current one. This axiom is associated with α's current guess $G[s]$.

Proof. 1. See the proof of Proposition 5.1.1 part (1).

2. The current marker is always defined as a fresh set of size ≥ 1 and its members have never been extracted from the set B. If a strategy extracts a member of $M(n)$ at stage s then it cancels this marker and it ceases to be current.

3. As d is α's threshold at stage s, $d \in \overline{K}[s]$. If α is in initial state at stage s then there is no axiom for d in $\Gamma[s]$.

Otherwise any axiom for d that the $\mathcal{S}$-strategy has cancelled at a previous stage is invalid. The $\mathcal{S}$-strategy always attempts to extract a member of the current B-marker before it is cancelled and by part (2) of this proposition it is always allowed to do this.

This member can never be reenumerated in the set B.

Any axiom for d that was used for a previous guess $G[t]$ at a stage $t < s$ is not valid. Whenever α changes the value of the guess it executes the actions under step 7 and the marker recorded in its previous value is not in the approximation to B. If α cancels the guess during steps 2 or 3 then it is allowed to do so and the marker recorded in it is either not a subset of B or one of its members is extracted from B. This marker will never be reenumerated in B and the axiom associated with the old value of the guess remains invalid forever. Whenever α cancels the current B-marker of the threshold at step 4 it extracts a member of it it from B invalidating the axiom associated with it.

Thus the only axioms for d that can be valid at stage s are the current one and the one used in the current guess $G[s]$. □

Now we can prove the desired statement.

Lemma 6.1.1. *Let n be any number. Let $a(n)$ and $M(n)$ be the current markers of the element n defined at stage s_0. If a strategy wishes to extract a member of $M(n)$ at any further stage then it is allowed to.*

Proof. The set $M(n)$ is of size $\sum_{x<a(n)} g(x) + 1$. We will show that if a member of the set $m \in M(n)$ is extracted at stage $s_1 \geq s_0$ and a strategy requires to extract a different member of the set at stage $s_2 \geq s_1$ then there is a stage t such that $s_1 \leq t < s_2$ and $A \upharpoonright a(n)[t] \neq A \upharpoonright a(n)[t+1]$. Thus there is a member of $M(n)$ available for extraction at stage s_2.

Assume inductively that the statement is true for elements $k < n$ and their axioms defined up until stage s_0. At the end of stage s_1 the marker $M(n)$ is not current for n by 6.1.1, part (2). Thus in order for a strategy to require to extract a member of $M(n)$ at stage t there is a stage s such that $s_1 < s < s_2$ at which m is reenumerated in the set $B[s]$. We have to consider different cases depending on which strategy extracts m at stage s_1.

If m is extracted at stage s_1 by an $\mathcal{N}$-strategy α, then n is α's threshold at stage s_1. We have the following cases:

- α executes step 2 or 3. Then α extracts m invalidating the guess. This member will never be reenumerated in the set B.
- α executes step 4 or step 7 and $M(n)$ is the marker enumerated in the current guess. Then $\langle n, A_n, B_n \rangle$ is the axiom associated with the current guess and $A_n \subseteq A[s_1]$. At stage s the strategy α enumerates m back in $B[s]$ executing step 5 hence $A_n \nsubseteq A[s]$.
- α extracts m cancelling the current marker for the threshold at step 4 then m will never be reenumerated in the set B.

If at stage s_1 the marker m is extracted by the $\mathcal{S}$-strategy then we have two cases:

- $n \in \overline{K}[s_1]$. Then $M(n)$ is the current marker for n at stage s_1 and the $\mathcal{S}$-strategy cancels it. The member m will never be reenumerated in B.
- $n \notin \overline{K}[s_1]$. Then at stage s_1 we have $A_n \subseteq A[s_1]$ and $B_n \subseteq B[s_1]$. Note that n is not the threshold to any strategy at stage s_2, thus at stage s_2 the $\mathcal{S}$-strategy requires to extract a new member of $M(n)$. At stage s_2 we have again $A_n \subseteq B[s_2]$ and $B_n \subseteq B[s_2]$.

  The marker m is reenumerated in B at stage $s < s_2$ by an $\mathcal{N}$-strategy α. This strategy is not initialized at stages t, $s_1 \leq t \leq s$ and always allowed to perform its actions. If $d \geq n$ then α would execute step 2 or step 3 cancelling the value of the guess G and will never be able to reenumerate m in the set B. Thus $d < n$ and by Proposition 6.1.1 there is an axiom for d valid at stage t, the axiom $\langle d, A_d, B_d \rangle$ with $A_d \subseteq A_n$ and $B_d \subset B_n$. Furthermore the member m is enumerated back in the set B at stage s. The only elements that α can enumerate in the set B at stage s, which do not belong to markers of its threshold are the ones in the set D, where $\langle y, D\rangle$ is recorded in the current guess $G[s]$. It follows that the guess is made at a stage $t < s_1$ before m is extracted from B. Then at stage t the strategy α cancelled the current B-markers for all elements $k \geq d$. Thus the axiom $\langle d, A_d, B_d\rangle$ is not the current axiom for d at stage s_1 and by Proposition 6.1.1 it is the one associated with the guess $G[s_1] = \langle A_d, \langle y, D\rangle, M, t\rangle$. At stage s_1 the strategy α will therefore require attention under step 4. It will enumerate m back in the set B (so $s_1 = s$) and it will extract a member of M from the set B. Note that $M \subseteq B_n$ and thus there is a further stage $s^* < s_2$ at which M is reenumerated in the set B. This can only be done by α under step 5 if $A_d \nsubseteq A[s^*]$. Thus $A_n \subseteq A[s_1]$ and $A_n \nsubseteq A[s^*]$.

□

We have established that the *check*-clauses for any strategy at any stage are always valid and no strategy will ever require attention under step 8. We can safely carry on with our proof.

Propostion 6.1.2. *Let α be an $\mathcal{N}$-strategy not initialized after stage s. Suppose the value of the current guess G is not cancelled at stages $t > s$. Denote by $U[t]$ the value of the first component of the current guess at stage t. Then $\{U[t]\}_{t>s}$ is a c.e. approximation to the set $\mathbb{U} = \bigcup_{t>s} U[t]$.*

Proof. See the proof of Proposition 5.1.2. □

Lemma 6.1.2. *1. Every strategy requires attention only at finitely many stages.*
2. Every $\mathcal{N}$-requirement is satisfied.

Proof. We prove both parts of this lemma with simultaneous induction: Suppose that the statement is true for strategies of higher priority than α and let s_1 be the least stage such that at stages $t \geq s_1$ strategies of higher priority than α do not require attention.

We note as in the proof of Lemma 5.1.1 that at stages $t > s_0$ only $\mathcal{S}$ and α can modify the approximation to $B[t] \upharpoonright b(d)[t] + 1$.

At stages $t \geq s_1$ the strategy α is not initialized. Hence stage s_1 is the last stage at which α requires attention under step 1. The set $\overline{K}$ is infinite hence this there is a stage $s_2 \geq s_1$ after which α does not execute step 2. The value of the threshold d and the A-markers for elements $n < d$ do not change after stage s_2. As the set A is Δ^0_2, the approximation to $A \upharpoonright \max_{n<d} a(n) + 1$ will stabilize and hence there is a least stage $s_3 \geq s_2$ such that the $\mathcal{S}$-strategy does not extract from the set B any members of markers for elements $n < d$ at stages $t > s_3$. So after stage s_3 the strategy α does not execute step 3 and the value of α's guess will not be cancelled. By Proposition 6.1.2 the set $\mathbb{U} = \bigcup_{t>s_3} U[t]$ is a c.e. set. As A is not, $A \neq \mathbb{U}$.

Suppose that there is an element $n \in \mathbb{U} \setminus A$, and let s be the least stage such that $n \in U[t]$ and $n \notin A[t]$ for all $t \geq s$. Then at stages $t \geq s$ we always have $U[t] \nsubseteq A[t]$ and steps 4 and 7 will never again be executed. Let $s_7 > s_3$ be the last stage at which step 7 is executed. Then at stage s_7 the final value of the guess $G = \langle U, \langle y, D\rangle, M, s_7\rangle$ is recorded, $\langle y, D\rangle \in \Phi_\alpha[s_7]$ is valid at stage s_7 and y is extracted from $E[s_7]$. At all stages $t \geq s_7$ we have $D \setminus M \subseteq B[t]$. As $s_7 > s_3$ and $d \in \overline{K}$ the $\mathcal{S}$-strategy does not modify $B \upharpoonright M + 1$. If it modifies B below $\max D$ at stage $t > s_7$ then it extracts a member of an old marker of an element $n > d$ for a valid axiom in $\Gamma[t]$. By Proposition 5.1.1 there is an axiom for the threshold d which is also valid at stage t and this is the one associated with the guess G. Thus $U \subseteq A[t]$, $M \in B[t]$ and α executes step 4, restoring $D \setminus M \subseteq B[t]$. As this does not happen after stage s, the $\mathcal{S}$-strategy does not modify B below $\max D$ at stages $t \geq s$. At stage s the strategy α enumerates the marker M back in the set $B[s]$ executing step 5 for the last time and ensuring that $D \subseteq B[s]$. Thus after stage s the strategy α can only require attention under step 6. Furthermore $B[t] \upharpoonright \max D + 1$ is not modified at all $t > s$ and thus the requirement associated with α is satisfied by $y \in \Phi^B_\alpha$ and $y \notin E$. As the current markers of the threshold d are not modified by α after stage s, there will be a least stage $s_6 > s$ at which the $\mathcal{S}$-strategy will modify the current axiom for d and the approximation to $B \upharpoonright b(d)+1$ for the last time, after $A \upharpoonright a(d)+1$ has stabilized. Then stage s_6 is the last stage at which α can execute step 6. The strategy will not require attention at further stages.

Suppose that $\mathbb{U} \subseteq A$. Then let n be a number such that $n \in A \backslash \mathbb{U}$. If we assume that α requires attention under step 7 at infinitely many stages we will reach a contradiction as the number n would eventually be enumerated in the set $\mathbb{U}$. Let s_7 be the last stage at which α executes step 7, $s_7 = s_3$ if α never executes step 7. The final value of the guess is $G = \langle U, \langle y, D\rangle, M, s_7\rangle$ or $\langle U = \emptyset, \uparrow, \uparrow, \uparrow\rangle$. Let s be the least stage such that $U \subseteq A[t]$ at all stages $t \geq s$. Then α can execute step 4 for the last time at stage s. After this $M \nsubseteq B[t]$ or $M \uparrow$ and α can only require attention under step 6. The current markers of the threshold d are not changed after stage s by α. As in the previous case there is a least stage $s_6 > s$ after which α does not execute step 6. At stages $t \geq s_6$ the witness x has a permanent value. As α does not execute step 7 after stage s_6 we have that $x \notin \Phi_\alpha^B[t]$ at all $t > s_6$. Thus the requirement is satisfied by $x \in E \setminus \Phi_\alpha^B$. The strategy does not require attention at stages $t > s_6$. □

Corollary 6.1.1. *The $\mathcal{S}$-requirement is satisfied.*

Proof. Let m be any element. Let α be an $\mathcal{N}$-strategy with permanent threshold $d > m$. By Lemma 6.1.2 every $\mathcal{N}$-strategy has a permanent threshold and by construction every $\mathcal{N}$-strategy chooses a threshold of value greater than any chosen before, thus this choice of α is satisfiable. There is a stage s such that α does not require attention at stages $t > s$. At stage s the $\mathcal{S}$-strategy ensures that $\Gamma^{A,B}[s](m) = \overline{K}(m)$. If at a later stage $t > s$ this equality is injured then $\mathcal{S}$ would extract a member of a marker $M(m)$ from the set B. If $m \in \overline{K}[t]$ then a member of the current marker for m would be extracted, otherwise a member of a marker of the valid axiom for m in Γ at stage t would be extracted. In both cases α would require attention under step 3 contradicting our choice of s. If $m \notin \overline{K}$ then there is no valid axiom for m in Γ at any stage $t > s$. If $m \in \overline{K}$ then the current axiom for m at stage s is valid at all stages $t > s$. $\square$

6.2 Non-cupping by a 3-c.e. enumeration degree

In this section we give a proof of Theorem 6.0.2. Given an incomplete Σ^0_2 enumeration degree **a** we will construct a nonzero 3-c.e. enumeration degree **b** which is not cupped by **a**.

Let A be a representative of the given Σ^0_2 enumeration degree. Let $\{A[s]\}_{s<\omega}$ be a good Σ^0_2 approximation to A, see Definition 1.4.1. We shall construct two 3-c.e. sets X and Y, so that ultimately the degree of one of them will have the requested properties. Consider the following requirements:

1. Let $\{\Theta_i\}_{i<\omega}$ and $\{\Psi_i\}_{i<\omega}$ be computable enumerations of all enumeration operators. For every i we will have a pair of requirements:
$$\mathcal{P}^0_i : \Theta_i^{A,X} \neq \overline{K} \quad \text{and} \quad \mathcal{P}^1_i : \Psi_i^{A,Y} \neq \overline{K}.$$

2. Let $\{W_e\}_{e<\omega}$ be a computable enumeration of all c.e. sets. For every natural number e we have a requirement:
$$\mathcal{N}_e : W_e \neq X \wedge W_e \neq Y.$$

We shall construct the sets X and Y so that for all e the requirement $\mathcal{N}_e$ is satisfied, thus both X and Y have nonzero e-degree, and if $\mathcal{P}^j_i$ is not satisfied for some i then for all i' the requirement $\mathcal{P}^{1-j}_{i'}$ is satisfied, thus the degree of at least one of the sets $A \oplus X$ or $A \oplus Y$ is incomplete.

6.2.1 Basic strategies

We shall describe the basic strategies as usual with the context of the tree in mind. The tree of strategies shall be designed so that each node shall be assigned either an $\mathcal{N}$-requirement or a pair of a $\mathcal{P}^0$- and a $\mathcal{P}^1$-requirement.

A $\mathcal{P}$-strategy

A $\mathcal{P}$-strategy α is associated with a pair of requirements, $\mathcal{P}^0_\alpha$ and $\mathcal{P}^1_\alpha$. It will attempt at proving that at least one of them is satisfied. To do this the strategy constructs an e-operator Γ_α, threatening to prove that $A \geq_e \overline{K}$. The outcomes of the strategy will be divided into two groups, *finitary*, i.e. requiring a finite number of actions, and *infinitary* outcomes, requiring an infinite number of actions. There will be infinitely many infinitary outcomes - two for each number n arranged from left to right by the order of the natural numbers: $\langle X, 0\rangle <_L \langle Y, 0\rangle < \langle X, 1\rangle \ldots$ Then there will be two finitary rightmost outcomes $\langle X, w\rangle <_L \langle Y, w\rangle$. Thus all the outcomes of a $\mathcal{P}$-node are arranged as follows:

$$\langle X, 0\rangle <_L \langle Y, 0\rangle <_L \cdots <_L \langle X, n\rangle <_L \langle Y, n\rangle \cdots <_L \langle X, w\rangle <_L \langle Y, w\rangle.$$

For each outcome the first element of the pair indicates which requirement has been satisfied. The next $\mathcal{P}$-strategy below outcomes $\langle X, -\rangle$ shall be associated with a new $\mathcal{P}^0$-requirement and the same $\mathcal{P}^1$-requirement. Similarly the next $\mathcal{P}$-strategy below outcomes $\langle Y, -\rangle$ will be associated with the same $\mathcal{P}^0$-requirement and a different $\mathcal{P}^1$-requirement. Thus if $\mathcal{P}^j_i$ never gets satisfied for some i then all $\mathcal{P}^{1-j}_{i'}$ must be.

Similarly to the $\mathcal{P}$-strategy described in Section 5.2 the strategy α performs cycles of increasing length. On the k-th cycle it examines all elements $n = 0, 1, \ldots, k$ in turn. While it examines an element n the strategy can choose to end its actions for the particular stage by selecting an outcome or move on to the next element in the cycle, possibly even starting a new cycle. Suppose α is examining the element n. If the element n currently belongs to $\overline{K}$ then the only possible outcomes that it can choose for this element are the infinitary $\langle X, n\rangle$ or $\langle Y, n\rangle$. If the element n is in both sets $\Theta^{A,X}_\alpha$ and $\Psi^{A,Y}_\alpha$ and has been there at all stages since α last looked at n then it will enumerate an axiom for n in Γ_α which comprises the A-parts of the two axioms for n in Θ_α and in Ψ_α that have been valid the longest, i.e. have least age defined in 1.2.1, and move on to the next element. Otherwise α will select the appropriate outcome corresponding to the set that has failed to provide a valid axiom and end its actions for this stage. When α is active again, it will start working with the next element of the cycle.

If the element n has left the approximation of $\overline{K}$ then for each axiom in Γ_α for this element the strategy shall restore one of the axioms in either Θ_α or Ψ_α by enumerating the corresponding X-part back in X or Y-part back in Y and have the corresponding finitary outcome $\langle X, w\rangle$ or $\langle Y, w\rangle$. The strategy shall then wait until it has observed a change in A that rectifies the operator Γ_α, i.e. it will not move on to the next element in the cycle until (if ever) this happens and it will keep having the same finitary outcome. Note that α will not only consider stages at which it is active, instead every time it is visited it will check if $\Gamma_\alpha^A(n)[s] = 0$ at any stage s since the last stage at which α was active.

As A is incomplete the strategy will eventually include in its cycles an element n such that $\Gamma_\alpha^A(n) \neq \overline{K}(n)$. If there is an element n such that $n \in \Gamma_\alpha^A \backslash \overline{K}$ then $n \in \Gamma_\alpha^A[s] \setminus \overline{K}[s]$ at all but finitely many stages s. Thus eventually $\Gamma_\alpha^A(n)$ will not be rectified by any change in A and α will have a finitary outcome proving the successful diagonalization. Otherwise α will have infinitely many cycles and each element n will be examined infinitely often. Consider the least n such that $n \in \overline{K} \backslash \Gamma_\alpha^A$. By the properties of a good approximation we have that at infinitely many stages s, in fact at all good stages, $n \notin \Gamma_\alpha^A[s]$. Thus infinitely often α will discover that at least one of the operators Θ_α or Ψ_α has failed to provide it with an axiom that is permanently valid, i.e. infinitely often α will have proof that $\Theta_\alpha^{A,X}(n) = 0$ or $\Psi_\alpha^{A,Y}(n) = 0$ and have outcome $\langle X, n\rangle$ or $\langle Y, n\rangle$ respectively.

An $\mathcal{N}$-strategy

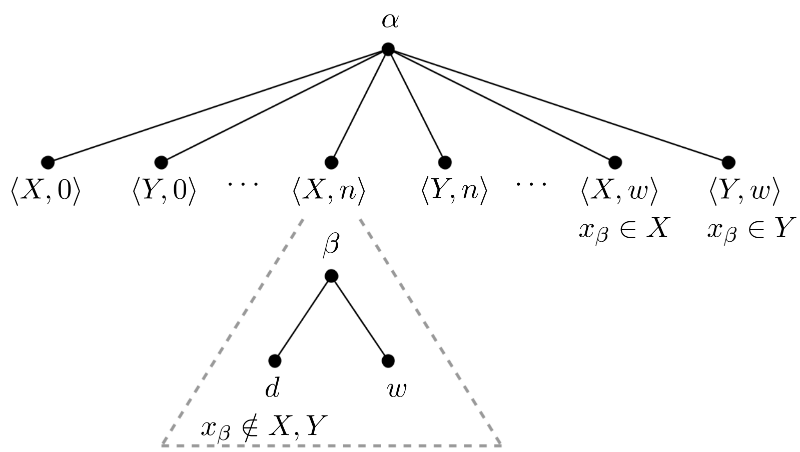
An $\mathcal{N}$-strategy β working on W_β would like to prove that $W_\beta \neq X$ and $W_\beta \neq Y$. The obvious strategy for β would be the one described in Section 5.2. It will select a witness

x_β and wait until $x_\beta \in W_\beta$. The sets X and Y will initially be approximated by $\mathbb{N}$, then during the construction the strategies extract or enumerate back elements in the sets. Thus if x_β never enters W_β the strategy will be successful and will have outcome w. If the element does enter W_β then the strategy will extract x_β from both sets X and Y, have outcome d, where $d <_L w$, and again will have proved a difference. This strategy is unfortunately incomplete as we shall see in the next section.

6.2.2 Elaborating the $\mathcal{N}$-strategy to avoid conflicts

The naive $\mathcal{N}$-strategy described in the previous section is in conflict with the need of higher priority $\mathcal{P}$-strategies to restore axioms by enumerating elements back in one of the sets X or Y, constructed as 3-c.e. sets. Therefore the strategy for an $\mathcal{N}$-node β will have to be more elaborate. This conflict justifies the introduction of nonuniformity.

The elaborated strategy will start off as the original strategy: select a witness x_β as a fresh number and wait until $x_\beta \in W_\beta$. If this never happens then the requirement will be satisfied with outcome w. Otherwise it will extract x_β from both sets X and Y. Suppose a higher priority strategy α requires that x_β be enumerated back in X or Y.



In this case β shall initialize all lower priority strategies, choose a new witness y_β that has not been used in any axiom so far, restrain X on x_β and let x_β be enumerated back in Y. From this point on any axiom that appears in the construction shall necessarily have $x_\beta \notin X$, thus x_β and y_β cannot appear in the same axiom. The strategy β will wait again with outcome w until y_β enters W_β and then extract it from Y with outcome d . Should a higher priority α require that y_β be enumerated back in one of the sets then β will only give permission to enumerate back in X.

This will resolve the central conflict between strategies. Note that as the only actions that the $\mathcal{P}$-strategies ever take is enumerating certain elements back in the sets X and Y, the $\mathcal{P}$-strategies are not in conflict with each other.

Possible conflicts between $\mathcal{N}$-strategies are resolved via initialization. Whenever a higher priority $\mathcal{N}$-strategy β decides to extract a number n from X or Y all strategies below outcome w are initialized and all strategies below outcome d are in initial state. Thus lower priority strategies will operate at further stages under the assumption that n is extracted, the axioms used by $\mathcal{P}$-strategies of lower priority will not include this element and the witnesses used by $\mathcal{N}$-strategies will be chosen as big numbers that do not appear in any axiom seen so far, thus cannot appear in an axiom that includes the element n.

6.2.3 Parameters and the tree of strategies

A $\mathcal{P}$-strategy α will have a parameter Γ_α, the e-operator that it will construct when visited. At initialization Γ_α is set to the empty set. The strategy will also have parameters k_α denoting the current cycle of the strategy and $n_\alpha \leq k_\alpha$ denoting the current element of the cycle that α is working with. On initialization the values of the parameters are set to $k_\alpha = 0$ and $n_\alpha = 0$. Furthermore for each element $n < \omega$ the strategy α shall have one more parameter $D_\alpha(n)$, a list of all pairs of X- and Y-parts of axioms from Θ_α and Ψ_α respectively, for which the A-parts are used in axioms for n in Γ_α. Initially the values of all such lists will be $\emptyset$. Finally it will have two parameters $Ax^\theta_\alpha(n)$ and $Ax^\psi_\alpha(n)$ denoting axioms in Θ_α and Ψ_α respectively which will be candidates for the construction of a new axiom in Γ_α, initially undefined.

An $\mathcal{N}$-strategy β shall have parameters x_β, y_β, initially undefined. Furthermore on initialization β will give up any restraint it has imposed so far.

Let $O_\mathcal{P}$ denote the set of all possible outcomes of a $\mathcal{P}$-strategy and $O_\mathcal{N} = \{d, w\}$. Let $O = O_\mathcal{P} \cup O_\mathcal{N}$ be the collection of all possible outcomes and R the collection of all requirements. The tree of strategies is a computable function T with domain a downwards closed subset of $O^{<\omega}$ and range a subset of $R^2 \cup R$ with the following inductive definition:

1. $T(\emptyset) = \langle \mathcal{P}^0_0, \mathcal{P}^1_0 \rangle$.

2. Let α be in the domain of T and α be a $\langle \mathcal{P}^0_i, \mathcal{P}^1_j \rangle$-node. Then $\alpha\hat{\ }o$, where $o \in O_\mathcal{P}$, is also in the domain of T and $T(\alpha\hat{\ }o) = \mathcal{N}_{|\alpha|/2}$.

3. Let β be an $\mathcal{N}$-node in the domain of T. Then $\beta = \alpha\hat{\ }o$, where α is a $\langle \mathcal{P}^0_i, \mathcal{P}^1_j \rangle$-node for some i and j. Then $\beta\hat{\ }o'$, where $o' \in O_\mathcal{N}$, is in the domain of T. If $o = \langle X, n \rangle$ for some $n \in \omega \cup \{w\}$ then $T(\beta\hat{\ }o') = \langle \mathcal{P}^0_{i+1}, \mathcal{P}^1_j \rangle$. If $o = \langle Y, n \rangle$ for some $n \in \omega \cup \{w\}$ then $T(\beta\hat{\ }o') = \langle \mathcal{P}^0_i, \mathcal{P}^1_{j+1} \rangle$.

6.2.4 Construction

We shall perform the construction in stages. At each stage s we shall approximate the sets X and Y by constructing cofinite sets $X[s]$ and $Y[s]$. We shall also construct a string $\delta[s]$ of length s through the domain of T. At true stages strategies will be allowed to modify their parameters and choose an outcome. At the end of stage s we shall initialize all nodes to the right of $\delta[s]$.

At stage 0 all nodes are initialized and $X[0] = Y[0] = \mathbb{N}$, $\delta[0] = \emptyset$.

Suppose we have constructed $\delta[t]$, $X[t]$ and $Y[t]$ for $t < s$. The sets $X[s]$ and $Y[s]$ shall be obtained by allowing the strategies visited at stage s to modify the approximations $X[s-1], Y[s-1]$ obtained at the previous stage. We construct $\delta[s](n)$ with an inductive definition. Define $\delta[s](0) = \emptyset$. Suppose that we have constructed $\delta[s] \upharpoonright n$. If $n = s$, we end this stage and move on to $s+1$. Otherwise we visit the strategy $\delta[s] \upharpoonright n$ and let it determine its outcome o. Then $\delta[s](n+1) = o$. We have two cases depending on the type of the node $\delta[s] \upharpoonright n$.

(I.) If $\delta[s] \upharpoonright n = \alpha$ is a $\mathcal{P}$-node, we perform the following actions:

Let s^- be the previous α-true stage if α has not been initialized since and $s^- = s$ otherwise. The strategy α will inherit the values of its parameters from stage s^- and during its actions it can change their values several times. Thus we will omit the subscript indicating the stage when we discuss α's parameters. If the current element n_α does not need further actions we shall move on to the next element. As this is a subroutine which is frequently performed in the construction, we define it here once and for all, and we refer to it with the phrase **reset the parameters**. Denote the current values of n_α by n and of k_α by k. We *reset the parameters* by changing the values of the parameters as follows: $n_\alpha := n+1$ if $n < k$, otherwise $n = k$ and we set $k_\alpha := k+1$, $n_\alpha := 0$. In both cases we initialize the strategies extending $\alpha\hat{}\langle X, w\rangle$ and $\alpha\hat{}\langle Y, w\rangle$.

1. Let $k = k_\alpha$ and $n = n_\alpha$. Let s_n^- be the previous stage when n was examined, if α has not been initialized since, $s_n^- = s$ otherwise.

2. If $n \in \overline{K}[s]$ and $n \in \Gamma_\alpha^A[t]$ for all stages t with $s_n^- < t \leq s$ then *reset the parameters* and go to step 1.

3. If $n \in \overline{K}[s]$, but $n \notin \Gamma_\alpha^A[t]$ at some stage t with $s_n^- < t \leq s$ then:

   a.X If $Ax_\alpha^\theta(n)$ is not defined, then define it as the axiom $\langle n, A_\theta, X_\theta\rangle$ with least age $a(A[s] \oplus X[s], A_\theta \oplus X_\theta, s) \leq s_n^-$ and move on to step $a.Y$. If there is no such axiom then let the outcome be $\langle X, n\rangle$ and *reset the parameters.*

   b.X If $Ax_\alpha^\theta(n)$ is defined but was not valid at some stage t with $s_n^- < t \leq s$ then cancel its value (make it undefined) and let the outcome be $\langle X, n\rangle$, *reset the parameters*. Otherwise go to step $a.Y$.

a.Y If $Ax^{\psi}_{\alpha}(n)$ is not defined, then define it as the axiom $\langle n, A_\psi, Y_\psi\rangle$ with least age $a(A[s] \oplus Y[s], A_\psi \oplus Y_\psi, s) \leq s^-_n$ and move on to step c. If there is no such axiom then let the outcome be $\langle Y, n\rangle$ and *reset the parameters.*

b.Y If $Ax^{\theta}_{\alpha}(n)$ is defined but was not valid at some stage t with $s^-_n < t \leq s$ then cancel its value (make it undefined) and let the outcome be $\langle Y, n\rangle$, *reset the parameters.* Otherwise go to step c.

c. If both $Ax^{\theta}_{\alpha}(n) = \langle n, A_\theta, X_\theta\rangle$ and $Ax^{\psi}_{\alpha}(n) = \langle n, A_\psi, Y_\psi\rangle$ are defined and have been valid at all stages t with $s^-_n < t \leq s$ then enumerate in Γ_α the axiom $\langle n, A_\theta \cup A_\psi\rangle$. Enumerate $\langle X_\theta, Y_\psi\rangle$ in $D_\alpha(n)$. *Reset the parameters* and go back to step 1.

4. If $n \notin \overline{K}[s]$ and $n \notin \Gamma^A_\alpha[t]$ at some stage t: $s^-_n < t \leq s$ *reset the parameters* and go back to step 1.

5. Suppose $n \notin \overline{K}[s]$ but $n \in \Gamma^A_\alpha[t]$ at all t such that $s^-_n < t \leq s$. For every pair $\langle X_\theta, Y_\psi\rangle \in D_\alpha(n)$ find the highest priority $\mathcal{N}$-strategy $\beta \supset \alpha$ that has permanently restrained an element $x \in X_\theta$ out of X or $y \in Y_\psi$ out of Y. If there is such a strategy β and it has a permanent restraint on X, enumerate Y_ψ in $Y[s]$; if it has a permanent restraint on Y, enumerate X_θ back in $X[s]$. Otherwise if there is no such strategy enumerate Y_ψ back in $Y[s]$. Choose the axiom $\langle n, A_\theta \cup A_\psi\rangle$ in Γ^A_α with least age $a(A[s], A_\theta \cup A_\psi, s)$. Let X_θ and Y_ψ be the corresponding X and Y parts of the axioms $\langle n, A_\theta, X_\theta\rangle \in \Theta_\alpha$ and $\langle n, A_\psi, Y_\psi\rangle \in \Psi_\alpha$.

a. If $X_\theta \subseteq X[s]$ then this will ensure that $n \in \Theta^{A,X}_\alpha[s]$. Let the outcome be $\langle X, w\rangle$. Note that we will not reset the parameters at this point, thus the construction will keep going through this step while there is no change in A.

b. If $X_\theta \not\subseteq X[s]$ then $Y_\psi \subseteq Y[s]$ and this will ensure that $n \in \Psi_\alpha^{A,Y}[s]$. Let the outcome be $\langle Y, w \rangle$.

(II.) If $\delta[s] \upharpoonright n = \beta$ is an $\mathcal{N}$-node, we perform the following actions:

Let s^- be the previous β-true stage if β has not been initialized since, go to the step indicated at stage s^-. Otherwise $s^- = s$ and go to step 1.

1. Define x_β as a fresh number, one that has not appeared in the construction so far and is bigger than s. Go to the next step.

2. If $x_\beta \notin W_\beta[s]$ then let the outcome be w, return to this step at the next stage. Otherwise go to the next step.

3. Extract x_β from $X[s]$ and $Y[s]$. Restrain permanently x_β out of X. Let the outcome be d, go to the next step at the next stage.

4. If $x_\beta \in Y[s]$ then define y_β as a fresh number, initialize all strategies of lower priority than β and go to the next step. Otherwise the outcome is d, return to this step at the next stage.

5. If $y_\beta \notin W_\beta$ then let the outcome be w. Return to this step at the next stage. Otherwise go to the next step.

6. If y_β is not yet restrained then restrain y_β permanently out of Y and extract y_β from $Y[s]$. Let the outcome be d, return to this step at the next stage.

This completes the construction.

6.2.5 Proof

The tree is infinitely branching and therefore there is a risk that there might not be a path in the tree that is visited infinitely often. However we shall start the proof by establishing some basic facts about the relationship between strategies.

For clarity we shall define one more notation. Let α be a $\mathcal{P}$-strategy. To every axiom $Ax = \langle n, A_\theta \cup A_\psi \rangle \in \Gamma_\alpha$ we shall associate a corresponding entry $\langle n, A_\theta, X_\theta, A_\psi, Y_\psi \rangle$ so that $\langle n, A_\theta, X_\theta \rangle \in \Theta_\alpha$ and $\langle n, A_\psi, Y_\psi \rangle \in \Psi_\alpha$ are the corresponding axioms used to construct Ax.

Lemma 6.2.1. *Let β be an $\mathcal{N}$-strategy, initialized for the last time at stage s_i. If β has a witness x_β that is extracted by β at stage $s_x > s_i$ then $x_\beta \notin X[t]$ at all $t \geq s_x$. If β has a witness y_β that is extracted from Y at stage $s_y > s_x$ then $y_\beta \notin Y[t]$ at all $t \geq s_y$.*

Proof. There are only finitely many $\mathcal{N}$-strategies of higher priority than β that are ever visited in the construction as after stage s_i no strategy to the left of β is visited. Every higher priority strategy $\beta' < \beta$ that is ever visited is not initialized after stage s_i, as otherwise β would be initialized after stage s_i contrary to our assumption. We can inductively assume that the statement is valid for every higher priority strategy β'.

Suppose β chooses the witness x_β at stage $s_1 > s_i$. We can furthermore prove the following:

Claim: Any witness which is permanently extracted by a higher priority strategy β' is extracted before or at stage s_1.

Indeed, suppose that β' permanently extracts a new witness at stage $s_2 > s_1$. Then at stage s_2 the strategy β' has outcome d. Thus if $\beta >_L \beta'$ or $\beta \supseteq \beta'\hat{\ }w$ then β would be initialized at stage s_2 contrary to assumption. This leaves us with the only possibility that $\beta \supseteq \beta'\hat{\ }d$. Then at stage s_1, as β was visited, β' was visited and had outcome d. As β' is not initialized after stage s_1 and permanently extracts a new witness at stage s_2 it must be the case that β' permanently extracts a witness $y_{\beta'}$ from Y and $x_{\beta'}$ was already extracted before or at stage s_1. It follows that between stages s_1 and s_2, β' has selected this new witness $y_{\beta'}$ passing through $II.4$ of the construction and initializing all lower priority strategies including β. This leads again to a contradiction with the assumption that β is not initialized after stage s_1 and hence the claim is correct.

Thus at stage s_1 all witnesses of higher priority strategies that are ever permanently restrained out of either set X or Y are already permanently restrained out of X or Y. At stage s_1 the strategy β selects x_β as a fresh number, i.e. one that has not appeared in the construction so far. And at stage s_x the witness x_β is permanently restrained out of X.

Now we will prove again inductively but this time on the stage t, that $x_\beta \notin X[t]$ at all stages $t \geq s_x$.

So suppose this is true for $t < s_3$ and that at stage $s_3 > s_x$ a $\mathcal{P}$-strategy α is visited and reaches point $I.5$ of the construction. Suppose α wants to enumerate X_θ or Y_ψ back in X or Y respectively for the axiom $\langle n, A_\theta \cup A_\psi\rangle$ in Γ_α with corresponding entry $\langle n, A_\theta, X_\theta, A_\psi, Y_\psi\rangle$. We have the following cases to consider:

1. Suppose $\alpha > \beta$. If $\alpha >_L \beta\hat{}d$ then α is initialized at stage s_x. If $\alpha \subset \beta\hat{}d$, then α was initialized at stage s_i and was not accessible before stage s_x. Thus in both cases the axiom $\langle n, A_\theta \cup A_\psi\rangle$ was enumerated in Γ_α at stage t with $s_x \leq t < s_3$, at which both $\langle n, A_\theta, X_\theta\rangle$ and $\langle n, A_\psi, Y_\psi\rangle$ were valid i.e. $X_\theta \subseteq X[t]$ and $Y_\psi \subseteq Y[t]$. By induction $x_\beta \notin X[t]$ hence $x_\beta \notin X_\theta$ and thus α does not enumerate x_β back in X.

2. Suppose $\alpha < \beta$. If $\alpha <_L \beta$ then β would be initialized at stage s_3, hence $\alpha \subset \beta$. Suppose the axiom $\langle n, A_\theta \cup A_\psi\rangle$ was enumerated in Γ_α at stage t. If $t \leq s_1$ then by the choice of x_β as a fresh number at stage s_1 we have that $x_\beta \notin X_\theta$. If $t > s_1$ then both $\langle n, A_\theta, X_\theta\rangle$ and $\langle n, A_\psi, Y_\psi\rangle$ were valid at stage t i.e. $X_\theta \subseteq X[t]$ and $Y_\psi \subseteq Y[t]$. By $I.5$ of the construction α will consider all $\mathcal{N}$-strategies that extend it and select the one with highest priority that has permanently restrained an element out of either set X or Y.

   Consider any $\beta' < \beta$. By our *Claim* any witness $x_{\beta'}$ or $y_{\beta'}$ of β' that is ever permanently restrained out of X or Y is already restrained out at stage s_1 and by induction at all stages $s \geq s_1$ including at stage t. Thus X_θ does not contain $x_{\beta'}$ and Y_ψ does not contain $y_{\beta'}$. As this is true for an arbitrary strategy β' of higher priority than β that is ever visited, if $x_\beta \in X_\theta$ then β will be the strategy selected by α and α will choose to enumerate Y_ψ back in Y. Thus again α does not enumerate x_β back in X.

To prove the second part of the lemma suppose y_β is selected at stage s_4 and extracted at stage s_y. Because $s_1 < s_4$ and all strategies of lower priority than β are initialized at stage s_4 the interactions between β and other strategies are dealt with in the same way as in the case when we were considering x_β. The only thing left for us to establish is that β does not come into conflict with itself. So suppose that at stage $s_5 > s_y$ a $\mathcal{P}$-strategy α is visited and reaches point $I.5$ of the construction. Suppose α wants to enumerate X_θ or Y_ψ back in X or Y respectively for the axiom $\langle n, A_\theta \cup A_\psi \rangle$ in Γ_α with corresponding entry $\langle n, A_\theta, X_\theta, A_\psi, Y_\psi \rangle$. We will prove that if $x_\beta \in X_\theta$ then $y_\beta \notin Y_\psi$. Let t be the stage at which the axiom $\langle n, A_\theta \cup A_\psi \rangle$ was enumerated in Γ_α. If $t < s_4$ then $y_\beta \notin Y_\psi$ by the choice of y_β at stage s_4 as a fresh number. If $t \geq s_4 > s_x$ then we have already proved that $x_\beta \notin X[t]$. The axiom $\langle n, A_\theta, X_\theta \rangle$ was valid at stage t, thus $X_\theta \subseteq X[t]$, and hence $x_\beta \notin X_\theta$.

This completes the induction step and the proof of the lemma. □

Lemma 6.2.2. *Let α be a $\mathcal{P}$-strategy, visited infinitely often and not initialized after stage s_i. If α performs finitely many cycles then:*

1. *There is a stage $s_n \geq s_i$ after which the value of n_α does not change.*
2. *At all α-true stages $t > s_n$, α has either outcome $\langle X, w \rangle$ or outcome $\langle Y, w \rangle$.*
3. *There is a stage $s_d \geq s_n$ such that at all α-true stages $t > s_d$, α has a fixed outcome o .*
4. *If $o = \langle X, w \rangle$ then $\Theta_\alpha^{A,X} \neq \overline{K}$ and if $o = \langle Y, w \rangle$ then $\Psi_\alpha^{A,Y} \neq \overline{K}$.*

Proof. It follows from the construction and the definition of the action *reset the param-*

eters that if the value of n_α changes infinitely often, then there will be infinitely many cycles. Thus part (1) of the lemma is true. Let s_n be the stage after which the value of n_α does not change. The only case when the value of the parameter $n_\alpha = n$ is not reset is when $n \notin \overline{K}$ and $n \in \Gamma_\alpha^A[t]$ at all stages t since the last time n was examined at stage s_n^-, thus α will have only outcomes $\langle X, w\rangle$ or $\langle Y, w\rangle$ at all stages after s_n and part (2) is true. It follows from step $I.5$ of the construction and the fact that n_α does not change any longer that at all stage $t > s_n$, $n \in \Gamma_\alpha^A[t]$. By the properties of a good approximation and under these circumstances $n \in \Gamma_\alpha^A$. Then there will be an axiom $\langle n, A_\theta \cup A_\psi\rangle \in \Gamma_\alpha$ which is valid at all but finitely many stages. Select the axiom with least limit age. This axiom has corresponding entry $\langle n, A_\theta, X_\theta, A_\psi, Y_\psi\rangle$. The strategy α will eventually be able to spot this precise axiom, after possibly finitely many wrong guesses. So after a stage $s_d \geq s_n$ the strategy α will consider this axiom to select its outcome.

At stage s_n either $X_\theta \subset X[s_n]$ or $Y_\psi \subset Y[s_n]$. As we initialize all strategies below outcomes $\langle X, w\rangle$ and $\langle Y, w\rangle$ whenever we reset the parameters, we can be sure that $\mathcal{N}$-strategies visited at stages $t > s_n$ of lower priority than α will not extract any elements of $X_\theta \cup Y_\psi$ from X or Y. Higher priority $\mathcal{N}$-strategies will not extract any elements at all, otherwise α would be initialized. Thus if $X_\theta \subseteq X[s_n]$ then for all stages $t \geq s_n$ we have $X_\theta \subseteq X[t]$ and similarly if $Y_\psi \subseteq Y[s_n]$ then for all stages $t \geq s_n$ we have $Y_\psi \subseteq Y[t]$.

Suppose $X_\theta \subseteq X[s_n]$. Then at stages $t \geq s_d$ the strategy α will always have outcome $\langle X, w\rangle$. The axiom $\langle n, A_\theta, X_\theta\rangle \in \Theta_\alpha$ will be valid at all stages $t \geq s_d$, thus $n \in \Theta^{A,X}$, and $n \notin \overline{K}$.

If $X_\theta \nsubseteq X[s_n]$ then there is a strategy $\beta \supset \alpha$ which is permanently restraining some element $x \in X_\theta$ out of X at stage s_n. Then $\beta <_L \alpha\hat{\ }\langle X, w\rangle$ as strategies extending $\alpha\hat{\ }\langle X, w\rangle$ or to the right of it are in initial state at stage s_n and do not have any restraints. This strategy β will not be initialized at stages $t \geq s_n$ according to part (2) of this lemma and the choice of $s_n > s_i$. By Lemma 6.2.1 $x \notin X[t]$ at all $t \geq s_n$. Hence case $I.5.b$ of the construction is valid at all $t \geq s_d$. Thus α will have outcome $\langle Y, w\rangle$ at all stages $t \geq s_d$ and $n \in \Psi^{A,Y}$. This proves parts (3) and (4) of the lemma. □

Propostion 6.2.1. *Let α be a $\mathcal{P}$-strategy, visited infinitely often and not initialized after stage s_i. If v is an element such that $\Gamma_\alpha^A(v) = \overline{K}(v)$ then there is a stage s_v after which the outcomes $\langle X, v\rangle$ and $\langle Y, v\rangle$ are not accessible any longer.*

Proof. If α has finitely many cycles then by Lemma 6.2.2 there will be a stage s_n after which $\langle X, v\rangle$ and $\langle Y, v\rangle$ are not accessible. Suppose there are infinitely many cycles.

If $v \notin \overline{K}$ then there is a stage s_v at which v exits $\overline{K}$. Then after stage s_v the outcomes $\langle X, v\rangle$ and $\langle Y, v\rangle$ are not accessible.

If $v \in \Gamma_\alpha^A$ then there is an axiom in Γ_α that is valid at all but finitely many stages, say at all stages $t \geq s'_v$. If α is on its k-th cycle during stage s'_v then let s_v be the beginning of the $(k+2)$-nd cycle. Then after stage s_v, whenever α considers v, part $I.2$ of the construction holds and hence α will never have outcome $\langle X, v\rangle$ or $\langle Y, v\rangle$. □

Lemma 6.2.3. *Let α be a $\mathcal{P}$-strategy, visited infinitely often and not initialized after stage s_i. If α performs infinitely many cycles, then there is leftmost outcome $o <_L \langle X, w\rangle$ that α has at infinitely many stages and*

1. *If $o = \langle X, u\rangle$ then $\Theta_\alpha^{A,X}(u) \neq \overline{K}(u)$.*
2. *If $o = \langle Y, u\rangle$ then $\Psi_\alpha^{A,Y}(u) \neq \overline{K}(u)$.*

Proof. The set A is not complete by assumption, hence $\Gamma_\alpha^A \neq \overline{K}$. Let u be the least difference between the sets. By Proposition 6.2.1 for every $v < u$ the outcomes $\langle X, v\rangle$ and $\langle Y, v\rangle$ are not visited at stages $t > s_v$. Let s_0 be a stage bigger than $\max_{v<u}(s_v)$. As α has infinitely many cycles there will be infinitely many stages $t > s_0$ at which $n_\alpha[t] = u$. If $u \notin \overline{K}$ and $u \in \Gamma_\alpha^A$ then there is a stage $s_1 > s$ such that at all stages $t > s_1$ we have $u \in \Gamma_\alpha^A[t]$ and $u \notin \overline{K}[t]$ and when α considers u at the first stage after s_1, it will never move on to the next element, and α would have finitely many cycles. Hence $u \in \overline{K}$ and $u \notin \Gamma_\alpha^A$.

1. If $u \notin \Theta_\alpha^{A,X}$ then all axioms for u in Θ_α are invalid at infinitely many stages. Let t be any stage greater than or equal to s_0. We will prove that there is a stage $t' \geq t$ at which α has outcome $\langle X, u\rangle$. As $u \notin \Gamma_\alpha^A$ and $\{A[s]\}_{s<\omega}$ is a good approximation to A there are infinitely many stages s at which $u \notin \Gamma_\alpha^A[s]$ and hence part $I.3$ of the construction holds at infinitely many stages at which we consider u. Let $t_1 \geq t$ at which $n_\alpha[t_1] = u$ and part $I.3$ of the construction is true. If $Ax_\alpha^\theta(u)$ is not defined and we are not able to define it as there is no appropriate axiom in Θ_α valid for long enough then α will have outcome $\langle X, u\rangle$ at stage t_1, hence $t' = t_1$ proves the claim. Otherwise $Ax_\alpha^\theta(u)$ is defined at stage t_1 and by assumption there are infinitely many stages $t \geq t_1$ at which it is invalid. Let $t_2 > t_1$ be the next stage when $Ax_\alpha^\theta(u)$ is invalid and let $t' \geq t_2$ be the first stage after t_2 at which again $n_\alpha[t'] = u$ and part $I.3$ of the construction is true. By $I.3.b.X$ of the construction α will have outcome $\langle X, u\rangle$ at stage t'.

2. Now assume that $u \in \Theta_\alpha^{A,X}$. Then there is an axiom $\langle u, A_\theta, X_\theta\rangle \in \Theta_\alpha$ valid at all but finitely many stages. Select the axiom, say Ax, with least limit age. Then $Ax_\alpha^\theta(u)$ will have a permanent value Ax after a certain stage s_1. It follows that $u \notin \Psi_\alpha^{A,Y}$ as otherwise we would be able to find an axiom in $\Psi_\alpha^{A,Y}$ valid at all but finitely many stages, and construct an axiom in Γ_α valid at all but finitely many stages. Now a similar argument as the one used in part (1) of this lemma proves that α will have outcome $\langle Y, u\rangle$ at infinitely many stages. □

As an immediate corollary from Lemmas 6.2.2 and 6.2.3 we obtain the existence of the true path:

Corollary 6.2.1. *There exists an infinite path through the tree of strategies with the following properties:*

1. $(\forall n)(\exists^\infty s)[h \upharpoonright n \subseteq \delta[s]]$;
2. $(\forall n)(\exists s_l(n))(\forall t > s_l(n))[\delta[t] \not<_L h \upharpoonright n]$;
3. $(\forall n)(\exists s_i(n))(\forall t > s_i(n))[h \upharpoonright n$ *is not initialized at stage* $t]$.

Proof. We will define the true path with induction on n and prove that it has the properties needed. The case $n = 0$ is trivial: $h \upharpoonright 0 = \emptyset$ is visited at every stage of the construction and is never initialized, $s_l(0) = s_i(0) = 0$. Suppose we have constructed $h \upharpoonright n$ with the required properties. We shall define $h \upharpoonright (n+1)$.

If $h \upharpoonright n = \beta$ is an $\mathcal{N}$-strategy then let $o \in \{d, w\}$ be the leftmost outcome that β has at infinitely many stages. The design of the strategy ensures that there is a stage $s_o > s_i(n)$ such that $\beta\hat{}o \subseteq \delta[t]$ at all $t \geq s_o$. If β does not define y_β after stage $s_i(n)$ then s_o is the first stage after $s_i(n)$ at which β has outcome o. If β defines y_β at stage s_y then s_o is the first stage after s_y at which β has outcome o. We define $h \upharpoonright (n+1) = \beta\hat{}o$ and $s_i(n+1) = s_o$.

Suppose $h \upharpoonright n = \alpha$ is a $\mathcal{P}$-strategy. If α performs finitely many cycles then by Lemma 6.2.2 there is a stage $s_o > s_i(n)$ after which α does not reset the parameters and has the same fixed outcome o. We define $h \upharpoonright (n+1) = \alpha\hat{}o$ and $s_i(n+1) = s_o$.

If α performs infinitely many cycles then by Lemma 6.2.3 there is a leftmost outcome $o <_L \langle X, w\rangle$ that α has at infinitely many stages. Let $s_o > s_i(n)$ be a stage such that at stages $t > s_o$ the strategy α does not have outcomes $o' <_L o$. Then $h \upharpoonright (n+1) = \alpha\hat{}o$ and $s_i(n+1) = s_o$. □

Corollary 6.2.2. *X and Y are not c.e.*

Proof. For every requirement $\mathcal{N}_e$ there is an $\mathcal{N}_e$-strategy β along the true path, visited infinitely often and not initialized at any stage $t > s_i$. Let x_β and y_β be the final values of β's witnesses. If $\beta\hat{}w \subset h$ then there is an element $u \in \{x_\beta, y_\beta\}$ that never enters W_e. The way each $\mathcal{N}_e$-strategy chooses its witnesses ensures that only β can extract u from either of the sets X or Y. The construction and the definition of the true path ensure that β does not extract u from X and Y at any stage. Hence $u \in X \cap Y$ and $u \notin W_e$.

If $\beta\hat{}d \subset h$ then $x_\beta \in W_e$ and there is a β-true stage s_x at which β extracts x_β from X and Y. By Lemma 6.2.1 $x_\beta \notin X[t]$ at all stages $t \geq s_x$. If at any stage $t \geq s_x$ we have that $x_\beta \in Y[t]$ then β selects y_β at its next true stage. As the true outcome is d, $y_\beta \in W_e[t']$ at some stage $t' \geq t$. Then at the next β-true stage $s_y \geq t'$ the strategy β will permanently restrain y_β out of Y and by Lemma 6.2.1, we have that $y_\beta \notin Y$. $\square$

Corollary 6.2.3. *$A \oplus X \not\equiv_e \overline{K}$ or $A \oplus Y \not\equiv_e \overline{K}$.*

Proof. Consider the $\mathcal{P}$-nodes on the true path. From the definition of the tree it follows that either for every $\mathcal{P}^0_e$-requirement there is a node on the tree α which is associated with $\mathcal{P}^0_e$ or there is a fixed requirement $\mathcal{P}^0_e$ associated with all but finitely many nodes. In the latter case there is a node on the true path for every $\mathcal{P}^1_e$-requirement.

Suppose there is a node on the tree for each $\mathcal{P}^0_e$-requirement. We can show that $A \oplus X \not\equiv_e \overline{K}$. Assume for a contradiction $\Theta^{A,X}_e = \overline{K}$ and let $\alpha \subset h$ be the last node associated with $\mathcal{P}^0_e$. Then α has true outcome $\langle X, u\rangle$ for some $u \in \omega \cup \{w\}$. It follows from Lemma 6.2.2 and Lemma 6.2.3 that $\Theta^{A,X}_e \neq \overline{K}$.

The case when there is a node for every $\mathcal{P}^1_e$-requirement yields by a similar argument that $A \oplus Y \not\equiv_e \overline{K}$. $\square$

Lemma 6.2.4. *The sets X and Y are 3-c.e.*

Proof. We can easily obtain a 3-c.e. approximation of each of the sets X and Y from the one constructed. Define $\widehat{X}[s] = X[s] \restriction s$ and $\widehat{Y}[s] = Y[s] \restriction s$.

It follows from the construction that elements extracted from X and Y are necessarily witnesses of $\mathcal{N}$-strategies. Suppose therefore that n is the witness x_β for an $\mathcal{N}$-strategy β. Then n appears in the defined approximations $\{\widehat{X}[s]\}_{s<\omega}$ and $\{\widehat{Y}[s]\}_{s<\omega}$ at stage $n+1$. If β never extracts x_β then we are done - as no other strategy can extract it. If β extracts x_β then it does so only once at stage s_x when it goes through $II.3$ and moves on to $II.4$ at the next stage. In order for β to return to step $II.3$ of the construction it will have to be initialized and will select new witnesses. Thus after its extraction at stage s_x from both $\widehat{X}[s_x]$ and $\widehat{Y}[s_x]$, the number x_β can only be enumerated back in either set and hence $|\{\ s|\ \widehat{X}[s-1](x_\beta) \neq \widehat{X}[s](x_\beta)\}| \leq 3$ and $|\{\ s|\ \widehat{Y}[s-1](x_\beta) \neq \widehat{Y}[s](x_\beta)\}| \leq 3$.

If n is the witness y_β then it will never be extracted from X. If it is ever extracted from Y it is extracted only once by β at the first stage it reaches step $II.6$. After that y_β is already restrained by β and whenever β executes step $II.6$ it will ignore the first sentence of the instruction and just have outcome $o = d$. Thus again $|\{\ s|\ \widehat{Y}[s-1](y_\beta) \neq \widehat{Y}[s](y_\beta)\}| \leq 3$.

This concludes the proof of the lemma and of the theorem. □

Chapter 7

A Non-splitting Theorem for the 3-c.e. Enumeration Degrees

The final chapter of this thesis returns us to the non-splitting theme which was the main topic of Chapter 2 and Chapter 3. Lachlan [Lac75] shows the existence of pairs of c.e. Turing degrees $\mathbf{a} > \mathbf{b}$ such that $\mathbf{a}$ cannot be split by a pair of c.e. degrees above $\mathbf{b}$. The significance of Lachlan's non-splitting theorem for the disclosure of the complexity of the c.e. Turing degrees is enough motivation to search for an analog of this result in the enumeration degrees. We have already seen that the Σ^0_2 enumeration degrees have an even stronger property, an analog of Harrington's non-splitting theorem, [Har80]. In Theorem 3.0.1 we showed that there is an incomplete Σ^0_2 enumeration degree $\mathbf{a}$ such that $\mathbf{0}'_e$ cannot be split by any pair of Σ^0_2 enumeration degrees above $\mathbf{a}$. On the other hand we have Ahmad and Lachlan's [AL98] proof of the existence of a nonzero Δ^0_2 enumeration degree $\mathbf{a}$ that is not the least upper bound of any pair of enumeration degrees below it. Thus we have a stronger analog of Lachlan's non-splitting theorem for the Δ^0_2 enumeration degrees in the other extreme: the second element in the pair can be taken to be $\mathbf{0}_e$.

Our classification of the Δ^0_2 enumeration degrees based on Ershov's hierarchy demands investigation of the non-splitting properties for every class of n-c.e. enumeration degrees, where $n \leq \omega$. Transferring the relativized version of Sack's splitting theorem [Sac63] we see that every Π^0_1 enumeration degree has a Δ^0_2-splitting above each lesser Π^0_1 enumeration degree, thus non-splitting fails for the class of the Π^0_1 enumeration degrees. We shall therefore consider the second class of enumeration degrees in our hierarchy - the class of all 3-c.e. enumeration degree. Arslanov and Sorbi [AS99] show that we cannot prove an analog of Harrington's result for any subclass of the Δ^0_2 enumeration degrees as there is a Δ^0_2 splitting of $\mathbf{0}'_e$ above every incomplete Δ^0_2 enumeration degree. Kalimullin [Kal02] shows that every nonzero n-c.e. degree has a non-trivial splitting. Thus we cannot have a strong non-splitting property for the class of 3-c.e. enumeration degrees in either extreme. In this chapter we prove that we nevertheless have an analog of Lachlan's non-splitting theorem for the 3-c.e. enumeration degrees.

Theorem 7.0.1. *There exists a pair of a Π^0_1 enumeration degree $\mathbf{a}$ and a 3-c.e. enumeration degree $\mathbf{b} < \mathbf{a}$ such that $\mathbf{a}$ cannot be split by a pair of enumeration degrees above $\mathbf{b}$.*

Kent has pointed out (in a private conversation) that his construction, [Ken05], of a nonzero degree that is not the least upper bound of any two lesser degrees actually gives an ω-c.e. enumeration degree. Thus the strong version of Lachlan's theorem, where the second element of the pair is $\mathbf{0}_e$, is valid for the ω-c.e. enumeration degrees.

The work presented in this chapter is joint with Marat Arslanov, S. Barry Cooper and Iskander Kalimullin. An extended abstract of Theorem 7.0.1 is published in [ACKS08], see Appendix A.3.

7.1 Requirements and strategies

Recall that Cooper [Coo90] showed that the class of the Π^0_1 enumeration degrees coincides with the class of the 2-c.e. enumeration degrees. We shall therefore construct a

2-c.e. set A and 3-c.e. set B satisfying the following list of requirements:

1. We have a global requirement which ensures that $B \leq_e A$ via an enumeration operator Ω constructed by us:
$$\mathcal{S} : B = \Omega^A.$$

2. To ensure the non-splitting property of the degree of A consider a computable enumeration of all triples of enumeration operators $\{(\Xi, \Psi, \Theta)_i\}_{i<\omega}$. We denote the members of the i-th triple by Ξ_i, Ψ_i and Θ_i. For every i we shall have a requirement:
$$\mathcal{P}_i : A = \Xi_i^{\Psi_i^A, \Theta_i^A} \Rightarrow (\exists \Gamma_i, \Lambda_i)[A = \Gamma_i^{\Psi_i^A, B} \vee A = \Lambda_i^{\Theta_i^A, B}].$$

3. Finally we need to ensure that the degree of A is strictly greater than the degree of B. Let $\{\Phi_e\}_{e<\omega}$ be a computable enumeration of all enumeration operators. For every e we shall have a requirement:
$$\mathcal{N}_e : A \neq \Phi_e^B.$$

The requirements shall as usual be given a priority ordering:
$$\mathcal{S} < \mathcal{P}_0 < \mathcal{N}_0 < \mathcal{P}_1 < \mathcal{N}_2 < \dots$$

In the course of the construction whenever we enumerate an element in the set B, we will enumerate a corresponding axiom in the set Ω. Whenever we extract an element from B, we invalidate the corresponding axiom by extracting an element from A. Thus the global requirement $\mathcal{S}$ shall be satisfied without an explicit strategy on the tree ensuring this. More precisely every element n that enters B will be assigned a marker $\omega(n)$ in A and an axiom $\langle n, \{\omega(n)\}\rangle$ in Ω. If n is extracted from B then we extract $\omega(n)$ from A. This can happen only once as we will be constructing a 3-c.e. approximation to the set B. If n is later re-enumerated in B, it will remain in B forever and we can just enumerate the axiom $\langle n, \emptyset\rangle$ in Ω.

To satisfy a $\mathcal{P}$-requirement working with the triple (Ξ, Ψ, Θ) we will initially attempt to reduce A to the set $\Psi^A \oplus B$ by constructing an e-operator Γ to witness this. In this case as well the enumeration of elements in A is always accompanied by an enumeration of axioms in Γ, and extraction of elements from A can be rectified via B-extractions.

The $\mathcal{N}$-strategies follow a variant of the Friedberg-Muĉnik strategy while at the same time respecting the rectification of the operators constructed by higher priority strategies. We shall use labels for $\mathcal{N}$-strategies which clarify with respect to which constructed operators they work. An $\mathcal{N}$-strategy working with respect to the initial $\mathcal{P}$-strategy, for example, shall be denoted by $(\mathcal{N}, \Gamma)$. The $(\mathcal{N}, \Gamma)$-strategy working with the operator Φ shall choose a witness x, enumerate it in A and then wait until $x \in \Phi^B$. If this happens it shall extract the element x from A while restraining $B \restriction use(\Phi, B, x)$ in B, see Definition 1.3.1.

The need to rectify Γ after the extraction of the witness x from A can be in conflict with the restraint on B. To resolve this conflict we try to obtain a change in the set Ψ^A which would enable us to rectify Γ without any extraction from the set B. We introduce an explicit $\mathcal{P}$-strategy on the tree whose only job will be to monitor the length of agreement $l(\Xi^{\Psi^A, \Theta^A}, A)[s]$, see Definition 2.1.1, at every stage s. The $(\mathcal{N}, \Gamma)$-strategy will proceed with actions directed at a particular witness once it is below the length of agreement. This ensures that the extraction of x from A will have one of the following consequences.

1. The length of agreement will never return to its previous value as long as at least one of the axioms that ensure $x \in \Xi^{\Psi^A, \Theta^A}$ remains valid. In this case the $\mathcal{P}$-requirement is satisfied and we can use the simple FM-strategy for $\mathcal{N}$.

2. The length of agreement returns and there is a useful extraction from the set Ψ^A rectifying Γ. The $\mathcal{P}$-strategy remains intact while the $(\mathcal{N}, \Gamma)$-strategy is successful.

3. The length of agreement returns and there is an extraction from the set Θ^A.

We will initially assume that the third consequence is true and commence a backup strategy $(\mathcal{N}, \Lambda)$ which is devoted to building an enumeration operator Λ attempting to reduce A to $\Theta^A \oplus B$. This strategy will work with the same witness which it receives from $(\mathcal{N}, \Gamma)$. It will use the change in Θ^A in order to satisfy its own requirement. Only when we are provided with evidence that our assumption is wrong will we return to the initial strategy $(\mathcal{N}, \Gamma)$-strategy.

7.2 Simple cases

To provide the reader with more intuition about the construction we shall discuss a few simpler cases before we proceed with the general construction. We start off with the simplest case of just one $\mathcal{N}$-requirement below one $\mathcal{P}$-requirement. Then we shall explain how we can deal with all $\mathcal{N}$-requirements below a single $\mathcal{P}$-requirement. Finally we will discuss how to handle an $\mathcal{N}$-requirement working with respect to two $\mathcal{P}$-requirements.

7.2.1 One $\mathcal{N}$-requirement below one $\mathcal{P}$-requirement

Consider a $\mathcal{P}$-requirement associated with the triple (Ξ, Ψ, Θ) and an $\mathcal{N}$-requirement associated with the enumeration operator Φ. We describe the strategies associated with each requirement and at the same time define the first few levels of the tree of strategies.

The $(\mathcal{P}, \Gamma)$-strategy

The root of the tree is associated with the $(\mathcal{P}, \Gamma)$-strategy. We will denote it by α. It will have two outcomes $e <_L l$. At stage s the strategy α will monitor all elements $x \notin A[s]$. If there is an element $x \notin A[s]$ such that $x \in \Gamma^{\Psi^A, B}[s]$ then the operator Γ cannot be rectified. We shall later see that this yields $x \in \Xi^{\Psi^A, \Theta^A}[s]$ and the $\mathcal{P}$-requirement is satisfied. The strategy α shall have outcome l in this case. Strategies working below this outcome will follow the simple FM-strategy described in Section 1.3. If for every element $x \notin A \Rightarrow x \in \Gamma^{\Psi^A, B}$ the strategy shall have outcome e and the $(\mathcal{N}, \Gamma)$-strategy shall be activated.

At stage s the strategy α acts as follows:

1. Scan all witnesses $x \notin A[s]$ defined at stages $t \leq s$.

2. If $x \in \Gamma^{\Psi^A, B}[s]$, then let the outcome be $o = l$.

3. If all witnesses are scanned and none has produced an outcome $o = l$, then let the outcome be $o = e$.

The $(\mathcal{N}, \Gamma)$-strategy

The $\mathcal{N}$-requirement below outcome e will be assigned to an $(\mathcal{N}, \Gamma)$-strategy denoted by β. It will have four outcomes: three finitary outcomes, f, w and l, and one infinitary outcome g, arranged in the following way: $g <_L f <_L w <_L l$.

The strategy first defines a witness x, enumerates it in the set A and then waits for this witness to enter the set Ξ^{Ψ^A, Θ^A}. While it waits the outcome is l indicating a global win for the $\mathcal{P}$-requirement as $A(x) \neq \Xi^{\Psi^A, \Theta^A}(x)$.

If the witness x enters the set Ξ^{Ψ^A, Θ^A} then there is a valid axiom of the form $\langle x, G(x) \oplus H(x) \rangle \in \Xi$ with $G(x) \subseteq \Psi^A$ and $H(x) \subseteq \Theta^A$. The strategy β shall then define a B-marker for x, $\gamma(x)$ and enumerate it in the set B. This is accompanied by enumerating a corresponding axiom for $\gamma(x)$ in Ω. Then it shall define a new axiom for x in Γ of the form $\langle x, G(x) \oplus (B \upharpoonright \gamma(x)+1) \rangle$. The strategy is now finally ready to execute the FM-strategy: while $x \notin \Phi^B$ it has outcome w. Finally if $x \in \Phi^B$ the strategy shall perform *capricious destruction* on the operator Γ by extracting the marker $\gamma(x)$ from B. Then instead of extracting the witness x from the set A, it shall *send* the witness x to a backup $(\mathcal{N}, \Lambda)$-strategy which will be described in detail later and have outcome g. After this β starts a new cycle with a new witness x_1. As the old witness x is still in the set A but has no valid axiom in the operator Γ, the strategy shall rectify the operator Γ at x, using the axiom that will be defined for the new witness x_1. If the old witness x is later returned by the backup strategy then it was extracted from the set A with no useful extraction from the set $H(x)$. Thus if $x \notin \Xi^{\Psi^A,\Theta^A}$ then there is a useful extraction in $G(x)$. The strategy β shall then restore the set B by reenumerating the marker $\gamma(x)$. If at the next stage the $(\mathcal{P},\Gamma)$-strategy α does not see a global win for its requirement then $G(x) \nsubseteq \Psi^A$, the operator Γ is rectified and β can successfully preserve $x \in \Phi^B \setminus A$ at further stages. It will have outcome f in this case.

Every witness or marker that we define shall be selected as a fresh number, one that has not yet appeared in the construction so far under any form.

At stage s the strategy β will initially start its work at *Setup* and then later from the step of the module indicated at the previous stage.

- **Setup:**
  1. Choose a new current witness x as a fresh number. Enumerate x in $A[s]$.
  2. If $x \notin \Xi^{\Psi^A,\Theta^A}[s]$ then let the outcome be l and return to this step at the next stage. Otherwise define $G(x)$ and $H(x)$ to be finite sets such that $x \in \Xi^{G(x),H(x)}[s]$, $G(x) \subseteq \Psi^A[s]$, $H(x) \subseteq \Theta^A[s]$. Go to the next step.
  3. Define the B-marker $\gamma(x)$, along with its A-marker $\omega(\gamma(x))$, as fresh numbers. Enumerate $\gamma(x)$ in $B[s]$ and $\omega(\gamma(x))$ in $A[s]$. Enumerate a new axiom $\langle \gamma(x), \{\omega(\gamma(x))\}\rangle$ in $\Omega[s]$.

     Enumerate each $\langle z, G_x \oplus (B \upharpoonright \gamma(x)+1)\rangle$ in Γ, where $z \in A[s]$ is either x, or $\omega(\gamma(x))$, or a witness from a previous cycle of the strategy for which there is no valid axiom in Γ. This axiom for x shall be called *the main axiom* for x in Γ. Let the outcome be $o = w$. Go to $Waiting$ at the next stage.
- **Waiting:** If $x \in \Phi^B[s]$ then go to *Attack*. Otherwise let the outcome be $o = w$ and return to $Waiting$ at the next stage.

- **Attack:**

  1. Check if any previously sent witness has been returned. If so go to *Result*. Otherwise go to the next step.

  2. Define $\lambda(x) = \max(use(\Phi, B, x)[s], \gamma(x) + 1)$ and $R[s] = \gamma(x)$. Extract $\gamma(x)$ from $B[s]$ and $\omega(\gamma(x))$ from $A[s]$. Note that the extraction of $\omega(\gamma(x))$ does not injure $x \in \Xi^{\Psi^A, \Theta^A}[s]$ as the marker is defined as a fresh number larger than $\max(use(\Psi, A, G(x)), use(\Theta, A, H(x)))$.

     Send x. Let the outcome be $o = g$. At the next stage start from *Setup*, choosing a new current witness. The strategy working below outcome g will work under the assumption that B does not change below the right boundary $R[s]$.

- **Result:** Let the returned witness be x. Enumerate $\gamma(x)$ back in $B[s]$ and $\langle \gamma(x), \emptyset \rangle$ in $\Omega[s]$. *Cancel* each witness $z \in A[s]$ of this strategy by enumerating the axiom $\langle z, \emptyset \rangle$ in $\Gamma[s]$. Let the outcome be $o = f$. Return to *Result* at the next stage.

The backup strategies

We have two backup strategies: a $(\mathcal{P}, \Lambda)$-strategy $\widehat{\alpha}$ and an $(\mathcal{N}, \Lambda)$-strategy $\widehat{\beta}$.

The $(\mathcal{P}, \Lambda)$-strategy $\widehat{\alpha}$ will only monitor the status of the sent witnesses. If it spots a witness that is ready to be sent back it will do so ending the stage prematurely. It has only one outcome e. At stage s it operates as follows:

1. Scan all sent witnesses $x \notin A[s]$.

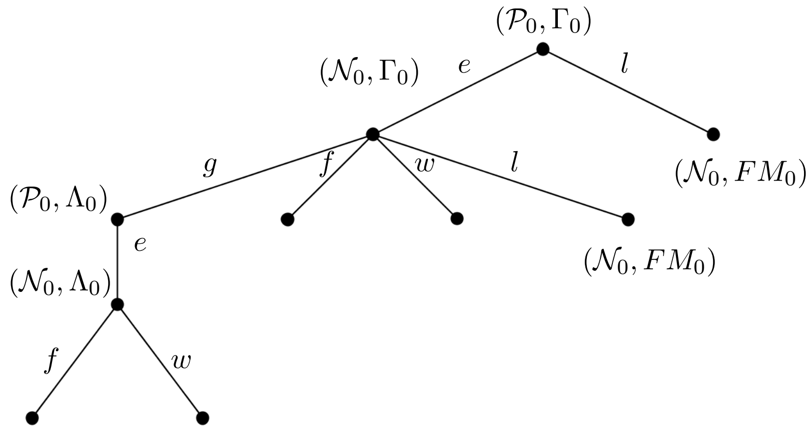
2. If $x \in \Lambda^{\Theta^A, B}[s]$ then return x. End this stage.

3. If all witnesses are scanned and none are returned then let the outcome be e.

The $(\mathcal{N}, \Lambda)$-strategy $\widehat{\beta}$ shall wait for an available witness x to be sent by β. It shall enumerate the axiom $\langle x, H(x) \oplus (B \upharpoonright \lambda(x)) \rangle$ in the operator Λ and carry on with the usual FM-strategy: wait for $x \in \Phi^B$ with outcome w, then extract x from A. If this does not entail a useful extraction from the set $H(x)$ then $\widehat{\alpha}$ shall send the witness x back and $\widehat{\beta}$ shall not be accessible at further stages. If $\widehat{\beta}$ is visited again then it shall have outcome f. At stage s the $(\mathcal{N}, \Lambda)$-strategy $\widehat{\beta}$ operates as follows:

- **Setup:** Let $x \in A[s]$ be a new witness which was sent by the $(\mathcal{N}, \Gamma)$-strategy. Now x becomes the *witness* of the $(\mathcal{N}, \Lambda)$-strategy. Enumerate $\langle x, H(x) \oplus (B[s] \upharpoonright \lambda(x) + 1)\rangle$ in $\Lambda[s]$. This is *the main axiom* for x in Λ. Go to $Waiting$.
- **Waiting:** If $x \in \Phi^B[s]$ and $use(\Phi, B, x)[s] < R[s]$ then go to $Attack$. Otherwise the outcome is $o = w$, return to $Waiting$ at the next stage.
- **Attack:** Extract x from $A[s]$. Go to *Result*.
- **Result:** Let the outcome be $o = f$. Return to *Result* at the next stage.

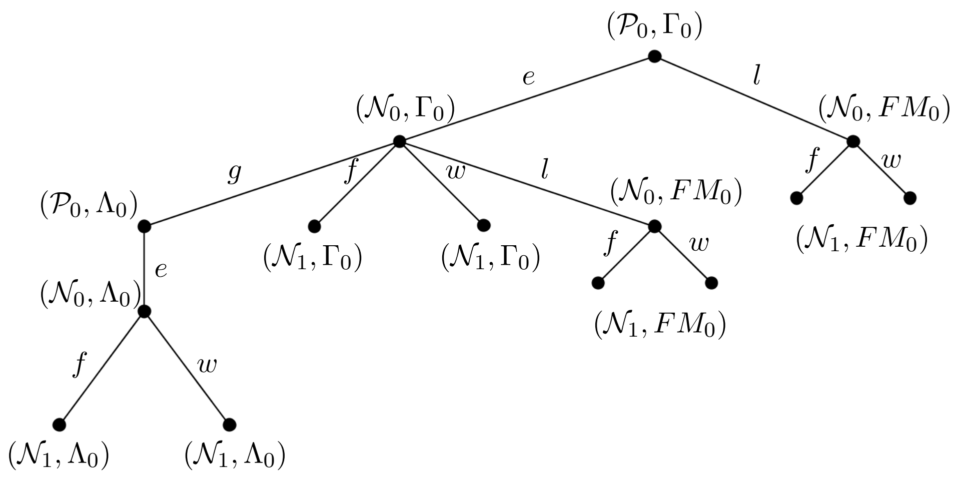
The next picture shows the first few levels of the tree of strategies:



When we inspect the tree in detail we notice that we might visit an $(\mathcal{N}, FM)$-strategy on several occasions, allow it to enumerate its own witness in the set A and then initialize it. In the design of the operators Γ and Λ we have neglected to enumerate axioms for such elements. If the $(\mathcal{N}, FM)$-strategy manages to extract from A its witness before it is initialized then this will not cause any errors in the constructed operators. If the element is still in A then we could have a problem. To avoid this every time we initialize an $(\mathcal{N}, FM)$-strategy we will enumerate axioms $\langle x, \emptyset \rangle$ in both Γ and Λ for every witness x of this strategy which is not extracted from the set A. This extra action will keep Γ and Λ always rectified.

7.2.2 Many $\mathcal{N}$-strategies below one $\mathcal{P}$-strategy

To incorporate a further $\mathcal{N}$-strategy in the construction described in the previous section we use the same basic ideas. The second $\mathcal{N}$-requirement $\mathcal{N}_1$ shall be assigned to an $(\mathcal{N}_1, FM)$-strategy below the l-outcomes of both α and β. Below β's outcomes w and f we have $(\mathcal{N}_1, \Gamma)$-strategies $\beta\hat{\ }w$ and $\beta\hat{\ }f$ which operate just like the strategy β described in Section 7.2.1. Similarly below the outcome f and w of the backup strategy $\widehat{\beta}$ we have $(\mathcal{N}_1, \Lambda)$-strategies $\widehat{\beta}\hat{\ }w$ and $\widehat{\beta}\hat{\ }f$ which operate just like the strategy $\widehat{\beta}$.



We only need to take extra care to keep the constructed operators Γ and Λ rectified at elements enumerated in A by strategies that are later initialized. Firstly we will use the initialization rule inspired by the $(\mathcal{N}, FM)$-strategy described in the previous section. Whenever we initialize an $\mathcal{N}$-strategy α we will enumerate axioms $\langle x, \emptyset \rangle$ in all operators constructed by higher priority strategies $\beta < \alpha$ for every witness x of α which is not extracted from the set A.

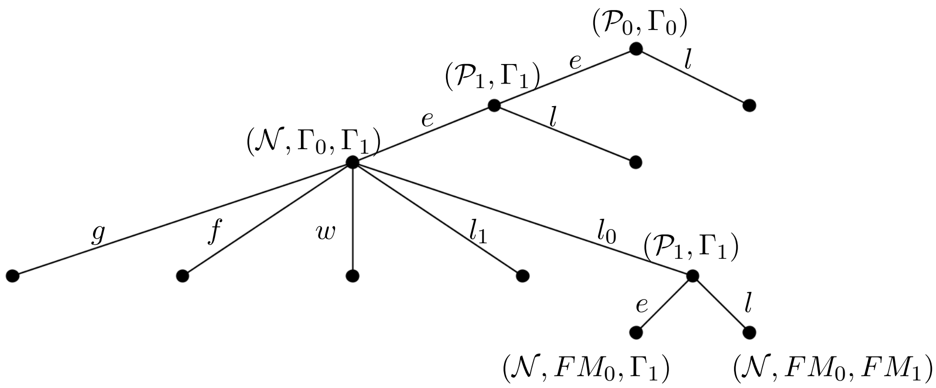
This action is sufficient if the initialized strategy does not enumerate axioms in any of the constructed operators. An $(\mathcal{N}, \Gamma)$-strategy such as $\beta\hat{\ }w$ or $\widehat{\beta}\hat{\ }w$ however enumerates axioms in the operator Γ. When it is initialized it will stop monitoring the correctness of Γ at its witnesses. We will therefore enumerate an axiom $\langle z, \emptyset \rangle$ in Γ if $z \in A$ is a witness of the initialized strategy or an Ω-marker defined by this strategy.

If a witness of the initialized strategy is already extracted from the set A we need to ensure that there are no valid axioms for it in Γ. We will modify the axioms a bit to ensure this. We will transfer the responsibility for the rectification of an operator at witnesses of initialized strategies to the strategy which initializes them. We notice that an $\mathcal{N}$-strategy such as β initializes the $(\mathcal{N}, \Gamma)$-strategies below its outcome w only when it invalidates an axiom for its witness. The axiom for this witness will continue to be invalid at all further stages at which β is visited. So whenever we define an axiom for a witness x of a strategy extending $\beta\hat{\ }w$ it shall have the form $\langle x, G(x) \oplus (B \restriction \gamma(x)+1) \cup U \rangle$, where U is the union of all sets D such that $\langle v, D \rangle$ is a valid axiom in Γ and $v \in A$ is a witness of a higher priority $(\mathcal{N}, \Gamma)$ strategy constructing the same operator Γ. Thus if β with current witness v initializes the strategies extending $\beta\hat{\ }w$ which had enumerated an axiom for a witness x, then this axiom contains an axiom for v which will be invalid at further stages, making the axiom for x invalid as well.

Similarly the axioms enumerated in Λ shall have the form $\langle x, (H(x) \oplus B \restriction \lambda(x)) \cup U \rangle$, where U is the union of all finite sets D such that $\langle v, D \rangle \in \Lambda$ and $v \in A$ is a witness of a higher priority $(\mathcal{N}, \Lambda)$-strategy, constructing the same operator Λ.

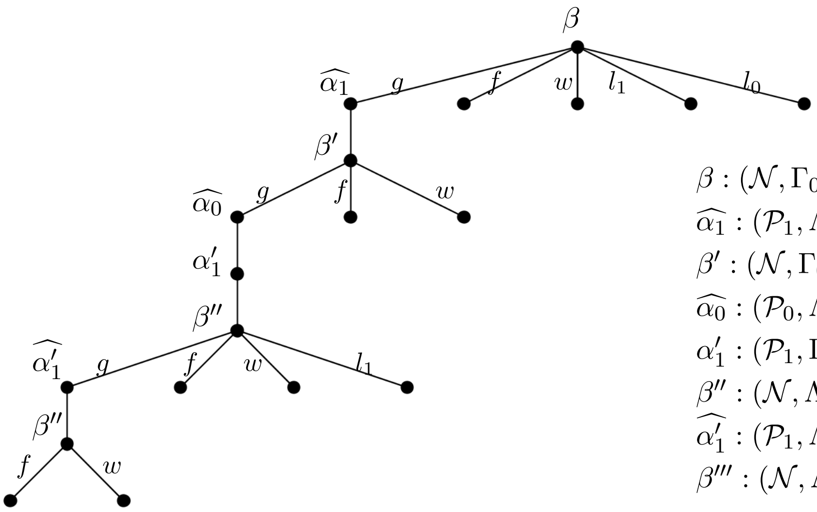
7.2.3 One $\mathcal{N}$-requirement below two $\mathcal{P}$-requirements

Before we present the full construction we shall discuss the design of an $\mathcal{N}$-strategy working with respect to two $\mathcal{P}$-requirements. Each new $\mathcal{P}_i$-requirement is initially assigned a $(\mathcal{P}_i, \Gamma_i)$-strategy as described in Section 7.2.1. Suppose we have two such successive strategies α_0 and α_1 working on the requirements $\mathcal{P}_0$ and $\mathcal{P}_1$ and with the operators Γ_0 and Γ_1, respectively. The most general of the strategies for an $\mathcal{N}$-requirement below $\mathcal{P}_0$ and $\mathcal{P}_1$ is the one placed below both e-outcomes, denote it by β. This is an $(\mathcal{N}, \Gamma_0, \Gamma_1)$-strategy which now needs to respect the rectification of both constructed operators Γ_0 and Γ_1.



The strategy β selects a witness x which is enumerated in A. Before x can start its journey along the tree β needs to setup its axioms in both operators Γ_0 and Γ_1. The setup module comes in two copies, one for each operator. The rectification of the operator Γ_0 has higher priority, so β first tries to find a valid axiom for x in $\Xi_0^{\Psi_0^A, \Theta_0^A}$. If the strategy is unsuccessful it has true outcome l_0 and $\mathcal{P}_0$ is globally satisfied. The operator Γ_1 will remain unrectified at this point and therefore we need to restart the $\mathcal{P}_1$-strategy below outcome l_0. Once the sets $G_0(x)$ and $H_0(x)$ are successfully defined the strategy defines the markers $\gamma_0(x)$ and $\omega(\gamma_0(x))$ and enumerates the necessary axioms in the operators Γ_0 and Ω. The strategy β then proceeds to search for a valid axiom for x in $\Xi_1^{\Psi_1^A, \Theta_1^A}$. If it cannot find such an axiom the outcome is l_1, $\mathcal{P}_1$ is satisfied and the operator Γ_0 is correct. After β has successfully defined the sets $G_1(x)$ and $H_1(x)$ as well it defines markers $\gamma_1(x)$ and $\omega(\gamma_1(x))$ and enumerates the necessary axioms in the operators Γ_1 and Ω for x and for both markers $\omega(\gamma_1(x))$ and $\omega(\gamma_0(x))$. Finally we need to enumerate an axiom in Γ_0 for the newly defined $\omega(\gamma_1(x))$. The marker $\omega(\gamma_1(x))$ belongs to A if and only if the marker $\gamma_1(x)$ belongs to B and x belongs to A. Thus we enumerate an axiom which reflects this - constructed from the axiom enumerated in Γ_0 for x by adding the marker $\gamma_1(x)$.

The strategy β then waits for x to enter Φ^B with outcome w while $x \notin \Phi^B$. Once x enters the set Φ^B the strategy β needs to ensure useful extractions from both sets $G_0(x)$ and $G_1(x)$. Of course the extraction of x from A might cause changes in any of the combinations $[G_0(x), G_1(x)]$, $[G_0(x), H_1(x)]$, $[H_0(x), G_1(x)]$, $[H_0(x), H_1(x)]$. Therefore we will need a backup strategy for each of these combinations.



$\beta : (\mathcal{N}, \Gamma_0, \Gamma_1)$
$\widehat{\alpha_1} : (\mathcal{P}_1, \Lambda_1)$
$\beta' : (\mathcal{N}, \Gamma_0, \Lambda_1)$
$\widehat{\alpha_0} : (\mathcal{P}_0, \Lambda_0)$
$\alpha_1' : (\mathcal{P}_1, \Gamma_1')$
$\beta'' : (\mathcal{N}, \Lambda_0, \Gamma_1')$
$\widehat{\alpha_1'} : (\mathcal{P}_1, \Lambda_1')$
$\beta''' : (\mathcal{N}, \Lambda_0, \Lambda_1')$

The strategy β performs capricious destruction only on the operator Γ_1 by extracting the marker $\gamma_1(x)$ from B and correspondingly $\omega(\gamma_1(x))$ from A. Note that this action does not injure $x \in \Xi_0^{\Psi_0^A, \Theta_0^A}$ as the marker $\omega(\gamma_1(x))$ is defined as fresh number after the definition of $G_0(x)$ and $H_0(x)$. The strategy then sends the witness x to the first backup strategy β', an $(\mathcal{N}, \Gamma_0, \Lambda_1)$-strategy which constructs the same operator Γ_0 and uses the set $H_1(x)$ to enumerate an axiom for x in the new operator Λ_1. This strategy requires for success the second combination of useful changes $[G_0(x), H_1(x))]$. If the witness x reappears in Φ^B the strategy β' performs capricious destruction on the operator Γ_0 and sends the witness further to a second backup strategy β''. Before the second backup strategy is activated we need to restart the $\mathcal{P}$-strategy on a node α'_1, as the original operator Λ_1 might be destroyed: β' extracts the marker $\omega(\gamma_0(x))$, possibly injuring $H_1(x) \subseteq \Theta_1(A)$. The second backup strategy has the form $(\mathcal{N}, \Lambda_0, \Gamma'_1)$ and constructs two new operators: Λ_0 using the set $H_0(x)$ to define an axiom for x and Γ'_1 for which the setup process is repeated and new finite sets $G'_1(x)$ and $H'_1(x)$ are defined if possible. Finally if x enters the set Φ^B again it is sent to the last backup strategy β''', which is of the form $(\mathcal{N}, \Lambda_0, \Lambda'_1)$. It is the strategy that will extract x from A if it reenters Φ^B for the third time.

Depending on the changes that this extraction causes we have the following cases:

- $H_0(x) \nsubseteq A \setminus \{x\}$: If there is no change in either $G'_1(x)$ or $H'_1(x)$, then $\mathcal{P}_1$ is satisfied and α'_1 will have outcome l forever. Otherwise the $\mathcal{N}$-requirement will be satisfied by β''' or β''.

- $H_0(x) \subseteq A \setminus \{x\}$: The witness x will be sent back to β' and the axiom for x in Γ_0 will be restored. If $G_0(x) \subseteq A \setminus \{x\}$ then the requirement $\mathcal{P}_0$ will be satisfied and α_0 will have outcome l. If $G_0(x) \nsubseteq A \setminus \{x\}$ then either $H_1(x) \nsubseteq A \setminus \{x\}$ and β' is successful or the witness x is sent back to β and the axiom for x in Γ_1 is restored. If $G_1(x) \subseteq A \setminus \{x\}$ then $\mathcal{P}_1$ is satisfied and α_1 will have outcome l forever, otherwise $G_1(x) \nsubseteq A \setminus \{x\}$ and β is successful.

Thus in every case we have made progress on the satisfaction of requirements as at least one of the considered strategies α_0, α_1, β, β', α'_1, β'' or β''' is successful.

We shall put all these ideas in techniques together to define the general construction.

7.3 All Requirements

As in Chapters 2 and 3, for every requirement we have different possible strategies along the tree. For every $\mathcal{P}$-requirement $\mathcal{P}_i$ we have two different strategies: $(\mathcal{P}_i, \Gamma_i)$ with outcomes $e <_L l$ and $(\mathcal{P}_i, \Lambda_i)$ with one outcome e. For every $\mathcal{N}$-requirement $\mathcal{N}_i$ we have strategies of the form $(\mathcal{N}_i, S_0, \dots, S_i)$, where $S_j \in \{\Gamma_j, \Lambda_j, FM_j\}$. We will call S_j the j-method of this strategy. The possible outcomes of an $(\mathcal{N}_i, S_0, \dots, S_i)$-strategy are

$$g <_L f <_L w <_L l_0 \cdots <_L l_i,$$

although not every strategy shall have all of these outcomes. Before we can make the outcomes precise we shall introduce the notion of dependence between $\mathcal{N}$-strategies:

Definition 7.3.1. *If α is a node in the tree of strategies labelled by an $(\mathcal{N}_i, S_0, \dots, S_i)$-strategy then let β be the largest node in the tree with $\beta\hat{\ }g \subset \alpha$. If there is no such node then we say that α is independent. Otherwise we say that α depends on β. We denote β by $ins(\alpha)$ and call it the instigator of α.*

A dependent strategy α will receive its witnesses from its instigator. The strategy $ins(\alpha)\hat{\ }g$ will be a $(\mathcal{P}, \Lambda_k)$-strategy for some $k \leq i$. We shall introduce a further parameter related to α, $k(\alpha)$ and its value will be the index of the requirement that $ins(\alpha)\hat{\ }g$ is working on. In this case $k(\alpha) = k$. If α is independent then $k(\alpha) = -1$. The methods that α works with will be divided into the following groups:

- If $S_j = FM_j$ we shall call it an *invisible* method.
- If $S_j \neq FM_j$ and $j < k$ then it is an *old visible* method.
- If $S_j \neq FM_j$ and $j \geq k$ then it is a *new visible* method.

The strategy α shall then have outcome g only if there is some $j \leq i$ such that $S_j = \Gamma_j$ and an outcome l_j for every new visible method $S_j = \Gamma_j$. Let $\mathbb{O}$ be the set of all possible outcomes and $\mathbb{S}$ be the set of all possible strategies.

7.3.1 The tree of strategies

The tree of strategies is a computable function $T : D(T) \subset \mathbb{O}^{<\omega} \to \mathbb{S}$ which has the following properties:

1. If $T(\alpha) = S$ and O_S is the set of outcomes for the strategy S then for every $o \in O_S$, $\alpha\hat{\ }o \in D(T)$.

2. The root of the tree is labelled by $(\mathcal{P}_0, \Gamma_0)$. The node e is labelled by $(\mathcal{N}_0, \Gamma_0)$ and the node l is labelled by $(\mathcal{N}_0, FM_0)$.

3. If $T(\alpha) = (\mathcal{N}_i, S_0, S_1, \ldots, S_i)$.

Below outcome g: $T(\alpha\hat{\ }g) = (\mathcal{P}_k, \Lambda_k)$, where $k \leq i$ is the largest index such that $S_k = \Gamma_k$. The next levels of the subtree with root $\alpha\hat{\ }g$ are assigned to $(\mathcal{P}_j, \Gamma_j)$-strategies for every j, $k < j \leq i$ such that S_j is visible. After this follows a level of $\mathcal{N}$-strategies $\beta = \alpha\hat{\ }g\hat{\ }e\ldots\hat{\ }o_j\ldots\hat{\ }o_i$, where $j > k$ and $o_j = \emptyset$ if $S_j = FM_j$, with the structure $(\mathcal{N}_i, S_0, \ldots, \Lambda_k, S'_{k+1}\ldots S'_i)$. For $j > k$ if $S_j = FM_j$ or $o_j = l$ then $S'_j = FM_j$ and otherwise $S'_j = \Gamma_j$.

Below outcomes f, w: $T(\alpha\hat{\ }o) = (\mathcal{P}_{i+1}, \Gamma_{i+1})$, where $o \in \{f, w\}$. $T(\alpha\hat{\ }o\hat{\ }e) = (\mathcal{N}_i, S_0, S_1, \ldots, S_i, \Gamma_{i+1})$ and $T(\alpha\hat{\ }o\hat{\ }l) = (\mathcal{N}_i, S_0, S_1, \ldots, S_i, FM_{i+1})$

Below outcome $\mathbf{l_k}$: The first levels of the subtree with root $\alpha\hat{\ }l_k$ are assigned to $(\mathcal{P}_j, \Gamma_j)$-strategies for every j, $k < j \leq i$ such that S_j is visible. After this follows a level of $\mathcal{N}$-strategies $\beta = \alpha\hat{\ }l_k\ldots\hat{\ }o_j\ldots\hat{\ }o_i$, where $j > k$ and $o_j = \emptyset$ if $S_j = FM_j$, with the structure $(\mathcal{N}_i, S_0, \ldots, \Lambda_k, S'_k, \ldots, S'_i)$. For $j > k$ if $S_j = FM_j$ or $o_j = l$ then $S'_j = FM_j$ and otherwise $S'_j = \Gamma_j$.

7.3.2 Construction

At each stage s we shall construct a finite path through the tree of outcomes $\delta[s]$ of length s starting from the root. The nodes that are visited at stage s shall perform activities as described below and modify their parameters. Each $\mathcal{N}$-node α shall have a right boundary R_α which will also be defined below. At all stages s the $\mathcal{N}$-strategies on the first level of the tree have $R_l[s] = R_e[s] = \infty$. After the stage is completed all $\sigma > \delta[s]$ will be initialized, their parameters including all their witnesses will be cancelled or set to their initial value $\emptyset$. Whenever we cancel a witness $x \in A[s]$ of a strategy σ we additionally enumerate an axiom $\langle x, \emptyset \rangle$ in every operator constructed by strategies $\delta \leq \sigma$. If $\omega(\gamma_j(x)) \in A[s]$ for any j then we will also enumerate the axiom $\langle \omega(\gamma_j(x)), \emptyset \rangle$ in these operators.

Suppose we have constructed $\delta[s] \upharpoonright n = \alpha$. If $n = s$ then the stage is finished and we move on to stage $s+1$. If $n < s$ then α is visited and the actions that α performs are as follows:

(I.) $T(\alpha) = (\mathcal{P}_i, \Gamma_i)$.

1. Scan all witnesses $x \notin A[s]$ for which there is an axiom in Γ_i starting from the least.
2. If $x \in \Gamma_i^{\Psi_i^{A,B}}[s]$ then let the outcome be $o = l$.
3. If all witnesses are scanned and none has produced an outcome $o = l$ then let the outcome be $o = e$.

(II.) $T(\alpha) = (\mathcal{P}_i, \Lambda_i)$.

1. Scan all sent witnesses $x \notin A[s]$ for which there is an axiom in Λ_i starting from the least.
2. If $x \in \Lambda_i^{\Theta_i^{A,B}}[s]$ with least valid axiom $\langle x, T_x \oplus B_x \rangle$ then define $L_i(x) = use(\Theta_i, A, T_x)[s]$. Restrain A on $L_i(x)$ and return x. End this stage.
3. If all witnesses are scanned and none are returned then let the outcome be e.

(III.) $T(\alpha) = (\mathcal{N}_i, S_0, \dots, S_i)$ with defined $k(\alpha)$, right boundary $R_\alpha[s]$ and possibly undefined $ins(\alpha)$. We will denote by s^- the previous α-true stage. If α has been initialized since its previous true stage or if it has never before been visited then $s^- = s$. The strategy starts at *Setup* if $s^- = s$, otherwise it goes to the step indicated at s^-. Unless otherwise stated $R_{\alpha\hat{\ }o}[s] = R_\alpha[s]$.

- **Setup:** If $ins(\alpha)\downarrow$ then wait for a witness x together with its marker $\lambda_{k(\alpha)}(x)$ to be assigned by $ins(\alpha)$. End this stage if there is no assigned witness and return to this step at the next stage. If $ins(\alpha)\uparrow$ choose a new witness x as a fresh number and enumerate it into $A[s]$. Once the witness is defined, for every $j \geq \max(k(\alpha), 0)$ such that S_j is visible perform *Setup*(j) starting from the least such j. Note that if $k(\alpha) \geq 0$ then $S_{k(\alpha)} = \Lambda_{k(\alpha)}$ and if $j > k(\alpha)$ then $S_j = \Gamma_j$.

  Setup(j**)** for $j = k(\alpha) \geq 0$:

  Enumerate in $\Lambda_j[s]$ an axiom $\langle z, H_j(x) \oplus (B[s] \upharpoonright \lambda_j(x) + 1) \cup U\rangle$, where

  - $z \in A[s]$, there is no valid axiom for z in $\Lambda_j[s]$ and z is x or a witness from a previous cycle of the strategy or z is a marker $\omega(\gamma_l(z'))$ for which there is no valid axiom in Λ_j and z' is x or a previous witness of the strategy.
  - U is the union of all finite sets D such that $\langle n, D\rangle \in \Lambda_j[s]$ is a valid axiom at stage s and $n < x$ is an uncancelled witness in $A[s]$.

  The axiom enumerated for x shall be called *the main axiom* for x in Λ_j. If $j < i$ go to *Setup*$(j+1)$. Otherwise let the outcome be $o = w$ and go to *Waiting* at the next stage.

  Setup(j) for $j > k(\alpha)$:

  1. If $x \notin \Xi_j^{\Psi_j^A, \Theta_j^A}[s]$ then let the outcome be $o = l_j$ and return to this step at the next stage. Otherwise go to the next step.
  2. Define $G_j(x)$, $H_j(x)$ as finite sets such that $G_j(x) \subseteq \Psi_j^A[s]$, $H_j(x) \subseteq \Theta_j^A[s]$ and $x \in \Xi_j^{H_j(x) \oplus G_j(x)}[s]$. Define $\gamma_j(x)$ and $\omega(\gamma_j(x))$ as fresh numbers. Enumerate $\gamma_j(x)$ in $B[s]$ and $\omega(\gamma_j(x))$ in $A[s]$. Define a new axiom

$\langle \gamma_j(x), \{\omega(\gamma_j(x))\}\rangle$ in $\Omega[s]$.

Enumerate in $\Gamma_j[s]$ an axiom $\langle z, G_j(x) \oplus (B[s] \upharpoonright \gamma_j(x)+1) \cup U\rangle$, where

- $z \in A[s]$, there is no valid axiom for z in $\Gamma_j[s]$ and z is either x, or a witness from a previous cycle of the strategy or $\omega(\gamma_l(z'))$, where $z' = x$ or z' is previous witness of the strategy.
- U is the collection of all finite sets D such that $\langle n, D\rangle \in \Gamma_j[s]$ is a valid axiom at stage s and $n < x$ is an uncancelled witness in $A[s]$.

The axiom enumerated for x shall be called *the main axiom* for x in Γ_j.

3. For all operators S_l, where $l < j$ with current axiom for x, say $\langle x, D_l\rangle$, enumerate the axiom $\langle \omega(\gamma_j(x)), D_l \cup \emptyset \oplus \{\gamma_j(x)\}\rangle$.

If $j < i$ then go to *Setup*$(j+1)$. Otherwise let the outcome be w and go to *Waiting*.

- **Waiting:** If $x \in \Phi_i^B[s]$ and the computation has use $u(\Phi_i, B, x)[s] < R_\alpha[s]$ then go to *Attack*. Otherwise let the outcome be $o = w$ and return to *Waiting* at the next stage.

- **Attack:**

1. If α does not have an outcome g then extract x from $A[s]$. Go to *Result* 2. Otherwise let j be the largest index such that $\Gamma_j = S_j$ and go to the next step.
2. If there is a returned witness from a previous cycle $\bar{x}$ then go to *Result*. Otherwise go to the next step.
3. Define $R_{\alpha\hat{}g}[s] = \gamma_j(x)$. Extract $\gamma_j(x)$ from $B[s]$ and $\omega(\gamma_j(x))$ from $A[s]$. Define $\lambda_j(x) = \max(\gamma_j(x), use(\Phi_i, B, x)[s])$. Let s_a^- be the previous stage when α sent a witness. *Send* x assigning it to the least strategy β such that $\alpha\hat{}g \subset \beta \subseteq \delta[s_a^-]$ which requires a witness. If this is the first witness then

assign it to the least strategy $\beta \supset \alpha\hat{\ }g$ which requires a witness. Let the outcome be $o = g$. At the next stage start from *Setup*.

- **Result:**

  1. Enumerate $\gamma_j(\bar{x})$ back in $B[s]$ and $\langle \omega(\gamma_j(\bar{x})), \emptyset \rangle$ in Ω. *Cancel* all witnesses $z \in A[s]$ of the strategy α. Restrain A on $L_j(\bar{x})$ defined by $\alpha\hat{\ }g$. Go to the next step.
  2. Let the outcome be $o = f$, return to this step at the next stage.

7.4 Proof

We start the proof with some of the more easier properties of the construction. We note that the sets A and B are constructed as a 2-c.e. and a 3-c.e. set respectively. It is straightforward to prove also that $B \leq_e A$.

Lemma 7.4.1. *The set B is enumeration reducible to the set A.*

Proof. We shall prove that $\Omega^A = B$. Fix any number n. If n is not a B-marker of a witness then $n \notin B$ and there is no axiom in Ω for n, so $n \notin \Omega^A$. Suppose n is a marker of a witness x defined by a strategy α at stage s then α enumerates $n \in B[s]$, $\omega(n) \in A[s]$ and an axiom $\langle n, \{\omega(n)\} \rangle$ in $\Omega[s]$. If n is not extracted from B at any stage then neither is $\omega(n)$ and hence the axiom is valid $n \in B \cap \Omega^A$. If n is extracted at stage s_1 then so is $\omega(n)$ and the axiom will remain invalid at all further stages. If n is not reenumerated in B then no further axioms for n are enumerated in Ω and hence $n \notin B \cup \Omega^A$. Otherwise n is reenumerated in B at stage s_2 at which the axiom $\langle \omega(n), \emptyset \rangle$ is enumerated in Ω. As n does not get extracted more than once, $n \in B \cap \Omega^A$. □

Another quite easy statement about the tree of strategies is that along each path there are finitely many $\mathcal{P}_i$- and $\mathcal{N}_i$-strategies for every i. We saw that this is the case for $i = 0, 1$ in sections 7.2.1 and 7.2.3. The rest of the statement follows with an easy induction using the fact that the method for $\mathcal{P}_i$ can be restarted only if the method for $\mathcal{P}_j$, where $j < i$ changes, and after that it can change at most once to Λ_i or to FM_i. The $\mathcal{N}_i$-strategy is restarted only if one of the $\mathcal{P}_j$ methods for $j \leq i$ changes.

The rest of the properties of the construction are quite harder to prove. The main difficulty will be to examine the construction of a certain operator as now many strategies define a single operator in contrast to most previous constructions. Furthermore the axioms for a witness in a fixed operator are related to the axioms of previous witnesses. We shall have to study in detail the interactions between strategies before we can prove that the construction is successful.

7.4.1 Properties of the witnesses

We will first try to establish some properties of the witnesses and the axioms defined for them. The first one is that every witness travels a finite path in the tree of strategies.

Propostion 7.4.1. *Each witness can be assigned to finitely many strategies.*

Proof. Suppose x is a witness defined by the $(\mathcal{N}_i, S_0, \dots, S_i)$-strategy β. Then β is an independent strategy. Suppose that x is β's first witness. If it is sent by β at stage s then it will be assigned to the first $\mathcal{N}$-strategy β_1 extending $\beta\hat{\ }g$. This is also an $\mathcal{N}_i$-strategy and x will also be β_1's first witness. As there are only finitely many $\mathcal{N}_i$-strategies along each path in the tree, the witness x will be assigned to finitely many strategies.

Suppose that x is β's n-th witness. Consider the sequence$\{(\beta_k, i_k, n_k)\}$, where β_k is the k-th strategy to which x is assigned, i_k denotes the index of the $\mathcal{N}$-requirement that β_k works with and n_k denotes that x is β_k's n_k-th witness. We know already that the sequence is finite if for some k we have $n_k = 1$. We will prove that:

Claim: If $i_{k+1} = i_k$ then $n_{k+1} \leq n_k$ and if $i_{k+1} > i_k$ then $n_{k+1} < n_k$.

Thus for almost all k we have $i_k = i_{k+1}$ and as there are only finitely many $\mathcal{N}_i$-strategies for every i, the sequence is finite and the statement follows.

The first part of the claim is quite obvious. The strategy β_{k+1} receives all its witnesses from β_k so $n_{k+1} \leq n_k$. Suppose that $i_{k+1} > i_k$. From the definition of the tree it follows that there is an $\mathcal{N}_{i_k}$-strategy σ such that $\beta_k \subset \sigma \subset \beta_{k+1}$. Then before the first witness is assigned to β_{k+1} one of β_k's witnesses must be assigned to σ, thus $n_{k+1} < n_k$. □

Propostion 7.4.2. *Suppose β is an $\mathcal{N}$-strategy.*

1. If β sends its witness at stage s then the next witness assigned to β is defined after stage s.

2. If β is initialized at stage s_i and β is not independent then the next witness that β works with will be defined after the next β-true stage $s > s_i$.

3. Suppose β is not initialized after stage s_i and visited at infinitely many stages. If at stage $s > s_i$ the strategy does not have an assigned witness then it will eventually be assigned a witness.

Proof. 1. This is obviously true for independent strategies. Let $\beta_0\hat{\ }g \subset \beta_1\hat{\ }g \ldots \beta_k + 1 = \beta$ be the strategies such that β_0 is independent and $ins(\beta_{i+1}) = \beta_i$ for $i < k$. Every witness assigned to β is defined by β_0.

Suppose that β sends its witness at stage s. Then at stage s all of these strategies have outcome g and send their witnesses. Thus the next witness that β_0 uses is defined after stage s. At stage $s+1$ each strategy β_{i+1} does not have a defined witness. It will receive its witness from β_i at the next stage $t \geq s+1$ at which β_i has outcome g and sends its witness.

2. If β is initialized at stage s_i then a strategy $\sigma \subset \beta$ has outcome o such that $\sigma\hat{\ }o <_L \beta$. If at stage s_i a witness is assigned to β then it is cancelled at stage s_i. Before the next witness is assigned to β there must be a stage s at which β is visited. Then at stage s the instigator $ins(\beta)$ sends its witness and by step 1. of this proposition its next witness will be defined after stage s.

3. This is again obviously true for independent strategies. Let $ins(\beta) = \delta$. Then $\delta\hat{}g$ is visited infinitely often and not initialized after stage s. There are finitely many strategies α such that $\delta\hat{}g \subset \alpha\hat{}o \subseteq \beta$ and for every such strategy $o \neq g$. Suppose at stage s the strategy α is the least such strategy that also has no witness. The strategy β is visited at stage $s_1 \geq s$. At the next $\delta\hat{}g$-true stage $s_2 > s_1$ if α still has no witness then the witness that δ sends at stage s_2 will be assigned to α. As β is not initialized at stages $t \geq s_i$ this will remain α's permanent witness. As there are finitely many such strategies α they will each be assigned a permanent witness eventually. After this a witness will finally be assigned to β. □

These two properties have a very important consequence which tells us a bit about the true path. It shows that the outcomes e and l of a $\mathcal{P}$-strategy are finitary. Thus the only infinitary outcome in this construction is the outcome g.

Corollary 7.4.1. *Let α be a $(\mathcal{P}_i, \Gamma_i)$-strategy initialized at stage s_1 and not initialized at stages t such that $s_1 < t < s_2$. If α has outcome l at a least stage s such that $s_1 \leq s < s_2$ then α has outcome l at all true stages t, $s < t < s_2$.*

Proof. Suppose this is true for higher priority strategies than α. Any strategy $\sigma \subset \alpha$ has outcome g at stage s or does not change its outcome at stages t, $s < t < s_2$. This follows from the induction hypothesis for $\mathcal{P}$-strategies. For $\mathcal{N}$-strategies with outcome $o \neq g$ it follows from the construction: σ is not initialized at stages $s < t < s_2$ so if it changes its outcome to o' at stage t then $o' <_L o$ and α would be initialized. Furthermore all of these strategies have a permanent witness for which they do not act by extracting elements at stages t, $s < t < s_2$. Strategies that have outcome g send their witnesses at stage s. A witness sent by σ is assigned to a strategy which was visited during σ's previous attack, thus is not assigned to a strategy extending $\alpha\hat{}l$. At stages t, $s < t < s_2$ accessible strategies have witnesses defined after stage s. This follows from Proposition 7.4.2 and the fact that all strategies $\delta \geq \alpha\hat{}l$ are in initial state at stage s. These witnesses together with their A- and B-markers are therefore larger then any number that has appeared in the construction until and including at stage s. At stage s the strategy α sees a valid axiom in Γ_i for a witness $x \notin A[s]$. This axiom remains valid at all further stages $t < s_2$ and whenever α is visited it will have outcome l. □

The next two properties will give us rules about the cancellation of a witness.

Propostion 7.4.3. *Suppose x is a witness that is defined at stage s_0 and sent or extracted at sub-stage s. If z is defined at substage t_0 with $s_0 < t_0 < s$ it is cancelled at the latest at stage s.*

Proof. Note that x is not cancelled until and at substage s. Let β_0 denote the strategy which defines x and δ_0 the strategy which defines z.

If $\beta_0 < \delta_0$ then $\beta_0\hat{\ }f <_L \delta_0$ as strategies below outcome $\beta_0\hat{\ }g$ do not define witnesses, rather they receive them from β_0 and strategies below outcome f are not accessible until x is extracted. Then δ_0 together with all its successors is initialized at stage s. The witness z, if not already cancelled, is assigned at stage s to a strategy extending δ_0 and hence is cancelled.

If $\delta_0 < \beta_0$ then similarly $\delta_0\hat{\ }g <_L \beta_0$. The witness z is defined at stage $t_0 > s_0$ so δ_0 is either in initial state at stage t_0 or at the previous δ_0-true stage t, $s_0 < t < t_0$, the strategy δ_0 sends its previous witness having outcome g. In all cases the strategy β_0 is in initial state at stage t_0 and x is cancelled contrary to assumption.

Finally suppose that $\delta_0 = \beta_0$. Let $\beta_0, \dots, \beta_k$ be all strategies to which x is assigned until stage s at stages $s_0 < s_1 < \cdots < s_k \leq s$ respectively. Then $t_0 > s_1$. At stage $s \geq t_0$ the witness x is extracted or sent by β_k thus every strategy β_i, $i < k$ has outcome g at stage s. It follows that z is sent by β at stage t_1 such that $s_1 < t_0 < t_1 \leq s$ and assigned to a strategy δ_1.

Again we have three cases. If $\beta_1 < \delta_1$ then δ_1 is initialized at stage s, z is cancelled.

If $\delta_1 <_L \beta_1$ then β_1 is in initial state at stage t_1 and x cancelled contrary to assumption. The final case is $\beta_1 = \delta_1$. Then $s_2 < t_1$. The same argument for $i = 1, 2, \dots, k-1$ proves that $\beta_i \leq \delta_i$ and if $\delta_i \neq \beta_i$ then z is cancelled at stage s, where δ_i denotes the i-th strategy to which z is assigned. If $\delta_i = \beta_i$ then $t_i > s_{i+1}$, where t_i denotes the stage at which z is assigned to β_i. Now as β_k extracts or sends x at stage s the witness z is sent by β_{k-1} at a stage t_k such that $s_k < t_k \leq s$. At stage t_k the strategy β_k does not require a witness. Thus if z is not cancelled already by stage s it is assigned to a strategy $\delta_k >_L \beta_k\hat{\ }f$ and hence z is cancelled at stage s at which β_k has outcome f or g. □

Propostion 7.4.4. *If x is a witness with marker $m_j(x)$, where m_j is either γ_j or λ_j, defined at stage s_0 and a marker $\gamma_l(z) < m_j(x)$ of a different witness $z \neq x$ is extracted from B at stage $s > s_0$ then x is cancelled.*

Proof. Any B-marker defined after stage s_0 is greater than $m_j(x)$. Suppose that the marker $\gamma_l(z)$ is defined at stage $t_0 \leq s_0$ and extracted by δ at stage s. Suppose that x is assigned to β at stage s.

If $\delta\hat{\ }g <_L \beta$ then β is initialized at stage s and x is cancelled.

If $\beta <_L \delta$ then δ is initialized at the last β-true stage $t < s$. The marker $m_j(x)$ must be defined before stage t, hence $s_0 < t$ otherwise it will be defined after stage s. The witness z must be defined after stage t by Proposition 7.4.2 hence $t < t_0$. Thus $s_0 < t < t_0$ contradicting the assumptions.

If $\beta\hat{\ }o \subset \delta$ we shall examine the different possibilities for o. If $o = g$ then at stage s the strategy β has outcome g, sends its witness and does not have a witness when δ is visited. In all other cases δ is in initial state when x is assigned to β. The marker $m_j(x)$ must be defined before the next δ-true stage t. Then the witness z is defined at $t_0 > t$ if δ is not independent by Proposition 7.4.2 or at stage $t_0 \geq t$ if δ is independent. Thus the marker $m_j(x)$ is defined before the marker $\gamma_l(z)$ contrary to assumption.

Finally suppose $\delta\hat{\ }g \subset \beta$. Any witness assigned to β must first be sent by δ. It follows that $z > x$ and δ has already sent the witness x at a previous stage $\delta\hat{\ }g$-true stage. By Proposition 7.4.2 the witness z is defined after the last $\delta\hat{\ }g$-true stage $t < s$ and this is the last stage when strategies to which x is assigned might be accessible to define the markers of x. Thus $s_0 \leq t < t_0$. □

7.4.2 Properties of the axioms

This section reveals some properties of the axioms in the constructed operators. Our main goal will be to prove that if a $\mathcal{P}$-strategy has outcome l at all but finitely many stages then the corresponding $\mathcal{P}$-requirement is satisfied. We shall need to investigate the axioms that are enumerated in an operator for elements x which are extracted from A. We shall prove three properties for the axioms. First we will show a connection between a witness x and a witness z such that an axiom for x is enumerated in an operator using the main axiom for z. This rather technical property will enable us to prove that the only axiom that can be valid for a witness $x \notin A[s]$ at an operator S_i is the main axiom for x in S_i. Finally we shall show that if the main axiom for a witness $x \notin A[s]$ is valid in S_i then $\Xi_i^{\Psi_i^A, \Theta_i^A} \neq A$.

Propostion 7.4.5. *Let α be a $(\mathcal{P}_i, S_i)$-strategy and x be a witness which is not cancelled until stage s and for which there is an axiom in the operator constructed by α. Suppose that δ invalidates the main axiom for x. Then every further axiom for x related to a different witness z remains valid at all stages $t \leq s$ or is invalidated by the same strategy δ, to which z is sent eventually.*

Proof. Suppose x is assigned to strategies $\beta_0 \subset \beta_1 \subset \beta_k$ at stages $s_0 < s_1 < \cdots < s_k \leq s$, where β_0 is the strategy which enumerates the main axiom for x in S_i at stage s_0. At stage s_0 all strategies $\sigma >_L \beta_0\hat{\ }g$ are in initial state and will work with witnesses defined after stage s_0. Strategies below $\beta_0\hat{\ }g$ are not accessible until stage s_1. At stage s_i the witness x is assigned to β_i strategies σ such that $\beta_{i-1} \subset \sigma \subset \beta_{i+1}$ have a defined witness which does not change and do not extract any numbers from A or B at stages $s_i \leq t \leq s_k$ or else x would be cancelled before stage s_k. Strategies $\sigma >_L \beta_i$ are in initial state at stage s_i and work with witnesses defined after stage s_i. Thus the only strategies that can invalidate the axiom for x are among $\beta_0, \ldots, \beta_k$.

If $\delta = \beta_k$ then it must extract x as otherwise x would be sent to a further strategy. Thus no new axioms will be enumerated in S_i.

Suppose $\delta = \beta_i$, $i < k$. Then δ has outcome g extracting a B-marker of x at stage t_0. At the next β_0-true stage t_1 the strategy β_0 defines a new axiom for x using its new current witness z. If this witness is never sent then the axiom remains valid at all stages $t \leq s$ as the only accessible strategies are in initial state at stage t_1. If this witness is sent it is assigned to the least strategy visited at stage t_0 which requires a witness. By the argument above this must be β_1. If β_1 does not send z then the axiom for z remains valid at all further stages otherwise β_1 sends z and it is assigned to β_2.

Thus eventually z will reach δ at stage t_2 with a valid main axiom in S_i. At all stages t with $t_1 < t \leq t_2$ there is a valid axiom for x in S_i - the one that uses main axiom for z, thus β_0 does not enumerate any further axioms for x. If the axiom for z is not invalidated by δ or it is invalidated at the same stage at which x extracted then no more axioms will be enumerated in S_i for x. Otherwise δ invalidates the axiom for z at stage t_3 and at the next β_0-true we have a very similar situation as at stage t_1: at stage t_3 all strategies $\beta_0, \ldots, \delta\hat{\ }g$ were visited and there is no valid axiom for x. The strategy β_0 will define a witness z' and enumerate an axiom for x and z in S_i using the main axiom for z'. If this axiom is invalidated then the witness z' must be sent to δ and δ invalidates it. □

Corollary 7.4.2. *Let x be any witness extracted from A at stage s and α be a $(\mathcal{P}_i, S_i)$-strategy such that there is an axiom for x in S_i. The only axiom in S_i that can be valid at a further stage $t > s$ is the main axiom for x.*

Proof. Suppose that there is a different axiom for x valid at stage $t > s$ and it uses the main axiom for $z > x$ defined before stage s. It follows from the proof of proposition 7.4.5 that this witness z is sent to the same strategy δ that invalidates the main axiom for x. Otherwise x could not be extracted at stage s. This strategy has greatest Γ-method with index $k \leq i$ and always extracts a B-marker $\gamma_k(y)$ when it sends its witness y. Before x is extracted it must send z at stage s_1 invalidating the axiom for z. If this axiom is valid at stage $t > s$ then z must be returned by $\delta\hat{}g$, constructing the operator Λ_k after stage s. We will prove that this is impossible.

At stage s_1 the witness z is assigned to the least strategy which requires a witness. Suppose δ_1 is the strategy to which x was assigned after it was sent by δ. Consider a strategy σ such that $\delta \subset \sigma\hat{}o \subseteq \delta_1$. Then $o \neq g$ as otherwise x would be assigned to σ. Furthermore σ works with the same operator Λ_k as this method can change only below a further g-outcome. Until x is extracted σ has the same outcome o or else x would be cancelled. Thus z is assigned to a strategy $\delta_1' \supseteq \delta_1$. And by the same argument both δ_1 and δ_1' construct the same operator Λ_k.

If $\delta_1' \neq \delta_1$ then at stage s_1 the strategy δ_1 has outcome $o \neq g, f$ and it has this outcome until δ_1' is cancelled. At all such stages there is a valid axiom for x in Λ_k defined by δ_1 which does not change and it is included in any axiom for z that δ_1' defines. The element z is cancelled at stage s at which δ_1 has outcome g or f.

If $\delta_1' = \delta_1$ then both x and z are witnesses for of δ_1. Every axiom enumerated in Λ_k for z either includes an axiom for x or otherwise the same axiom is enumerated for x and all axioms for z are enumerated before stage s as z is cancelled at stage s by Proposition 7.4.3.

Thus in both cases if z can be returned by $\delta\hat{}g$ at stage s_z then there is a valid axiom for both x and z in Λ_k. If we assume that $s_z \leq s$ then x could not be extracted at stage s as $\delta\hat{}g$ ends stage s_z prematurely and δ would have outcome f at all stages $t > s_z$ until it is initialized. Thus $s < s_z$, the witness x is already extracted from $A[s_z]$ and $\delta\hat{}g$ will return x instead of z. $\square$

Propostion 7.4.6. *Let α be a $(\mathcal{P}_i, \Gamma_i)$-strategy and let $\beta \supseteq \alpha\hat{\ }e$ be a strategy such that $S_i = \Gamma_i$ and this is the largest Γ-method at β. Suppose a witness x is returned to β at stage s and β restrains A on $L_i(x)$. If this restraint is injured at stage $s_1 > s$ then there is no valid axiom for x in Γ_i at all stages $t > s_1$ or else $\Xi_i^{\Psi_i^A, \Theta_i^A} \neq A$.*

Proof. Suppose the lemma is true inductively for witnesses $z < x$.

If α is initialized at stage s_1 then there will be no valid axiom for x in Γ_i at any further stage. Suppose that α is not initialized at stages t, $s \leq t \leq s_1$.

Any strategy that at stage s is in initial state or does not have an assigned witness will not injure the restraint by Proposition 7.4.2. The restraint is therefore injured by a strategy $\delta_1 \supseteq \alpha\hat{\ }e$ such that $\delta_1 \leq \beta$. In order for this strategy to be accessible there must be a strategy $\delta \supseteq \delta_1$ such that $\alpha\hat{\ }e \subset \delta\hat{\ }o \subset \beta$, $o \neq g$, and which has outcome g at stage s_1.

The strategy δ has the same witness $y < x$ and the same outcome o at all stages at which it is visited from the stage s_0 at which x is assigned to β until and including at stage s. Furthermore it works with the same operator Γ_i and the main axiom for y is not yet invalidated. The main axiom for x includes a valid axiom for every one of δ's witnesses $z \leq y$ and every B-marker defined for such a witness before stage s_0. Any further B-marker for a witness of δ is defined after stage s and the corresponding A-marker respects the restraint.

At stage s_1 the strategy δ_1 injures the restraint on A. Therefore it must extract from A a witness $z \leq y$ defined before stage s_0 or an A-marker $\omega(\gamma_l(z))$ together with $\gamma_l(z)$ for a witness $z \leq y$ both defined before stage s_0. If $z \in A$ then δ_1 extracts $\gamma_l(z)$ which invalidates all axioms for x and this marker is never reenumerated in B.

If $z \notin A$ and there is a valid axiom for z in Γ_i then by Proposition 7.4.2 this is the main axiom for z and by the induction hypothesis $H_i(z) \subseteq \Theta_i(A)$ hence $z \in \Xi_i^{\Psi_i^A, \Theta_i^A}$. Otherwise there is no valid axiom for z and hence no valid axiom for x. □

7.4.3 Satisfaction of the requirements

As usual we define the true path h to be the leftmost path in the tree such that the strategies along it are visited at infinitely many stages. As in two cases of the construction a strategy can end a stage prematurely we will need to prove that the so defined path is infinite. Once we have established that this is true we can prove that all $\mathcal{N}$- and $\mathcal{P}$-requirements are satisfied.

Lemma 7.4.2. *There is an infinite path h in the tree of strategies with the following properties:*

1. $(\forall n)(\exists^{\infty} s)[h \upharpoonright n \subseteq \delta[s]]$.

2. $(\forall n)(\exists s_l(n))(\forall s > s_l(n))[\delta[s] \geq h \upharpoonright n]$, *i.e. $h \upharpoonright n$ is not initialized after stage $s_l(n)$.*

Proof. We prove the statement with induction on n. The case $n = 0$ is trivial: $h \upharpoonright 0 = \emptyset$ is visited at every stage of the construction and is never initialized, $s_l(0) = 0$.

Suppose the statement is true for $h \upharpoonright n = \alpha$. If α is a $(\mathcal{P}_i, \Gamma_i)$-strategy by Corollary 7.4.1 either α has outcome e at every α-true stage in which case $h(n+1) = e$ and $s_l(n+1) = s_l(n)$, or there is a stage $s > s_l(n)$ such that α has outcome l at every true stage $t > s$, so $h(n+1) = l$ and $s_l(n+1) = s$.

If $\alpha = \beta\hat{\ }g$ is a $(\mathcal{P}_i, \Lambda_i)$-strategy then α does not returns a witness after stage $s_l(n)$. Otherwise β will have outcome f at almost all true stages contradicting the assumption that α is visited at infinitely many stages. Thus α has outcome e at every true stage $t \geq s_l(n)$ and $h(n+1) = e$, $s_l(n+1) = s_l(n)$.

If α is an $(\mathcal{N}_i, S_0, \dots, S_i)$ then we have the following cases:

- α has outcome g at infinitely many stages. Then $h(n+1) = g$, $s_l(n+1) = s_l(n)$.

- There is a stage $s > s_l(n)$ at which α receives back a witness. Then α has outcome f at all further stages, $h(n+1) = f$, $s_l(n+1) = s$.

- There is a stage s at which α attacks for the last time. By Proposition 7.4.2 α will be assigned a new witness x at a stage $s_1 > s$. If α enters *Setup*(j) at stage $s_2 > s$ and never completes it then α has outcome l_j at all stages $t > s_2$, $h(n+1) = l_j$, $s_l(n+1) = s$. Otherwise there is a stage s_3 at which α enters *Waiting* and then α has outcome w at all stages $t > s_3$, $h(n+1) = w$, $s_l(n+1) = s$.

□

Lemma 7.4.3. *Every $\mathcal{N}$-requirement is satisfied.*

Proof. Let β be the last $\mathcal{N}_i$-strategy along the true path. Then $\beta\hat{\ }w \subset h$ or $\beta\hat{\ }f \subset h$ as along all paths below every other outcome of β there is another $\mathcal{N}_i$-strategy. By Lemma 7.4.2 the strategy β has a permanent witness x at stages $t \geq s_l(|\beta|+1)$. If $\beta\hat{\ }w \subset h$ then $x \in A$ and at every true stage $t > s_l(|\beta|+1)$ if $x \in \Phi_i^B[t]$ then $use(\Phi_i, B, x)[t] > R_\beta[t]$. If β is independent then $R_\beta[t] = \infty$. Otherwise at every stage t the right boundary is defined by $ins(\beta) = \alpha$. If α has witness z at stage t then $R_\beta[t] = \gamma_{k(\beta)}(z)$. The next witness that α uses is defined after stage t and its B-markers are of value greater than $R_\beta[t]$. Thus $\lim_t R_\beta[t] = \infty$ and $x \notin \Phi_i^B$.

Suppose $\beta\hat{\ }f \subset h$. If β has an outcome g the witness x is returned by $\beta\hat{\ }g = \alpha$ which is a $(\mathcal{P}_j, \Lambda_j)$-strategy at stage $s = s_l(|\beta|+1)$. When β sent this witness at stage $s_0 < s$ we had $x \in \Phi_i^B[s_0]$. The strategy then defined the marker $\lambda_j(x) \geq use(\Phi_i, B, x)[s_0]$. As x is not cancelled at any stage by Proposition 7.4.4 no B-marker $b < \lambda_j(x)$ for a different witness $z \neq x$ is extracted at any stage $t \geq s_0$.

At stage s_0 the main axiom for x, say $\langle x, A_x \oplus B_x \rangle$ is enumerated in the operator Λ_j constructed at α and $B[s_0] \upharpoonright \lambda_j(x) \setminus \{\gamma_j(x)\} \subseteq B_x$. The strategy α returns this witness at stage s as it is the least $x \in \Lambda_j^{\Theta_j^A, B} \setminus A[s]$. By Corollary 7.4.2 the only axiom that can be valid at stage s is the main axiom for x in Λ_j. So $B[s_0] \upharpoonright \lambda_j(x) \setminus \{\gamma_j(x)\} \subseteq B[s]$, no more markers for x are extracted at any stage $t > s$, and at stage $s_l(|\beta|+1)$ the strategy β enumerates $\gamma_j(x)$ back in the set B. So $x \in \Phi_i^B[t]$ at all stages $t \geq s_l(|\beta|+1)$

and hence $x \in \Phi_i^B \setminus A$.

Suppose β does not have an outcome g. Then at stage $s_l(n+1) = s$ the strategy sees $x \in \Phi_i^B[s]$ and extracts x from the set A. Let $u = use(\Phi_i, B, x)[s]$. Strategies $\sigma\hat{\ }o \subset \beta$ with $o \neq g$ do not extract any markers from the set B. Strategies $\sigma\hat{\ }g \subset \beta$ have just sent their witness and by Proposition 7.4.2 will not extract any markers that are less than u. Strategies $\delta \geq \beta\hat{\ }f$ are in initial state at stage s and by the same proposition will not extract markers of value less than u. Thus $B[s] \upharpoonright u \subseteq B[t]$ at all $t \geq s$ and hence $x \in \Phi_i^B \setminus A$. □

Lemma 7.4.4. *Every $\mathcal{P}$-requirement is satisfied.*

Proof. Let α be the last $(\mathcal{P}_i, S_i)$-strategy along the true path.

If $\alpha\hat{\ }l \subseteq h$ then α is a $(\mathcal{P}_i, \Gamma_i)$-strategy. Let $x \notin A$ be the witness such that $x \in \Gamma_i^{\Psi^A, B}$. There is a least strategy $\beta \supseteq \alpha\hat{\ }e$ such that x is assigned to and whose greatest Γ-method is Γ_i. Before x is extracted from A the marker $\gamma_i(x)$ is extracted from B. As $x \in \Gamma_i^{\Psi^A, B}$ then by Corollary 7.4.2 the main axiom for x in Γ_i is valid and hence $\gamma_i(x)$ is enumerated back in B by β on a stage s at which β restrained $H_i(x)$ in Θ_i^A. By Proposition 7.4.6 if this restraint is injured then $\Xi_i^{\Psi_i^A, \Theta_i^A} \neq A$. If this restraint is not injured then $G_i(x) \oplus H_i(x) \subset \Psi_i^A \oplus \Theta_i^A$ and again $\Xi_i^{\Psi_i^A, \Theta_i^A} \neq A$ as $x \in \Xi_i^{\Psi_i^A, \Theta_i^A} \setminus A$.

Suppose α is a $(\mathcal{P}_i, \Gamma_i)$-strategy such that there is an $\mathcal{N}$-strategy β working with i-th method Γ_i and $\beta\hat{\ }l_i \subset h$. Then β has a permanent witness x such that $x \in A \setminus \Xi_i^{\Psi_i^A, \Theta_i^A}[t]$ at all β-true stages $t > s_l(|\beta| + 1)$. The requirement is satisfied by $A \neq \Xi_i^{\Psi_i^A, \Theta_i^A}$.

For all other cases denote by U the set Ψ_i^A if $S_i = \Gamma_i$ and Θ_i^A if $S_i = \Lambda_i$. We will prove that for all elements n enumerated in A at stages $t > s_l(n)$ we have $S_i^{U,B}(n) = A(n)$. Thus $A \leq_e U \oplus B$ and the requirement $\mathcal{P}_i$ is satisfied.

Let $n \notin A$ be a witness. If n is extraccted at stage s_n then at all α-true stages $t > \max(s_l(n), s_n)$ we have $n \notin S_i^{U,B}[t]$. Otherwise if $S_i = \Gamma_i$ then by Corollary 7.4.1 the strategy α would have true outcome l and if $S_i = \Lambda_i$ the witness n would be returned by α which is impossible as we saw in the proof of Lemma 7.4.2. Thus $n \notin S_i^{U,B}$.

Let $n \notin A$ be an A-marker $\omega(\gamma_l(z))$. Every axiom for n in S_i is of the form $\langle n, D \cup \{\gamma_l(z)\}\rangle$ and there is similar axiom $\langle z, D\rangle$ for z in S_i. As $n \notin A$ the marker $\gamma_l(z)$ is extracted from B. If an axiom for n is valid at a further stage then $\gamma_l(z)$ is reenumerated in B and hence $z \notin A$. By the argument above there is no valid axiom for z and hence for n in S_i at any α-true stage.

If $n \in A$ and n is cancelled then there is valid axiom $\langle n, \emptyset\rangle \in S_i$. Thus $A(n) = S_i^{U,B}(n)$. Suppose n is a witness that is never cancelled. We will prove that there is a valid axiom for n in S_i. Let $\beta_0, \ldots, \beta_k$ be all strategies to which n gets assigned in the course of the construction. As n is not cancelled $h \not<_L \beta_k$. Furthermore $\beta_k \supseteq \alpha\hat{\ }e$. Otherwise β_k would not be visited after stage $s_l(|\alpha|)$ and hence the witness x must be assigned to β_k before or at this stage. We are however dealing with witnesses that are defined after stage $s_l(|\alpha|)$.

Consider the least strategy $\beta_j \supseteq \alpha\hat{\ }e$. First we observe that $\beta_j \subset h$. If we assume otherwise then there is a strategy σ such that $\alpha\hat{\ }e \subset \sigma\hat{\ }o_1 \subset h$ and $\beta_j \supseteq \sigma\hat{\ }o_2$ and $o_2 <_L o_1$. Then $o_2 = g$ or else β_j is initialized before stage $s_l(|\sigma|)$ and not accessible after this stage and x is cancelled. But if $o_2 = g$ then β_j receives n from σ, so $\sigma = \beta_{j-1}$ and this contradicts our choice of β_j as the least strategy below $\alpha\hat{\ }e$.

The i-method of β_j is hence new and is S_i, as no strategy σ along the true path has outcome l_i and there is no strategy between α and β_j has outcome g, the only cases when the i-method changes. Thus β_j will enumerate axioms for n at all β_j-true stages at which there is no valid axiom in S_i.

If the main axiom $\langle n, D\rangle$ for n enumerated by β_j is never invalidated then $n \in S_i^{U,B}$. For every A-marker of n that is never extracted and is defined by stage $s_l(|\beta_j|)$, the strategy β_j enumerates an axiom in S_i using the current axiom for n. If a further A-marker $m = \omega(\gamma_k(n))$ for n is defined after this stage by a strategy β then $\beta \supseteq \beta_j$ and β has the same method S_l as β_j for $l \leq i$ otherwise the main axiom for n would be invalidated. As β can define a marker only for a new method, $k > i$ and β enumerates a new axiom for m of the form $\langle m, D \cup \emptyset \oplus \{\gamma_k\}\rangle$ in S_i. If $m \in A$ then $\gamma_k(n) \in B$ and this axiom is valid at all further stages.

Suppose that the main axiom for n in S_i is invalidated by δ at stage $s_0 > s_l(|\beta_j|)$. By Proposition 7.4.5 this is done by a strategy β_l, $l > j$. At the next true stage β_j enumerates an axiom for x using the main axiom for its current witness z. If this axiom is invalidated at all, it is invalidated by β_l. Now as β_l extracts a B-marker for a method with index less than i. It follows that $\beta_l\hat{\ }g$ is not on the true path, as otherwise there would be a further $\mathcal{P}_i$-strategy along the true path. Let s be the last $\beta_l\hat{\ }g$-true stage. Then the axiom for n enumerated at the first β_j-true stage after s will remain valid forever. Any A-marker of n, $m = \omega(\gamma_l(n)) \in A$ must be defined before stage s. Then if there is no valid axiom for m at the first β-true stage after s then an axiom is enumerated for m during $Setup(i)$. The axiom for m in S_i valid at this stage will remains valid forever.

This concludes the proof of the lemma and the theorem. □

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Appendix A

Published Articles

A.1 Genericity and Nonbounding in the Enumeration degrees

Authors: Mariya Soskova
The author is partially supported by Marie Curie Early Training Site MATHLOGAPS(MEST-CT-2004-504029).

Status: Journal of Logic and Computation, **17**, 1235–1255.

Abstract: The structure of the semi lattice of enumeration degrees has been investigated from many aspects. One aspect is the bounding and nonbounding properties of generic degrees. Copestake proved that every 2-generic enumeration degree bounds a minimal pair and conjectured that there exists a 1-generic degree that does not bound a minimal pair. In this paper we verify this longstanding conjecture by constructing such a degree using an infinite injury priority argument.

Key words: Enumeration reducibility, Nonbounding degrees, Generic degrees, Minimal pairs

Introduction

In contrast to the Turing case where every 1-generic degree bounds a minimal pair as proved in [5] we construct a 1-generic set whose e-degree does not bound a minimal pair in the semi-lattice of the enumeration degrees.

In her paper [1] Copestake examines the n-generic degrees for every $n < \omega$. She proves that every 2-generic enumeration degree bounds a minimal pair and states that there is a 1-generic enumeration degree that does not bound a minimal pair. Her proof of the statement does not appear in the academic press. Later Cooper, Li, Sorbi and Yang show in [2] that every Δ^0_2

enumeration degree bounds a minimal pair and construct a Σ^0_2 enumeration degree that does not bound a minimal pair. In the same paper the authors state that their construction can be used to build a 1-generic degree that does not bound a minimal pair. Initially the goal of this paper was to build a 1-generic enumeration degree with the needed properties by following the construction from [2]. In the working process it turned out that significant modifications of the construction had to be made in order to get the desired 1-generic degree. The enumeration degree that is constructed is also Σ^0_2 and generalizes the result from [2].

Constructing a 1-generic degree that does not bound a minimal pair

Definition A.1.1. *A set A is* enumeration reducible *to a set B if there is a c.e. set Φ such that:*

$$n \in A \Leftrightarrow \exists u(\langle n, u\rangle \in \Phi \wedge D_u \subset B)$$

where D_u denotes the finite set with code u under the standard coding of finite sets. The c.e. set Φ is an enumeration operator *and its elements will be called* axioms*. We will write $A \leq_e B$ to denote that A is enumeration reducible to B and $A = \Phi(B)$ to denote the fact that A is enumeration reducible to B via the enumeration operator Φ.*

We will denote enumeration operators by capital Greek letters $\Phi, \Theta \ldots$.

As with Turing reducibility, enumeration reducibility gives rise to a degree structure. Note that all c.e. sets have degree 0, the least enumeration degree.

We will use lower case Greek letters (especially ρ, τ) for finite binary strings and let $\tau \subseteq \rho$ indicate that τ is an initial segment of ρ. When A is a set $\tau \subset A$ means that τ is an initial segment of $A's$ characteristic function χ_A considered as an infinite binary sequence.

Definition A.1.2. *A set A is* 1-generic *if for every c.e. set X of finite binary strings*

$$\exists \tau \subset A(\tau \in X \vee \forall \rho \supseteq \tau(\rho \notin X)).$$

An enumeration degree is 1-generic if it contains a 1-generic set.

Definition A.1.3. *A pair of enumeration degrees a and b form a* minimal pair *in the semi-lattice of the enumeration degrees if:*

1. *$a > 0$ and $b > 0$.*
2. *For every enumeration degree c ($c \leq a \wedge c \leq b \rightarrow c = 0$).*

Theorem A.1.1. *There exists a 1-generic enumeration degree a that does not bound a minimal pair in the semi-lattice of the enumeration degrees.*

We will use the priority method with infinite injury to build a set A whose e-degree will have the intended properties. The construction involves a priority tree of strategies. For further definitions of both computability theoretic and tree notations and terminology we refer the reader to [3] and [4].

The Requirements

We will construct a set A satisfying the following requirements:

1. A is generic. Therefore for all c.e. sets W we have a requirement:
$$G^W : \exists\tau \subseteq A(\tau \in W \vee \forall\mu \supseteq \tau(\mu \notin W)),$$
where τ and μ are finite binary strings.

   Let Req^G be the set of all G^W requirements.

2. A does not bound a minimal pair. For each pair of enumeration operators Θ_0 and Θ_1 we will have a requirement:
$$R^{\Theta_0\Theta_1} : \Theta_0(A) \text{ is c.e.} \vee \Theta_1(A) \text{ is c.e.}\vee$$
$$\vee\exists D(D \leq_e \Theta_0(A) \wedge D \leq_e \Theta_1(A) \wedge D \text{ is not c.e.}).$$
Let Req^R be the set of all $R^{\Theta_0\Theta_1}$ requirements.

   Fix a requirement $R^{\Theta_0\Theta_1}$. Let $X = \Theta_0(A)$ and $Y = \Theta_1(A)$. This requirement is too complicated to be satisfied at once and we will break it up into subrequirements:
$$R^{\Theta_0\Theta_1} : (\exists\Phi_0)(\exists\Phi_1)(\forall\text{c.e. sets } W)[S^W]$$
where S^W is the subrequirement:
$$S^W : X \text{ is c.e.} \vee Y \text{ is c.e.} \vee [\Phi_0(X) = \Phi_1(Y) = D \wedge \exists d(W(d) \neq D(d))].$$
Let $Req^S_{R^{\Theta_0\Theta_1}}$ be the set of all S^W subrequirements of $R^{\Theta_0\Theta_1}$.

Let $Req = Req^G \cup Req^R \cup (\bigcup_{R\in Req_R} Req^S_R)$.

Priority Tree of Strategies

The set Req is linearly ordered with order type ω and requirements in earlier positions have higher priority. Each particular requirement can be satisfied in more than one way. We connect to each such way an outcome. The choice of the correct way to satisfy a certain requirement depends on the outcomes of higher priority requirements. Therefore we represent the set of all possible sequences of outcomes as a *tree of strategies*. Each node α on the tree is labelled by a requirement $P \in Req$ and the node α will be referred to as a P-strategy. The children of α correspond to each of α's possible outcomes. So, although each of those nodes will be labelled by the same requirement, each may have a different approach to satisfying its requirement depending on what it "believes" to be the outcome of α.

The set of all possible outcomes for each requirement will be linearly ordered ($<_L$, defined below) and the nodes of the tree of strategies will be ordered by the induced lexicographical ordering $\leq$. The construction is by stages; in each stage s we construct a set A_s approximating A and a string δ_s of length s in the tree of strategies. The initial segments $\delta \subseteq \delta_s$ are the nodes of the tree visited during stage s of the construction; they are the strategies that might act to satisfy their requirements. The intent is that there will be a true path, a leftmost path of nodes visited infinitely often, such that all nodes along the true path are able to satisfy their requirements. If the node β is visited on stage s, we say that s is a β-true stage.

Each node (say of length n) will build its own approximation A_s^n, so $A_s = A_s^s$; nodes will obey restrictions on A and $\overline{A}$ set by higher priority requirements. Ultimately A will be the set of all natural numbers a such that

$$(\exists t_a)(\forall t > t_a)[a \in A_t].$$

At the end of stage s we *initialize* all strategies $\delta > \delta_s$ by setting all parameters to their initial values and *cancelling* any witnesses.

We will proceed to describe what general actions the different types of strategies, corresponding to the different types of requirements, will make.

1. Let γ be a G^W-strategy. The actions that γ makes when visited are the following:

   (a) γ chooses a finite string λ_γ according to rules that ensure compatibility with strategies of higher priority.

   (b) Then it searches for a string μ such that $\lambda_\gamma\hat{\ }\mu \in W$. If it finds such a string then γ remembers the shortest one, μ_γ, and has outcome 0. If not then $\mu_\gamma = \varnothing$ and the outcome is 1. The order between the two outcomes is $0 <_L 1$. The strategy will be successful if $\lambda_\gamma\hat{\ }\mu_\gamma \subseteq A$. γ will restrain some elements out of and in A to ensure this.

2. Let α be a $R^{\Theta_0\Theta_1}$-strategy. It acts as a mother strategy to all its substrategies ensuring that they work correctly. We assume that on this level the two enumeration operators Φ_0 and Φ_1 are built. They are common to all substrategies of α. This strategy has only one outcome: 0.

3. Let β be a S^W-strategy. It is a substrategy of one fixed $R^{\Theta_0\Theta_1}$-strategy $\alpha \subset \beta$. The actions that β makes are the following:

   (a) First it tries to prove that the set X is c.e. by building a c.e. set U which should turn out to be equal to X. On each stage it adds elements to U and then looks if any errors have occurred in the set. While there are no errors the outcome is ∞_X.

   (b) If an error occurs then some element that was assumed to be in the set X has been extracted from X. The strategy can not fix the error by extracting the element from U because we want U to remain c.e. In this case β gives up on its desire to make X c.e. It finds the smallest error $k \in U \backslash X$ and forms a set E_k which is called an agitator set for k. The agitator contains an element a for every axiom for k in the current approximation of Θ_0, say $\langle k, D_k\rangle$, such that $a \in D_k$. So extracting the agitator set from A will make sure that each axiom for k in Θ_0 will not be valid for $\Theta_0(A) = X$, that is will make sure that $k \notin X$. On the other hand with some additional actions we will make sure that if the agitator is a subset of A then $k \in X$. And so the agitator will have the following property which we will refer to as the control property:

   $$k \in X \Leftrightarrow E_k \subseteq A.$$

   The strategy now turns its attention to Y. It tries to prove that it is c.e. by constructing a similar set V_k that would turn out to be equal to Y. It makes similar actions, checking at the same time if the agitator for k preserves its control property. Note that the agitator will lose this property if a new axiom for k is enumerated in Θ_0. While there are no errors in V_k the outcome is $\langle \infty_Y, k\rangle$.

(c) If an error is found in V_k, the strategy chooses the least $l \in V_k \backslash Y$ and forms an agitator F_l^k for l in a similar way. F_l^k now has the control property:

$$l \in Y \Leftrightarrow F_l^k \subseteq A.$$

Now β has some control over the sets X and Y, namely using the agitators it can determine whether or not $k \in X$ and $l \in Y$. It adds axioms $\langle d, \{k\}\rangle \in \Phi_0$ and $\langle d, \{l\}\rangle \in \Phi_1$ for some witness d, constructing a difference between D and W. If $d \in W$ the outcome is $\langle l, k\rangle$ and the agitators are kept out of A. If $d \notin W$ then the agitators are enumerated back in A, so $d \in D$ and the outcome is the symbol d_0.

The possible outcomes of a S^W-strategy are:

$$\infty_X <_L T_0 <_L T_1 <_L \cdots <_L T_k <_L \cdots <_L d_0$$

where T_k is the following group of outcomes:

$$\langle \infty_Y, k\rangle <_L \langle 0, k\rangle <_L \langle 1, k\rangle <_L \cdots <_L \langle l, k\rangle \ <_L \ldots$$

The priority tree of strategies is a computable function T with $\mathrm{Dom}(T) \subseteq \{0, 1, \infty_X, \langle \infty_Y, k\rangle, \langle l, k\rangle, d_0 | k, l \in \mathbb{N}\}^{<\omega}$ and $\mathrm{Range}(T) = Req$ for which the following properties hold:

1. If $\alpha \in \mathrm{Dom}(T)$ and $T(\alpha) \in Req^R$ then $\alpha\hat{\ }0 \in \mathrm{Dom}(T)$.
2. If $\gamma \in \mathrm{Dom}(T)$ and $T(\gamma) \in Req^G$ then $\gamma\hat{\ }o \in \mathrm{Dom}(T)$ where $o \in \{0, 1\}$.
3. If $\beta \in \mathrm{Dom}(T)$ and $T(\beta) \in Req_R^S$ then $\beta\hat{\ }o \in \mathrm{Dom}(T)$ where $o \in \{\infty_X, \langle \infty_Y, k\rangle, \langle k, l\rangle, d_0 | k, l \in \mathbb{N}\}$.
4. For all $\delta \in \mathrm{Dom}(T)$ such that the length $lh(\delta)$ is even $T(\delta) \in Req^G$.
5. If $\alpha \in \mathrm{Dom}(T)$ is a R-strategy then for each subrequirement S^W there is a S^W-strategy $\beta \in \mathrm{Dom}(T)$, a substrategy of α, such that $\alpha \subset \beta$.
6. If β is a S^W-strategy, a substrategy of α, then $\alpha \subseteq \beta$ and under $\beta\hat{\ }\infty_X$ and $\beta\hat{\ }\langle \infty_Y, k\rangle$ there aren't any other substrategies of α.
7. For each infinite path h in T and each R- or G-requirement there is a node $h \upharpoonright n$ along the path which is a R- or G-strategy respectively. For every S^W-requirement, subrequirement of R, there is also a node $h \upharpoonright n$ which is an S^W-strategy, unless there is already a higher priority S^W-strategy $h \upharpoonright m$ belonging to the same requirement R and $h(m+1) = \infty_X$ or $h(m+1) = \langle \infty_Y, k\rangle$.

Interaction between strategies

In order to have any organization whatsoever we make use of a global parameter, a counter b, whose value will be an upper bound to the numbers that have appeared in the construction up to the current moment.

1. First we will examine the interaction between an S^W-strategy β and a G^W-strategy γ. The interesting cases are when $\gamma \supseteq \beta\hat{}\infty_X$ and similarly when $\gamma \supseteq \beta\hat{}\langle\infty_Y, k\rangle$.

   Let $\gamma \supseteq \beta\hat{}\infty_X$ and suppose β is of length n. Suppose β is visited on stage $s > n$ and adds an element k to the set U. There is an axiom $\langle k, E'\rangle \in \Theta_0$ which is currently valid, i.e. $E' \subseteq A^n_s$. The strategy β will keep a list $\mathbb{U}$ of the axioms from Θ_0 that it assumes to be valid when enumerating new elements in U. It is possible that later (even on the same stage) γ chooses a string μ_γ and extracts a member of E' from A. If there aren't any other axioms for k in the corresponding approximation of Θ_0, we have an error in U. On the next β-true stage, s_1 say, β will find this error, choose an agitator for k and move on to the right with outcome $\langle\infty_Y, k\rangle$. It is possible that later a new axiom for k is enumerated in the corresponding approximation of Θ_0 and thus the error in U is corrected. On the next β-true stage s_2, β returns to its initial aim to prove that X is c.e. But then another G^W-strategy $\gamma_1 \supseteq \gamma$ chooses a string μ_{γ_1} and again takes k out of U by extracting an element that invalidates the new axiom for k. If this situation appears infinitely many times, ultimately we will claim to have $X = U$ but k will be taken out of X infinitely many times and thus our claim would be wrong. Then this S^W requirement will not be satisfied. This is why we will have to ensure some sort of stability for the elements that we put in U, more precisely for the corresponding axioms in $\mathbb{U}$ that we assume to be valid. This is how the idea for *applying an axiom* arises. We apply an axiom $\langle k, E'\rangle$ by changing the value of the global parameter b so that it is larger than the elements of the axiom and then by initializing those strategies that might take k out of X.

   The first thing that comes to mind is to initialize *all* strategies $\delta \supseteq \beta\hat{}\infty_X$. This way we would avoid errors at all. If the set X is infinite though, we would never give a chance to strategies $\delta \supseteq \beta\hat{}\infty_X$ to satisfy their requirements. This problem is solved with the notion of local priority. Every G^W-strategy $\gamma \supseteq \beta\hat{}\infty_X$ will have a fixed local priority regarding β. This priority is given by a computable bijection $\sigma_\beta : \Gamma \to \mathrm{N}$ where Γ is the set of all G^W-strategies in the subtree of $\beta\hat{}\infty_X$. If $\gamma \subset \gamma_1$ then $\sigma_\beta(\gamma) < \sigma_\beta(\gamma_1)$. A strategy $\gamma \supseteq \beta\hat{}\infty_X$ has local priority $\sigma_\beta(\gamma)$ in relation to β. When we apply the axiom $\langle k, E'\rangle$ only strategies γ with $\sigma_\beta(\gamma)$ greater than k will be initialized. Then as the stages grow so do the elements that we put into U and with them grows the number of G^W-strategies that we preserve. Ultimately all strategies will get a chance to satisfy their requirements.

2. Now let us examine the interactions between two S^W-strategies β and β_1. The interesting case is $\beta \supseteq \beta_1\hat{}\infty_X$ and $\alpha \subset \alpha_1$ where α and α_1 are the corresponding mother strategies. Suppose that on stage s_1 the strategy β chooses its agitators E_k and F^k_l and takes them out of A. Note that it is important to keep both agitators in A or both agitators out of A to preserve the equality in the sets $\Phi^\alpha_0(X)$ and $\Phi^\alpha_1(Y)$ constructed at level α. Suppose now that on the next β_1-true stage s_2 the strategy β_1 decides to build its own agitators and in them it includes members from only one of the agitators that β selected at stage s_1, causing a difference in the sets $\Phi^\alpha_0(X)$ and $\Phi^\alpha_1(Y)$. To avoid this β_1 will choose

its agitators carefully: along with the elements needed to form the agitator with the requested control property it will add also all elements of all agitators that were chosen and out of A on the previous β_1-true stage s_1. Thus the two agitators of β will not be separated and will not cause an error such as $d \notin \Phi_0^\alpha(X)$ and $d \in \Phi_1^\alpha(Y)$.

Unfortunately this will not solve the problem completely. It is possible that on a later stage s_3 a new axiom is enumerated in Θ_0 for k or a new axiom is enumerated in Θ_1 for l, causing one of the agitators to lose its control property and creating a difference between the sets $\Phi_0^\alpha(X)$ and $\Phi_1^\alpha(Y)$ at the element d. If β is visited again then it would fix this mistake by discarding the false witness d. If not, the error would stay unfixed and the R-strategy α might not satisfy its requirement. Therefore we will attach a new parameter to α: a list $Watched_\alpha$ through which α will keep track of all its S^W-substrategies. The list will contain entries for all substrategies including information on what their agitators are. If α sees that one of the agitators loses its control property then it will go ahead with the actions on discarding the false witness and correcting the mistake in the operators Φ_0 and Φ_1 in advance. This action will not interfere with β's work. In fact if β is ever visited again it will cancel the witness and give up the agitator that has lost its control property. In that sense α is just pre-empting the actions of β.

The Construction

We will begin the description of the construction by listing again all parameters that are connected with each strategy. Their purpose was explained intuitively in the previous two sections. While describing the parameters we will suppress the superscripts that indicate the strategy to which they belong. The superscripts will appear only when more than one strategies are involved in a discussion and we need to distinguish between their parameters.

We have one global parameter b, common to all strategies, which is an upper bound to all elements that have appeared so far in the construction. Its initial value is 0.

In addition every strategy δ visited on stage s will have two more parameters E_s and F_s. The set E_s contains all elements restrained out of A on this stage s by strategies $\delta' \subset \delta$. The set F_s contains all elements that are restrained in A by strategies of higher priority $\delta'' < \delta$. Note that these elements may have been restrained on a previous stage.

Each G^W-strategy γ will have two parameters: finite binary strings λ and μ, with initial value the empty string $\varnothing$.

Each R-strategy α has a list $Watched$ with entries of the form $\langle \beta : \langle E, E_k, F_l^k \rangle, d \rangle$ where β is a substrategy of α, E_k and F_l^k are β's current agitators, the set E contains information needed to assess if the agitators still have the control property and d is the witness that must be *cancelled* in case one of the agitators loses its control property. The initial value of the list is $\emptyset$. Also α has the parameters Φ_0 and Φ_1, the enumeration operators that α and all its substrategies β construct together. Their initial value is $\emptyset$ as well.

Each S-strategy β inherits the two parameters Φ_0 and Φ_1 from its mother strategy. In addition it has c.e. sets U and V_k for all k, initially all empty. Then corresponding to them lists

$\mathbb{U}$ and $\mathbb{V}_k$, with initial values the empty list. During the construction β might form agitators E_k for all k and F_l^k for all k and l or choose a witness d, but initially the agitators are empty and the witness is undefined.

On stage $s = 0$ all nodes of the tree are initialized, $b_0 = 0$, $\delta_0 = \varnothing$ and $A_0 = \mathbb{N}$.

On each stage $s > 0$ we will have $A_s^0 = \mathbb{N}$, $\delta_s^0 = \varnothing$ and $b_s^0 = b_{s-1}^{s-1}$.

Let's assume that we have already built δ_s^n, A_s^n and b_s^n. If $n = s$ then go on to the next stage $s+1$. Otherwise $n < s$ the strategy δ_s^n makes some actions as described below and has an outcome o. Then $\delta_s^{n+1} = \delta_s^n\hat{\ }o$.

I. δ_s^n is a G^W-strategy γ.

   (a) If $\lambda = \varnothing$ then define λ to be the binary string of length $b_s^n + 1$ such that
   $$\lambda(a) \simeq 0 \text{ iff } a \in E_s$$
   and increase the value of the counter to $b_s^{n+1} = b_s^n + 1$.

   (b) If $\mu = \varnothing$ then ask if $\exists\mu(\ \lambda\hat{\ }\mu\ \in W\ )$. If the answer is **No** then $A_s^{n+1} = A_s^n$. All elements for which $\lambda(a) = 1$ are restrained by γ in A and the outcome is $o = 1$. If the answer is **Yes** then let μ be the least such binary string so that $\lambda\hat{\ }\mu \in W$ and increase the value of the counter to
   $b_s^{n+1} = \max(b_s^{n+1}, lh(\lambda\hat{\ }\mu) + 1)$.
   Now μ is defined and $\lambda\hat{\ }\mu \in W$. All a such that $\lambda\hat{\ }\mu(a) = 1$ are restrained in A by γ. All a such that $a \geq lh(\lambda)$ and $\lambda\hat{\ }\mu(a) = 0$ are restrained out of A by γ. Let $A_s^{n+1} = A_s^n \setminus \{a | a \text{ is restrained out of } A \text{ by } \gamma\}$ and the outcome be $o = 0$.

II. δ_s^n is a R-strategy α.

   Then scan all entries in the list $Watched_\alpha$. For each $\langle \beta : \langle E, E_k, F_l^k\rangle, d\rangle \in Watched$ check if there is an axiom $\langle k, E'\rangle \in \Theta_0$ such that $E' \cap (E \cup E_k) = \emptyset$ or $\langle l, F'\rangle \in \Theta_1$ such that $F' \cap (E \cup E_k \cup F_l^k) = \emptyset$. If there is such an axiom then **cancel** d by enumerating in both sets Φ_0 and Φ_1 the axiom $\langle d, \emptyset\rangle$. $A_s^{n+1} = A_s^n$ and $o = 0$.

III. δ_s^n is a S^W-strategy β, a substrategy of α .

   First check if β is watched by α and delete the corresponding entry from $Watched_\alpha$ if there is one. Unless otherwise specified $b_s^{n+1} = b_s^n$. The actions that β makes depend on the outcome o^- that it had on the previous β-true stage s^-. If this is the first β-true stage in the construction, let $o^- = \infty_X$

   (a) The outcome o^- is ∞_X.

      1. Choose the least $k \in X \setminus U$. Here $X = \Theta_0^s(A_s^n)$. If there is such an element then there is an axiom $\langle k, E'\rangle \in \Theta_0^s$ with $E' \subseteq A_s^n$. Enumerate k in the set U and its relevant axiom $\langle k, E'\rangle$ in the list $\mathbb{U}$. **Apply** this axiom by initializing all strategies $\delta \supseteq \beta\hat{\ }\infty_X$ such that there is a G^W-strategy $\gamma \subseteq \delta$ of local $\beta\hat{\ }\infty_X$-priority with value greater than k and by setting $b_s^{n+1} = \max(b_s^n, E')$.

2. Proceed through the elements of U until an element draws attention or until all elements are scanned. An element $k \in U$ draws attention if there isn't an applicable axiom for it.

   Definition A.1.4. *An axiom $\langle k, E' \rangle \in \Theta_0$ is* applicable *if:*
   1. $E' \cap E_s^\beta = \emptyset$,
   2. $E' \cap Out1_s = \emptyset$ where $Out1_s$ is the set of all elements restrained out of A by some strategy $\delta \supseteq \beta\hat{}\infty_X$ such that:

   i. $\delta \subseteq \delta_{s^-}$,

   ii. All G^W-strategies $\gamma \subseteq \delta$ have local $\beta\hat{}\infty_X$-priority with value less than k (the ones that cannot be initialized when applying an axiom for k).

   The intuition behind this definition is that it is plausible that the axiom will end up valid. Note that the set Out_1 includes all elements that are restrained by G^W strategies with local priority less than k along what seems to be the true path. When we find a valid axiom that has not been applied, we will apply it thereby initializing all strategies below the first G^W-strategy with local priority greater than k along each path. We will not however initialize S^W-strategies above some G^W-strategy with local priority less than k. These S^W-strategies may have already chosen an agitator that may remain permanent. Therefore we must respect their choice and ask that an applicable axiom does not include any such elements.
   For each element $k \in U$ act as follows:

   - If k doesn't draw attention, find an applicable axiom $\langle k, E' \rangle$ for k that has minimal code. If the entry for k in $\mathbb{U}$ is different, replace it with $\langle k, E' \rangle$. If the axiom $\langle k, E' \rangle$ is not yet applied, **apply** it.
     If there aren't any elements k that draw attention then let $A_s^{n+1} = A_s^n$ and $o = \infty_X$.
   - If k draws attention:

     A. Initialize all strategies $\delta \supseteq \beta\hat{}\infty_X$ such there is a G^W-strategy $\gamma \subseteq \delta$ of local $\beta\hat{}\infty_X$-priority with value greater than k.

     B. Examine all strategies in the subtree with root $\beta\hat{}\infty_X$. If β' was visited on stage s^-, had outcome $\langle l', k' \rangle$ and was not initialized after stage s^- then add to the list $Watched_{\alpha'}$ where α' is the mother strategy of β' an element of the following structure:
     $< \beta' : \langle E_{s^-}^{\beta'}, E_{k'}^{\beta'}, F_{l'}^{k',\beta'} \rangle, d^{\beta'} >$.
     Then define the agitator for k as $E_k = Out1_s \backslash E_s^\beta$. All elements $a \in E_k$ are restrained out of A by β. Let $A_s^{n+1} = A_s^n \backslash E_k$ and $o = \langle \infty_Y, k \rangle$.

(b) The outcome o^- is $\langle \infty_Y, k \rangle$.

1. Check if there is an axiom $\langle k, E' \rangle \in \Theta_0$ such that $E' \cap (E_s^\beta \cup E_k) = \emptyset$. If so then act as in d.1.

2. Choose the least element $l \in Y \backslash V_k$. If there is such an element then there is $\langle l, F' \rangle \in \Theta_1^s$ with $F' \subseteq A_s^n \backslash E_k$. Enumerate the element l in V_k and its corresponding axiom $\langle l, F' \rangle$ in $\mathbb{V}_k$. **Apply** this axiom by initializing all strategies $\delta \supseteq \beta\hat{}\langle \infty_Y, k \rangle$ such there is a G^W-strategy $\gamma \subseteq \delta$ of local $\beta\hat{}\langle \infty_Y, k \rangle$-priority with value greater than l and by setting $b_s^{n+1} = \max(b_s^n, F')$.
3. Proceed through the elements of V_k until all are scanned or until an element draws attention. An axiom $\langle l, F' \rangle \in \Theta_1$ is defined to be applicable similarly to case a.2:

   Definition A.1.5. *An axiom* $\langle l, F' \rangle \in \Theta_1$ *is* applicable *if:*
   1. $F' \cap E_s^\beta = \emptyset$*,*
   2. $F' \cap Out2_s = \emptyset$ *where* $Out2_s$ *is the set of all elements restrained out of* A *by some strategy* $\delta \supseteq \beta\hat{}\langle \infty_Y, k \rangle$ *such that:*
   i. $\delta \subseteq \delta_{s^-}$*,*
   ii. All G^W*-strategies* $\gamma \subseteq \delta$ *have local* $\beta\hat{}\langle \infty_Y, k \rangle$*-priority with value less than* l*,*
   3. $F' \cap E_k = \emptyset$*.*

   For each element $l \in V_k$ act as follows

   - If l doesn't draw attention, find an applicable axiom with minimal code $\langle l, F' \rangle$. If the entry for l in $\mathbb{V}_k$ is different, replace it with $\langle l, F' \rangle$. If the axiom $\langle l, F' \rangle$ is not yet applied, **apply** it.
     If none of the elements draw attention then let $A_s^{n+1} = A_s^n \backslash E_k$ and $o = \langle \infty_Y, k \rangle$.
   - If l draws attention:
     A. Initialize all strategies $\delta \supseteq \beta\hat{}\langle \infty_Y, k \rangle$ such that there is a G^W-strategy $\gamma \subseteq \delta$ of local $\beta\hat{}\langle \infty_Y, k \rangle$-priority with value greater than l.
     B. Examine all strategies in the subtree with root $\beta\hat{}\langle \infty_Y, k \rangle$. If β' was visited on stage s^-, had outcome $\langle l', k' \rangle$ and was not initialized after stage s^- then add to the list $Watched_{\alpha'}$ where α' is the mother strategy of β' an element of the following structure: $< \beta' : \langle E_{s^-}^{\beta'}, E_{k'}^{\beta'}, F_{l'}^{k',\beta'} \rangle, d^{\beta'} >$.
     The agitator for l is $F_l^k = Out2_s \backslash (E_s^\beta \cup E_k)$. All elements $a \in (E_k \cup F_l^k)$ are restrained in A by β.
     Find the least element d that has not been used in the definition of Φ_0 yet. This will be a witness β. Enumerate the axiom $\langle d, \{k\} \rangle$ in Φ_0 and the axiom $\langle d, \{l\} \rangle$ in Φ_1. Let $A_s^{n+1} = A_s^n$ and $o = d_0$.

(c) The outcome o^- is d_0. Check if the witness d has been enumerated in the c.e. set W. That is, check if $d \in W_s$.

If the answer is **Yes** then β restrains all elements $a \in (E_k \cup F_l^k)$ out of A. Let $A_s^{n+1} = A_s^n \backslash (E_k \cup F_l^k)$ and $o = \langle l, k \rangle$.

If the answer is **No** then let $A_s^{n+1} = A_s^n$ and $o = d_0$.

(d) The outcome o^- is $\langle l,k\rangle$. Then the agitators E_k and F_l^k and the witness d are defined.

1. Check for an axiom $\langle k,E'\rangle \in \Theta_0$ such that $E' \cap (E_s^\beta \cup E_k) = \emptyset$, that is E_k has lost its control property. If there is one then **cancel** d and let $V_k = \mathbb{V}_k = E_k = F_l^k = \emptyset$. Replace the entry for k in $\mathbb{U}$ with $\langle k,E'\rangle$. **Apply** the axiom $\langle k,E'\rangle$. The strategy β stops restraining elements $a \in E_k \cup F_l^k$. Let $A_s^{n+1} = A_s^n$ and $o = \beta\hat{}\infty_X$.
2. Check for an axiom $\langle l,F'\rangle \in \Theta_1$ such that $F' \cap (E_s^\beta \cup E_k \cup F_l^k) = \emptyset$. If there is one then **cancel** d and let $F_l^k = \emptyset$. Replace the entry for l in $\mathbb{V}_k$ with $\langle l,F'\rangle$. **Apply** the axiom $\langle l,F'\rangle$. The strategy β stops restraining elements $a \in F_l^k$. Let $A_s^{n+1} = A_s^n \backslash E_k$ and $o = \beta\hat{}\langle\infty_Y,k\rangle$.
3. If neither of the above two cases hold, hence both agitators still have their control property, then let $A_s^{n+1} = A_s^n \backslash (E_k \cup F_l^k)$ and $o = \langle l,k\rangle$.

Proof

The proof of the theorem is divided into four groups of lemmas. The first group is about the restrictions. It gives a clear idea about which elements are restrained at different stages. The second group of lemmas is about the agitator sets. Its purpose is to prove that the agitators have the intended control properties that we claim. Then follows the group of lemmas about the true path. Finally we prove that the requirements are indeed satisfied.

Restriction Lemmas

The restriction lemmas are basic tools for the rest of the proof. We will establish some basic rules about the restriction that will help us later determine properties of the characteristic function of A. Note that, since the tree is infinitely branching, we could have infinite activity to the right of the true path. The following lemmas ensure that this activity does not have any undesired effect on A.

We start off with a simple property of the agitator sets that will be helpful for the rest of the restriction lemmas.

Propostion A.1.1. *Let β be a strategy that is visited and chooses an agitator Ag on stage s. Then the elements of the agitator Ag were restrained out of A by some G^W-strategy $\gamma \supset \beta$ on some previous stage $s_0 < s$ after β was last initialized.*

Proof. The proof is by induction on s. Suppose the lemma is true for all strategies visited on stages $t < s$ and let β be visited on stage s. Assume β chooses its agitator and let a be an element from this agitator. Finally let s' be the stage on which β was last initialized before stage s. We will concentrate on the case when β chooses E_k; the case when it chooses F_l^k is similar. Then $a \in Out1_s$ and hence is restrained out of A on stage s^- by some strategy in the subtree with root $\beta\hat{}\infty_X$. Obviously $s^- > s'$, otherwise $Out1_s = \emptyset$ because all strategies that extend β would also be initialized and would not restrain any elements out of A.

If a was restrained out of A by a G^W-strategy on stage s^- the lemma is proved. Suppose a was restrained by a S^W-strategy $\beta' \supseteq \beta\hat{}\infty_X$. Then a is in the agitator $Ag^{\beta'}$ of β'. This agitator was chosen on some stage $t \leq s^-$ and β' was not initialized on stages between t and s^- as otherwise the agitator would be cancelled. So if we assume that β' was cancelled for the last time before stage s on stage t', we have $s' \leq t' < t \leq s^-$.

Then according to our induction hypothesis for t we have that a was restrained by a G^W-strategy on stage t_0 such that $t' < t_0 < t$. In particular $s' < t_0 < s$. This concludes the induction and the proof of the lemma. □

Lemma A.1.1 (Preserving the Restrictions Lemma)**.** *Let s_1 and s_2 be two consecutive δ-true stages. If δ is not initialized on any intermediate stage t such that $s_1 < t \leq s_2$ then $E^\delta_{s_1} = E^\delta_{s_2}$.*

Proof. We will prove the lemma by induction on the length of δ. If δ is of length 0 then $\delta = \varnothing$ and $E^\delta_{s_1} = E^\delta_{s_2} = \emptyset$. So let us assume that the statement is true for strategies δ of length n. We will prove that it holds for $\delta\hat{}o$.

Suppose $\delta_1 = \delta\hat{}o$ is visited on stages s_1 and s_2 and not initialized on stages t such that $s_1 < t \leq s_2$. Then δ is also visited on stages s_1 and s_2 and is not initialized on any stage t such that $s_1 < t \leq s_2$. The induction hypothesis gives us $E^\delta_{s_1} = E^\delta_{s_2}$. So we only need to prove that the elements that δ restrains on stages s_1 and s_2 are the same. Indeed on each stage the set E_{δ_1} is obtained from E_δ by adding the elements that δ restrains out of A on that stage.

We will examine the different cases:

Case 1. If δ is a R-strategy, a G^W-strategy with $o = 1$ or a S^W-strategy with $o = \infty_X$ or $o = d_0$ then δ does not restrain any elements on stages s_1 and s_2.

Case 2. Suppose δ is a G^W-strategy with outcome $o = 0$. Then the value of δ's parameters λ and μ are the same on stages s_1 and s_2, as they can change only after initialization. Therefore the elements that δ restrains on both stages s_1 and s_2 are the same as well, namely the elements $a > lh(\lambda^\delta)$ such that $\lambda^\delta\hat{}\mu^\delta(a) = 0$.

Case 3. Suppose δ is a S^W-strategy with outcome $o = \langle \infty_Y, k\rangle$. Then the elements that δ restrains out of A on stages s_1 and s_2 are the ones in $(E_k)_{s_1}$ and $(E_k)_{s_2}$ respectively. If we assume that $(E_k)_{s_1} \neq (E_k)_{s_2}$ then on some stage t such that $s_1 < t \leq s_2$ we would have had an outcome $o = \infty_X$. Indeed δ can only choose a value for its agitator E_k if it had outcome ∞_X on the previous true stage. Once the value is chosen it can only be changed if the strategy is initialized or if the agitator loses its control property. In the latter case δ would have outcome ∞_X. But $\infty_X <_L \langle \infty_Y, k\rangle$ and therefore δ_1 would be initialized on stage t.

Case 4. Suppose δ is a S^W-strategy with outcome $o = \langle l, k\rangle$. Then the elements that δ restrains on stages s_1 and s_2 are the ones in $(E_k)_{s_1} \cup (F^k_l)_{s_1}$ and $(E_k)_{s_2} \cup (F^k_l)_{s_2}$ respectively. If we assume that $(E_k)_{s_1} \neq (E_k)_{s_2}$ or $(F^k_l)_{s_1} \neq (F^k_l)_{s_2}$ then on some stage t such that $s_1 < t \leq s_2$ we would have had an outcome $o' = \infty_X$ or $o' = \langle \infty_Y, k\rangle$ to the left of o and δ_1 would be initialized on stage t.

□

Propostion A.1.2. *If s is a δ-true stage and $a \in E^\delta_s$ then δ cannot restrain a (in or out of A) on this stage.*

Proof. 1. Let δ be a G^W-strategy. Let $s_0 \leq s$ be the earliest stage on which δ is visited such that δ is not initialized between stages s_0 and s. According to Lemma A.1.1, $E^\delta_{s_0} = E^\delta_s$ and therefore $a \in E^\delta_{s_0}$. The value of the parameter λ^δ is chosen on stage s_0 and remains the same until stage s. Then $a < lh(\lambda_\delta)$ and $\lambda_\delta(a) = 0$, hence δ does not restrain a on stage s.

2. Let δ be a S^W-strategy. Then δ restrains only elements in its agitators. Let $s_0 \leq s$ be the stage on which δ is visited and chooses an agitator Ag. According to Lemma A.1.1 $E^\delta_{s_0} = E^\delta_s$ and therefore $a \in E^\delta_{s_0}$. According to the construction $Ag \cap E^\delta_{s_0} = \emptyset$. Therefore δ does not restrain a. □

Lemma A.1.2. *If s is a δ-true stage and $a \in F^\delta_s$ then δ can not restrain a out of A on stage s.*

Proof. Assume that a is restrained in A by $\delta_1 < \delta$ on stage $s_1 \leq s$. Note that $a \in F^\delta_s$ until δ_1 is initialized or is visited and stops restraining a in A. Hence δ_1 is not initialized until stage s. Let $s_2 \geq s_1$ be the first stage after the imposition of the restraint on which δ is visited. We will prove that s_2 is the first visit of δ after an initialization.

Case 1. $\delta_1 <_L \delta$. Then δ is initialized on stage s_1.

Case 2. $\delta_1 \subset \delta$.

a. δ_1 is a G^W-strategy. Then s_1 is the earliest stage after δ_1's last initialization, say on stage t, on which it picks a value for one of it parameters λ or μ.

   If δ_1 chooses λ^{δ_1} on stage s_1 then s_1 is the first stage after the initialization on stage t on which δ_1 is visited. But δ was also initialized on stage t. If δ_1 chooses μ^{δ_1} on stage s_1 then it has outcome 0 and will have outcome 0 on each visit until it is initialized again (if ever). As δ is visited on stage s we can conclude that $\delta \supseteq \delta_1\hat{}0$. On the other hand the nodes that extend $\delta_1\hat{}0$ are visited for the first time after δ_1's last initialization on stage t not sooner than on stage s_1.

b. δ_1 is a S^W-strategy then on stage s_1 it has outcome d_0. This is the only case when a S^W-strategy restrains elements in A. Furthermore δ_1 had outcome $\langle \infty_Y, k\rangle$ on its previous visit on stage s_1^- and has outcome d_0 on each visit after s_1 while it is restraining the element in A. In particular it has this outcome on stage s. Hence $\delta \supseteq \delta_1\hat{}d_0$ and was initialized on stage s_1^-, when δ_1 had outcome $\langle \infty_Y, k\rangle$.

So, if $\gamma \supseteq \delta$ is a G^W-strategy then for any λ_γ that γ chooses on stages after stage s_1 we have $a < lh(\lambda_\gamma)$ and γ cannot restrain a out of A.

If δ is a S^W-strategy and we assume that δ restrains a out of A then a is included in some agitator Ag. As we proved in Proposition 1, any element of the agitator has been restrained out of A by some G_W-strategy $\gamma \supset \delta$ after $\delta's$ last initialization. But we just proved that no such γ restrains a out of A. Hence $a \notin Ag$. □

Lemma A.1.3. *Suppose that on stage s we visit δ_1. Suppose that δ_1 restrains out of A an element a that is currently restrained in A by a lower priority strategy $\delta_2 \supset \delta_1$. Then δ_2 is initialized on stage s.*

Proof. We will make the proof by induction on the distance $d(\delta_1, \delta_2) = lh(\delta_2) - lh(\delta_1)$. We know that $d > 0$. Let us assume that the statement is true for all pairs of strategies with distance $d < n$. Let $d(\delta_1, \delta_2) = n$.

On stage s_0 we have visited δ_2 which restrained a in A. Then from stage s_0 up until the substage on which we visit δ_1, the element is still restrained in A, hence δ_2 has not been initialized since stage s_0. Then neither is the strategy δ_1.

It follows from Proposition A.1.2 that a was not restrained out of A by δ_1 on stage s_0. So on stage s the elements that δ_1 restrains out of A are different from the ones it restrained on stage s_0.

If δ_1 is a G^W-strategy, this could only happen if it had outcome 1 on stage s_0 and outcome 0 on stage s. The parameter λ^{δ_1} does not change value between stages s_0 and s, as δ_1 is not initialized. So only if the parameter μ changed value, could δ_1 restrain new elements out of A. But this means that $\delta_2 \supseteq \delta_1\hat{}1$ and is initialized on stage s.

If δ_1 is a S^W-strategy then a is included in some agitator Ag. This agitator was chosen on stage $t \leq s$ and is extracted from A on stage s, but was not extracted from A on stage s_0.

The easy case is $\delta_2 \supseteq \delta_1\hat{}d_0$. Then on stage s, δ_1 has outcome $\langle l, k\rangle$ and initializes δ_2.

Whenever δ_1 has outcome $\langle l, k\rangle$ both agitators are extracted from A. In particular if this is the outcome on s_0, as the elements extracted by δ_1 on stages s_0 and s are different, δ_1 must have had outcome ∞_X or $\langle\infty_Y, k\rangle$ on an intermediate stage when it changed the values of at least one of the agitators. On that stage δ_2 would be initialized.

This leaves us with $\delta_2 \supseteq \delta_1\hat{}\infty_X$ or $\delta_2 \supseteq \delta_1\hat{}\langle\infty_Y, k\rangle$. In the first case $Ag = E_k$, as elements that enter F_l^k are restrained by G^W-strategies below $\delta_1\hat{}\langle\infty_Y, k\rangle$ by Proposition 1. These are initialized on stage s_0 and can not restrain a out of A by Lemma A.1.2. In the second case $Ag = F_l^k$ as E_k is already extracted from A on stage s_0 and does not change until stage s, or δ_1 would have outcome ∞_X on an intermediate stage and δ_2 would be initialized.

In both cases the agitator is chosen on stage $t > s_0$ and after that δ_1 has outcome to the right. Then by the definition of an agitator the element a was restrained out of A by some $\sigma \supset \delta_1$ on stage $t^- \geq s_0$. We claim that $\sigma \subset \delta_2$ and $s_0 < t^-$ so by the induction hypothesis δ_2 would be initialized on stage t^-, contradicting our assumptions.

Indeed $\sigma <_L \delta_2$ would initialize δ_2 on stage t^- and $\delta_2 < \sigma$ would not allow σ to restrain a out of A. So $\sigma \subset \delta_2$ and furthermore $s_0 \neq t^-$ or by Proposition A.1.2 δ_2 cannot restrain a at all on stage s_0. □

Corollary A.1.1. $\forall s \forall \delta (E_s^\delta \cap F_s^\delta = \emptyset)$.

Proof. Assume for a contradiction that $\exists s \exists \delta (E_s^\delta \cap F_s^\delta \neq \emptyset)$. Let s be the least stage and δ be the least strategy for which our assumption holds. Let $a \in E_s^\delta \cap F_s^\delta$. Then when we visit δ, a is restrained out of A by $\delta_1 \subset \delta$ and a is restrained in A by $\delta_2 < \delta$. We will examine the possible positions of δ_1 and δ_2:

1. $\delta_1 > \delta_2$. But then $a \in F_s^{\delta_1}$ and δ_1 cannot restrain a out of A.
2. $\delta_1 < \delta_2$. Then $\delta_1 \subset \delta_2$. We know that δ_1 restrains a on stage s. According to Lemma A.1.3 δ_2 is initialized on stage s. But then it stops restraining elements and a is not restrained by δ_2 when we visit δ. This contradicts our choice of δ_2.

□

Lemmas about the Agitators

Let's take a closer look at the agitators. Suppose β chooses an agitator at stage s. Then $o^- = \infty_X$, in which case $Ag = Out1_s \backslash E_s^\beta$, or $o^- = \langle \infty_Y, k \rangle$, in which case $Ag = Out2_s \backslash (E_k \cup E_s^\beta)$. It follows from Lemma A.1.1 that $E_s^\beta = E_{s^-}^\beta$. Any element a in $Out1$ or $Out2$ was restrained on stage s^- by a strategy $\delta \supset \beta$ and hence $E_{s^-}^\delta \supseteq E_{s^-}^\beta$. So $Out1_s \cap E_s^\beta = Out2_s \cap E_s^\beta = \emptyset$. Also in the second case $E_{s^-}^\delta \supseteq E_k$, so $Out2_s \cap E_k = \emptyset$. Hence the agitators have a simpler definition, namely $Ag = Out1_s$ in the first case and $Ag = Out2_s$ in the second case.

Suppose $\beta' \supset \beta$ is a S^W-strategy and on stage s^- it was visited and had outcome $\langle l', k' \rangle$. Then let $E_{\beta'} = E_{\beta'}^\beta \cup E_{k'} \cup F_{l'}^{k'}$ where $E_{\beta'}^\beta = E_{s^-}^{\beta'} \backslash E_{s^-}^\beta$: the elements that are restrained out of A by strategies below β, but above β'. If β' is not initialized on stage s then $E_{\beta'} \subset Ag$.

Similarly if $\beta' \supset \beta$ is a S^W-strategy and on stage s^- it was visited and had outcome $\langle \infty_y, k' \rangle$ then let $E_{\beta'} = E_{\beta'}^\beta \cup E_{k'}$ where $E_{\beta'}^\beta = E_{s^-}^{\beta'} \backslash E_{s^-}^\beta$. If β' is not initialized on stage s then $E_{\beta'} \subset Ag$.

Now that we have established these basic facts about the agitators we can proceed with the proof of some of their more complicated properties. Note that every S^W-strategy may have influence on the operators Φ_0 and Φ_1 that it helps construct, even if it is visited finitely many times. The following lemmas give us information on what that influence might be.

Lemma A.1.4. *1. Let β be a strategy that is visited on stage t_0 and chooses an agitator E_k for k. If the node $\beta\hat{\ }\infty_X$ is never again initialized or visited on any stage $t > t_0$ and $E_k \subseteq A$ then $k \in X$.*

2. Let β be a strategy that is visited on stage t_0 and chooses an agitator F_l^k for l. If the node $\beta\hat{\ }\langle \infty_Y, k \rangle$ is not initialized or visited on any stage $t > t_0$ and $F_l^k \subseteq A$ then $l \in Y$.

Proof. We will prove the first clause of the lemma; the second clause is proved similarly. To prove that $k \in X = \Theta_0(A)$ we need to find an axiom $\langle k, E' \rangle \in \Theta_0$ with $E' \subset A$.

Consider the axiom $\langle k, E' \rangle$ for k listed in $\mathbb{U}$ on stage t_0. We will prove that it has that property. It was applied not later than on stage t_0. Furthermore it was valid when it entered $\mathbb{U}$ hence $E' \cap E_{t_0}^\beta = \emptyset$ according to Lemma A.1.1.

The strategy β chooses an agitator for k on stage t_0. Hence we initialize all strategies δ such that $\beta <_L \delta$. Furthermore $o_{t_0^-} = \infty_X$, hence on stage t_0^- we have initialized all strategies δ' such that $\beta\hat{}\infty_X <_L \delta'$. The strategies $\delta' \supset \beta$ such that $\beta\hat{}\infty_X <_L \delta'$ are initialized on stage t_0^- and are not visited again before stage t_0.

Therefore all nodes δ such that $\beta\hat{}\infty_X <_L \delta$ cannot restrain elements from E' out of A. And the only strategy that can extract elements from E' out of A on stage t_0 is β.

For a contradiction assume that an element $a \in E'$ is extracted from A on infinitely many stages t. Let t_1 be the first stage after t_0 on which $a \notin A_{t_1}$. Let δ restrain a out of A on stage t_1. We know that $\beta\hat{}\infty_X \not<_L \delta$. Also our assumptions on β, namely that $\beta\hat{}\infty_X$ is never again visited or initialized, give us that $\delta \not\supseteq \beta\hat{}\infty_X$ and $\delta \not<_L \beta\hat{}\infty_X$. This leaves us with the following two possibilities:

a. $\delta = \beta$. If β itself extracts the element a out of A, then a must be an element of one of β's agitators. The F_l^k agitators are all empty at stage t_0, and when they are defined at later stages they will contain elements restrained out of A by strategies extending $\beta\hat{}\langle\infty_Y, k\rangle$. We have already established that such elements cannot be from the set E', so a must be in some version of E_k defined at or after stage t_0. However, E_k will not change its value after t_0, because otherwise we will have a $\beta\hat{}\infty_X$-true stage, contradicting our assumption. As we have also assumed $E_k \subset A$, we have reached the desired contradiction.

b. $\delta \subset \beta$. We treat G^W and S^W-strategies separately.

If δ is a G^W-strategy then in order to restrain elements out of A on stage t_1 it must have outcome $o = 0$. It cannot be that $\beta \supseteq \delta\hat{}1$ or β would be initialized on stage t_1. Hence $\delta\hat{}0 \subseteq \beta$ and δ is not initialized on stages t such that $t_0 < t \leq t_1$. Therefore $a \in E_{t_0}^{\beta}$ and $a \notin E'$.

If δ is a S^W-strategy then a is included in some agitator Ag which is taken out of A on stage t_1. Whenever a S^W-strategy chooses an agitator it moves on to the right. If the agitator is formed on stage $t \leq t_0$ then, since on stage t_0 the strategy β is visited and sees a in A, we can conclude that $\beta \supseteq \delta\hat{}d_0$, but then on stage t_1 it must be initialized.

Suppose Ag is formed on stage $t > t_0$. Then a was extracted from A on the previous δ-true stage t^- by one of the strategies extending δ. Our choice of t_1 as the first stage after t_0 on which a is extracted from A guarantees that $t^- = t_0$. But we know that the only strategy that can extract a on stage t_0 is β, hence $a \in E_k \subset A$. □

Lemma A.1.5. *Let $\beta\hat{}\langle l, k\rangle$ be visited on stage t_0. If β is not initialized or visited on stages $t > t_0$ and $(E_k \cup F_l^k) \not\subset A$ then $(E_k \cup F_l^k \cup E_{t_0}^{\beta}) \cap A = \emptyset$.*

Proof. Let $(E_k \cup F_l^k) \not\subset A$. First we will prove that $(E_k \cup F_l^k) \cap A = \emptyset$. Let $a \in E_k \cup F_l^k$. Then a is restrained out of A by some G^W-strategy $\gamma \supset \beta$ on some stage $t' < t_0$ after $\beta's$ last initialization as we established in Proposition A.1.1. As β is not initialized or visited anymore, no other G^W-strategy can restrain the element a out of A. Indeed G^W-strategies of higher priority than β would initialize β if they restrained a new element. The ones to the right of β are initialized on stage t' and choose their parameter λ to be of length greater than a. So if $a \notin A_t$ then a is restrained out of A by some S^W-strategy $\delta \subset \beta$. We can even say that $\delta\hat{}\infty_X \subseteq \beta$, if a is included in some agitator $E_{k'}$, and $\delta\hat{}\langle\infty_Y, k\rangle \subseteq \beta$, if a is included in some agitator $F_{l'}^{k'}$, again using the result from Proposition A.1.1. Moreover the agitator is chosen on stage $t_1 > t_0$, as after the strategy δ chooses its agitator it has outcomes to the right of β until the agitator is cancelled.

Suppose a is taken out of A on stage $t > t_0$ by $\beta_1 \subset \beta$. Then a is included in the agitator Ag_1 of β_1, chosen on stage $t_1 > t_0$. So $a \notin A_{t_1^-}$ and $t_1^- \geq t_0$. If $t_1^- = t_0$ then $E_k \cup F_l^k \subseteq Ag_1$. If $t_1^- > t_0$ then there is another strategy β_2 such that $\beta_1 \subset \beta_2 \subset \beta$ and a is included in one of its agitators Ag_2. With a similar argument we get a monotone decreasing sequence of stages $t_1 > t_2 > \dots$ bounded by t_0, hence finite.

Therefore always when $a \notin A_t$, we have a finite sequence of S^W-strategies $\beta_1 \subset \beta_2 \subset \cdots \subset \beta$ and a corresponding monotone sequence of their agitators $Ag_1 \supset Ag_2 \supset \cdots \supset (E_k \cup F_l^k)$ such that Ag_1 is restrained out of A on stage t. If $a \notin A_t$ and $t > t_0$ then $(E_k \cup F_l^k) \cap A_t = \emptyset$ and ultimately $(E_k \cup F_l^k) \cap A = \emptyset$.

Let us assume now that $b \in E_{t_0}^{\beta} \cap A \neq \emptyset$. Then there is a stage t_b such that $b \in A_t$ for all $t > t_b$. Let t' be a stage for which $(E_k \cup F_l^k) \cap A_t = \emptyset$ and $t' > t_b$. Then there is a series of S^W strategies $\beta_1 \subset \beta_2 \subset \cdots \subset \beta_n \subset \beta$ and a corresponding series of their agitators $Ag_1 \supset Ag_2 \supset \cdots \supset (E_k \cup F_l^k)$. According to Lemma A.1.1, we can express $E_{t_0}^{\beta}$ in the following way:

$E_{t_0}^{\beta} = E_{t_1}^{\beta_{t'}} \cup (E_{\beta_2}^{\beta_1})_{t_2} \cup \cdots \cup (E_{\beta}^{\beta_n})_{t_0}$.

If $b \in (E_{\beta_2}^{\beta_1})_{t_2} \cup \cdots \cup (E_{\beta}^{\beta_n})_{t_0}$ then $b \in Ag_1$ and therefore $b \notin A_{t'}$ contradicting the choice of $t' > t_b$. Therefore $E_{t_0}^{\beta} \cap A = \emptyset$. □

The True Path

The true path will ultimately be the path along which each strategy satisfies its requirement. It will be as usual the leftmost path visited infinitely often. It is not obvious that such a path exists, as our tree of strategies is infinitely branching. Fortunately we can prove the following:

Lemma A.1.6. *There exists an infinite path f in T with the following properties:*

1. $\forall n \exists^{\infty} t (f \restriction n \subseteq \delta_t)$ *- the infinite property,*
2. $\forall n \exists t_n \forall t > t_n (f \restriction n \not<_L \delta_t)$ *- the leftmost property.*

Proof. We will define f by induction on n and simultaneously prove that it has the desired properties. First $f \restriction 0 = \varnothing$ obviously has both properties. It is visited on every stage and $t_0 = 0$. Now let's assume we have defined $f \restriction n$ with the desired properties. We will define $f \restriction n+1 = (f \restriction n)\hat{\ }o$ where o is an outcome of the strategy $f \restriction n$. We will refer to this outcome as *the true outcome.*

I. If $f \restriction n$ is a R-strategy then $o = 0$. We always visit $f \restriction (n+1)$ when we visit $f \restriction n$, hence infinitely often and $t_{n+1} = t_n$.

II. If $f \restriction n$ is a G^W-strategy then the possible outcomes are 0 and 1. As we visit $f \restriction n$ infinitely many times, at least one of the two outcomes will also be visited infinitely many times. If

$$\exists^\infty t[(f \upharpoonright n)\hat{}0 \subseteq \delta_t]$$

then $o = 0$. As this is the leftmost possible outcome $t_{n+1} = t_n$.

Otherwise $(f \upharpoonright n)\hat{}0$ is visited only finitely many times and there exists a stage t_1 such that $\forall t > t_1[f \upharpoonright n \subseteq \delta_t \Rightarrow (f \upharpoonright n)\hat{}1 \subseteq \delta_t]$. Then $o = 1$ and $t_{n+1} = \max(t_n, t_1)$.

III. If $f \upharpoonright n$ is a S^W-strategy then:

(a) If

$$\exists^\infty t[(f \upharpoonright n)\hat{}\infty_X \subseteq \delta_t]$$

then $o = \infty_X$ and $t_{n+1} = t_n$.

Otherwise there exists a least $f \upharpoonright n$-true stage t_1 such that

$$\forall t \geq t_1[f \upharpoonright n \subseteq \delta_t \Rightarrow (f \upharpoonright n)\hat{}\infty_X \not\subseteq \delta_t].$$

On stage t_1 the strategy $f \upharpoonright n$ chooses an agitator E_k and has outcome $\langle \infty_Y, k\rangle$. Then for all stages greater than t_1 the possible outcomes are $\langle \infty_Y, k\rangle$, $\{\langle l, k\rangle | l \in \mathbb{N}\}$ and d_0.

(b) If

$$\exists^\infty t[(f \upharpoonright n)\hat{}\langle \infty_Y, k\rangle \subseteq \delta_t]$$

then $o = \langle \infty_Y, k\rangle$ and $t_{n+1} = \max(t_n, t_1)$.

Otherwise there exists a least $f \upharpoonright n$-true stage $t_2 \geq t_1$ such that

$$\forall t \geq t_2[f \upharpoonright n \subseteq \delta_t \Rightarrow (f \upharpoonright n)\hat{}\infty_X \not\subseteq \delta_t \wedge (f \upharpoonright n)\hat{}\langle \infty_Y, k\rangle \not\subseteq \delta_t].$$

Then on stage t_2 the strategy $f \upharpoonright n$ chooses a second agitator F_l^k and has outcome d_0. For all stages $t > t_2$ the possible outcomes are d_0 and $\langle l, k\rangle$.

If on some stage $t_3 > t_2$ we have an outcome $\langle l, k\rangle$ then on all stages $t \geq t_3$ we would have this outcome, because you can't return from outcome $\langle l, k\rangle$ back to d_0 without passing through $\langle \infty_Y, k\rangle$ or ∞_X.

(c) If the outcome $\langle l, k\rangle$ never occurs, that is

$$\forall t \geq t_2[f \upharpoonright n \subseteq \delta_t \Rightarrow (f \upharpoonright n)\hat{}d_0 \subseteq \delta_t],$$

then $o = d_0$ and $t_{n+1} = \max(t_n, t_2)$.

(d) Otherwise there is a stage t_3 such that

$$\forall t \geq t_3[f \upharpoonright n \subseteq \delta_t \Rightarrow (f \upharpoonright n)\hat{}\langle l, k\rangle \subseteq \delta_t].$$

Then $o = \langle l, k\rangle$ and $t_{n+1} = \max(t_n, t_3)$.

□

Unfortunately the leftmost property does not guarantee that the strategies along the true path will be initialized only finitely many times and will be able to satisfy their requirements eventually. This is due to the second case of initialization. That is why we need to prove this separately.

Lemma A.1.7 (Stability Lemma)**.** *For every S^W-strategy β the following statement is true:*

1. If $\beta\hat{}\infty_X \subseteq f$ then for every $k \in U$ there exists an axiom $\langle k, E'\rangle \in \Theta_0$ and a stage t_k such that if $t > t_k$ and β is visited on t with $o^- = \infty_X$ then $\langle k, E'\rangle$ is applicable for k and therefore k does not draw attention. Furthermore $E' \subseteq A$.

2. If $\beta\hat{}\langle\infty_Y, k\rangle \subseteq f$ then for every $l \in V_k$ there exists an axiom $\langle l, F'\rangle \in \Theta_1$ and a stage t_l such that if $t > t_l$ and β is visited on t with $o^- = \langle\infty_Y, k\rangle$ then $\langle l, F'\rangle$ is applicable for l and therefore l does not draw attention. Furthermore $F' \subseteq A$.

Proof. Assume that this is not the case and choose $\beta \subseteq f$ as the least strategy for which the proposition is false. Suppose $\beta\hat{}\infty_X \subseteq f$. The case $\beta\hat{}\langle\infty_Y, k\rangle \subseteq f$ is similar. Let $k \in U^\beta$ be the least element such that k draws attention infinitely many times.

Let $\Gamma = \{\gamma \supseteq \beta\hat{}\infty_X | \gamma$ is a G^W-strategy with local priority less than k $\}$.

We choose a stage t so big that:

a. If $\beta' \subset \beta$ is a S^W-strategy such that $\beta'\hat{}\infty_X \subseteq \beta$ then the elements of $U_{\beta'}$ which are less than or equal to the local β'-priority of any $\gamma \in \Gamma$ are already in $U_{\beta'}$ and do not draw attention any more. For these elements there is an applicable axiom and let all axioms with a smaller code that get applied at some stage be already applied. According to our choice of β as the least strategy for which the proposition is not true, this choice of t is satisfiable.

b. Similarly if $\beta' \subset \beta$ is a S^W-strategy such that $\beta'\hat{}\langle\infty_y, k'\rangle \subseteq \beta$ then the elements of $V_{k'}^{\beta'}$ which are less than or equal than the local β'-priority of any $\gamma \in \Gamma$ are already in $V_{k'}^{\beta'}$, do not draw attention anymore and do not apply any new axioms.

c. For all elements $m \in U$ such that $m \leq k$ we have $m \in U_t$.

d. All elements $m < k$ do not draw attention on stages $s > t$ and do not apply any axioms.

e. Let $M = \max\{lh(\gamma)|\gamma \in \Gamma\} + 2$. Let t_M be the stage for which $\forall s > t_M(\delta_s \not<_L f \upharpoonright M)$ from the leftmost property of f. Then $t > t_M$.

According to our choice of t, precisely conditions a, b and e, it is true that for all $s > t$, β does not get initialized on stage s. Then Lemma A.1.1 gives us that E_s^β is the same on all β-true stages $s > t$. We can therefore omit the index s in further discussions and refer to this set as E^β.

Let $t_1 > t$ be a stage on which $f \upharpoonright M$ is visited. On the next β-true stage t_1^+ the previous outcome is ∞_X. We scan the elements of U and change their corresponding elements in $\mathbb{U}$ if needed. The elements $m < k$ do not draw attention anymore, but it is still possible that k draws attention.

1. If k does not draw attention then for the axiom $\langle k, E'\rangle$ in $\mathbb{U}$ we have that:

   (a) $E' \cap E^\beta = \emptyset$,

   (b) $E' \cap Out1_{t_1^+} = \emptyset$.

2. If k does draw attention on stage t_1^+ then we define an agitator $E_k = Out1_{t_1^+}$ and move to the right of the true path. Let t_2 be the next stage on which $\beta\hat{}\infty_X$ is visited. On this stage we must have found an axiom $\langle k, E''\rangle$ for which again:
   (a) $E'' \cap E^\beta = \emptyset$,
   (b) $E'' \cap Out1_{t_1^+} = \emptyset$.

In both cases we have an axiom $\langle k, E^0\rangle$ for which the two conditions hold. Let $t_3 > t_1$ be a $f \upharpoonright M$-true stage by which this strategy is applied. We will prove that no strategy extracts elements from E^0 on stages $s > t_3$. Hence this axiom will be the one we are searching for.

Note that after this axiom has been applied, none of the strategies that have been initialized during or after this application can ever restrain any elements of E^0 out of A, including all strategies below $f \upharpoonright M$. At stage t_1 all axioms to the right of $f \upharpoonright M$ have been initialized. In the first case the axiom is applied not later than on stage t_1^+. The strategies to the right of $\beta\hat{}\infty_X$ are initialized on that stage and the strategies below $\beta\hat{}\infty_X$ that are to the right of $f \upharpoonright M$ are not visited after their initialization until t_1^+.

In the second case the axiom is applied on stage t_2 and again strategies to the right of $\beta\hat{}\infty_X$ are initialized on that stage and the strategies below $\beta\hat{}\infty_X$ that are to the right of $f \upharpoonright M$ are not visited after their initialization until t_2.

Strategies to the left of $f \upharpoonright M$ are not visited after stage $t_1 < t_3$ and can not restrain elements from E^0 out of A at any later stage.

The only danger is that a strategy δ along $f \upharpoonright M$ restrains an element from E^0 out of A on stage $s > t_3$. We will prove that this also does not happen.

First of all if δ is a G^W-strategy, by stage t_1 its outcome is final and so are all elements that it restrains out of A. These elements are in E^β if $\delta \subset \beta$ or in $Out1_{t_1^+}$ if $\delta \supset \beta$. In particular a is not restrained by δ out of A on any stage $s > t_3$.

If δ is an S^W-strategy then the elements it restrains out of A are the ones in its agitators. We need to consider the possible ways that such agitators might be constructed. So suppose that δ has an agitator Ag that it extracts on stage $s > t_3$.

Notice first that our approximation of the true path δ_s never goes left of $f \upharpoonright M$ after stage t_1. Thus δ does not have outcomes to the left of the outcome it had on stage t_1.

Suppose that δ had already chosen this agitator Ag by stage t_1, that is Ag has already a value on stage t_1 and does not change its value until stage s on which it is out of A. If $Ag \subset A_{t_1}$ then δ has outcome d_0 on stage t_1. This is the rightmost outcome and as δ does not have outcomes to the left of it on further stages it will not extract Ag on stage s. Thus Ag is restrained out of A on stage t_1. Hence $Ag \subset E^\beta \cup Out1^\beta_{t_1^+}$ and does not contain elements from E^0.

We are left with the case when δ chooses Ag after stage t_1. This limits the possibilities for the true outcome of δ. We can have $\delta\hat{}\infty_X \subset f$ in which case each agitator that δ ever chooses is eventually cancelled. We can also have $\delta\hat{}\langle\infty_Y, k\rangle \subset f$ in which case the agitator E_k is chosen before stage t_1 and does not contain elements from A, as we have just established, and each agitator F^k_l is eventually cancelled.

We will prove that agitators formed after stage t_3 cannot contain elements from E^0. This will show that the elements from E^0 can be extracted from A only finitely many times and hence $E^0 \subset A$.

It is convenient to consider each S^W-strategy $\delta \subset f \upharpoonright M$ in order of its length, starting from the longest. The reason is that strategies of lower priority determine the elements that enter agitators of higher priority strategies.

Let δ be the longest S^W-strategy along $f \upharpoonright M$. Suppose δ chooses an agitator Ag on stage $s > t_3$. All of Ag's elements were restrained by strategies extending δ on the previous δ-true stage $s^- \leq t_3$. These are either strategies that were initialized when the axiom $\langle k, E^0 \rangle$ was applied and hence cannot restrain elements from E^0, or G^W-strategies $\gamma \subset f \upharpoonright M$ which as we already proved do not restrain elements from E^0.

By induction we can prove the same for the shorter S^W-strategies. □

Corollary A.1.2. *Every strategy along the true path is eventually not initialized.*

Proof. We will prove by induction on n that there is a $f \upharpoonright n$-true stage t^*_n such that $f \upharpoonright n$ is not initialized on any stage $t > t^*_n$. We will refer to this stage t^*_n in the rest of the proof.

The case $n = 0$ is trivial because $f \upharpoonright 0 = \varnothing$ is never initialized and is visited on every stage, so $t^*_0 = 0$.

Assume that $f \upharpoonright n$ is visited on stage t^*_n and not initialized on stages $t > t^*_n$. If $f \upharpoonright (n+1)$ is a R- or S^W-strategy then $f \upharpoonright n$ is a G^W-strategy and it does not initialize strategies in its subtree at all. So let t^*_{n+1} be the first stage on which $f \upharpoonright (n+1)$ is visited after $\max\{t^*_n, t_{n+1}\}$ where t_{n+1} is the stage from the leftmost property of the true path (second property of LemmaA.1.6). Then $f \upharpoonright (n+1)$ is not initialized on stages $t > t^*_{n+1}$.

If $f \upharpoonright (n+1)$ is a G^W-strategy then we choose t^*_{n+1} so that the following conditions hold

1. $t^*_{n+1} > t^*_n$.

2. $t^*_{n+1} > t_{n+1}$ where t_{n+1} is the stage from the leftmost property of the true path.

3. For every S^W-strategy β with $\beta\hat{\ }\infty_X \subseteq f \upharpoonright (n+1)$ and every $k \in U_\beta$ less than the local β-priority of $f \upharpoonright (n+1)$, we have an applicable axiom $\langle k, E_0 \rangle$ which is applicable on every stage after t_k. There are finitely many axioms with a code that is less than that of E_0. Let t^*_{n+1} be so big that all axioms with a code that is smaller than the code of E_0 and that get applied at some point are already applied.

4. For every S^W-strategy β with $\beta\hat{\ }\langle \infty_Y, k \rangle \subseteq f \upharpoonright (n+1)$ and every $l \in V^\beta_k$ less than the local β-priority of $f \upharpoonright (n+1)$, we have an applicable axiom $\langle l, F_0 \rangle$ which is applicable on every stage after t_l. There are finitely many axioms with a code that is less than that of F_0. Let t^*_{n+1} be so big that all axioms with a code that is smaller than the code of F_0 and that get applied at some stage are already applied.

5. $f \upharpoonright (n+1)$ is visited on stage t^*_{n+1}.

It follows from Lemmas A.1.6 and A.1.7 that this choice of t^*_{n+1} is satisfiable. Clause 2 ensures that $f \upharpoonright (n+1)$ will not be initialized by strategies to the left. Clause 1 ensures that it won't initialized due to the initialization of G^W-strategies that $f \upharpoonright (n+1)$ extends and finally clauses 3 and 4 ensure that $f \upharpoonright (n+1)$ won't be initialized due to S^W-strategies that it extends. □

Satisfaction of The Requirements

Lemma A.1.8. *Every R requirement is satisfied.*

Proof. Fix a R-requirement. Let α be the corresponding R-strategy on the true path. We will prove that $\Theta_0(A) = X$ and $\Theta_1(A) = Y$ do not form a minimal pair. The proof is divided into the following three cases depending on the true outcomes of the S^W-substrategies of α along the true path:

1. All S^W-strategies $\hat{\beta} \subset f$, substrategies of α, have true outcomes d_0 or $\langle l, k\rangle$. Then we will prove that $\Phi_0(X) = \Phi_1(Y) = D$ and D is not c.e.
2. There is a S^W-strategy $\hat{\beta} \subset f$, substrategy of α, with true outcome ∞_X. Then X will be c.e.
3. There is a S^W-strategy $\hat{\beta} \subset f$, substrategy of α, with true outcome $\langle \infty_Y, k\rangle$. Then Y will be c.e.

We will treat each case separately.

1. For all S^W strategies $\hat{\beta} \subset f$, substrategies of α,
$$\exists k \exists l (\hat{\beta}\hat{\ }\langle l, k\rangle \subset f) \vee \hat{\beta}\hat{\ } d_0 \subset f.$$
We start by proving that $\Phi^\alpha_0(X) = \Phi^\alpha_1(Y)$. Now the properties of the agitators proved in Section 2.6 will play an important role as the operators Φ_0 and Φ_1 are constructed by *all* of α's substrategies, not only the ones along the true path. So we have to prove that $\Phi_0(X)(d^\beta) = \Phi_1(Y)(d^\beta)$, for every witness d^β that any substrategy β has ever used.

   We automatically have this equality for any witness d^β that is cancelled. Cancelling the witness involves enumerating the axiom $\langle d^\beta, \emptyset\rangle$ in both operators. So $\Phi_0(X)(d^\beta) = \Phi_1(Y)(d^\beta) = 1$.

   This means that strategies β to the right of the true path will not cause problems. Strategies to the left of and on the true path may have witnesses that are never cancelled. So let β be a substrategy of α and d^β be a witness chosen on stage t_0 that is never cancelled. Then β has outcome d_0 on stage t_0. After stage t_0 the strategy β is not initialized and does not have outcomes ∞_X or $\langle \infty_Y, k\rangle$, as in those cases we would cancel β's witness d. Let the corresponding agitators for d be E_k and F^k_l, so we have axioms $\langle d, \{k\}\rangle \in \Phi_0$ and $\langle d, \{l\}\rangle \in \Phi_1$. Also note that the length of the node β is necessarily less than t_0, as according to the construction a strategy acts only on stages s greater than its length.

We have the following three possibilities:

(a) $\beta <_L f$. Then let $t \geq t_0$ be the last stage on which β is visited.

If $\beta\hat{}\langle l,k\rangle \subseteq \delta_t$ then the conditions of Lemma A.1.5 are true. Therefore if $(E_k \cup F_l^k) \not\subseteq A$ then $(E_k \cup F_l^k \cup E_t^\beta) \cap A = \emptyset$. If $(E_k \cup F_l^k) \subseteq A$ then according to Lemma A.1.4 we have $k \in X$ and $l \in Y$ and therefore $\Phi_0(X)(d) = \Phi_1(Y)(d) = 1$.

If $(E_k \cup F_l^k \cup E_t^\beta) \cap A = \emptyset$ then from the proof of Lemma A.1.5 we can conclude that there is an entry $\langle \beta : \langle E_t, E_k, F_l^k\rangle, d\rangle \in Watched_\alpha$. In this case we claim $\Phi_0(X)(d) = \Phi_1(Y)(d) = 0$. Suppose for a contradiction that this is not true, say $\Phi_0(X)(d) = 1$. Then the only axiom in Φ_1 for d is true, so $k \in X = \Theta_0(A)$. Therefore there is an axiom $\langle k, E'\rangle \in \Theta_0$ such that $E' \subseteq A$ and hence $E' \cap (E_k \cup E_t^\beta) = \emptyset$. It appears in Θ_0^s on some stage s. The strategy $\alpha \subset f$ is visited on some stage $s' > s$. According to the construction α will spot this axiom while examining the entry for β in $Watched$ and cancel d. Similarly we may prove that $\Phi_1(Y)(d) = 0$.

If $\beta\hat{}d_0 \subseteq \delta_t$, as β is not initialized on stages $s' > t$, we have that $E_k \cup F_l^k$ is restrained in A by β. From Lemmas A.1.2 and A.1.3 it follows that $E_k \cup F_l^k \subset A$. Lemma A.1.4 gives us $k \in X$ and $l \in Y$. Hence $\Phi_0(X)(d) = \Phi_1(Y)(d) = 1$.

(b) Suppose $\beta\hat{}d_0 \subseteq f$. By Lemma A.1.6 and the fact that d is not cancelled whenever we visit β outcome d_0 from stage t_0 on. Therefore by Lemmas A.1.2 and A.1.3 $E_k \cup F_l^k \subset A$. Lemma A.1.4 gives us $k \in X$ and $l \in Y$. Hence $\Phi_0(X)(d) = \Phi_1(Y)(d) = 1$.

(c) If $\beta\hat{}\langle l,k\rangle \subseteq f$ then by Lemma A.1.6 there is a stage $t_1 > t_0$ such that on β-true stages $t > t_1$ the strategy β always has this outcome and $E_k \cup F_l^k$ is extracted from A_t. Also by Lemma A.1.1 $E_t^\beta = E_{t_1}^\beta$ for all β-true stages $t > t_1$ and we will refer to this set as E^β. As β is visited on infinitely many stages $(E_k \cup F_l^k \cup E^\beta) \cap A = \emptyset$. We claim that in this case $\Phi_0(X)(d) = \Phi_1(Y)(d) = 0$.

Assume for a contradiction that this is not true, say $\Phi_0(X)(d) = 1$. Then there is an axiom $\langle k, E'\rangle \in \Theta_0$ with $E' \subseteq A$ and therefore $E' \cap (E_k \cup F_l^k \cup E^\beta) = \emptyset$. The axiom appears in $\Theta_0^{s_1}$ on some stage s_1 . Let s be a β-true stage such that $s > \max(s_1, t_1)$. According to the construction on stage s the strategy β will have outcome ∞_X contradicting the choice of t_1. Similarly we may prove that $\Phi_1(Y)(d)$ cannot equal 1.

This gives us a set $D = \Phi_0(X) = \Phi_1(Y)$. To prove that D is not c.e. let W be any c.e. set and consider the S^W-substrategy $\hat{\beta}$ along the true path. Let $n = lh(\hat{\beta})$. After stage t_{n+1} from Lemma A.1.6 $\hat{\beta}$ always has its true outcome whenever it is visited and a permanent witness $\hat{d}$. This witness will prove $W \neq D$.

If $\hat{\beta}\hat{}\langle l,k\rangle \subset f$ then $W(\hat{d}) = 1$. The witness $\hat{d}$ is not cancelled by α. Indeed if α cancels the witness at stage t due to some axiom $\langle k, E'\rangle \in \Theta_0$ or $\langle l, F'\rangle \in \Theta_1$ then when we visit $\hat{\beta}$ on stage $t_1 \geq \max(t, t_{n+1})$ the strategy $\hat{\beta}$ would see this axiom and have outcome

∞_X or $\langle\infty_Y, k\rangle$ contradicting our choice of stage t_1. We just proved that in this case $D(\hat{d}) = 0$. Therefore $D \neq W$.

If $\hat{\beta}\hat{\ }d_0 \subset f$. Then the witness will not be cancelled by α as there will not be an entry for it in the list $Watched_\alpha$. We proved that $D(\hat{d}) = 1$. It follows that $W(\hat{d}) = 0$ as otherwise there would be a stage $s > t_{lh(n)+1}$ on which $\hat{d} \in W_s$. Then on the next $\hat{\beta}$-true stage we would have an outcome $\langle l, k\rangle <_L d_0$. Therefore $D \neq W$.

2. There is a strategy $\hat{\beta}$, substrategy of α, with $\hat{\beta}\hat{\ }\infty_X \subseteq f$. Let $n = lh(\hat{\beta})$. We will prove that $U_{\hat{\beta}} = X$ and so X is c.e. Assume for a contradiction that there is an element $k \in X\backslash U$ and choose the least one. Then there is an axiom $\langle k, E'\rangle \in \Theta_0$ such that $E' \subset A$. Note that on all $\hat{\beta}$-true stages $s > t^*_{n+1}$ by Lemma A.1.1 $E^{\hat{\beta}}_s = E^{\hat{\beta}}_{t^*_{n+1}}$ and $E^{\hat{\beta}}_s \subset \overline{A}$. So $E' \cap E^{\hat{\beta}}_s = \emptyset$. Let $t > t^*_{n+1}$ be a stage on which all elements smaller than k that ever enter U are already in U and all elements that are in E' are not taken out of A anymore. Then k will enter U on the next $\hat{\beta}$-true stage on which $o^- = \infty_X$, if not before.

   According to Lemma A.1.7 for every $k \in U$ there is an axiom $\langle k, E'\rangle \in \Theta_0$ for which $E' \subseteq A$, therefore $k \in X$ and $U \subseteq X$. Ultimately we get $X = U$.

3. There is a S^W-strategy $\hat{\beta}$ which is a substrategy of α with $\hat{\beta}\hat{\ }\langle\infty_Y, k\rangle \subset f$ for some k. We show in this case that $V^{\hat{\beta}}_k = Y$ and therefore Y is c.e. The proof is similar to part 2.

□

Lemma A.1.9. *Every G^W requirement is satisfied.*

Proof. Fix a c.e. set W and consider the G^W-strategy $\gamma \subset f$. Let $n = lh(\gamma)$. Let λ and μ denote the values of γ's parameters on stage t^*_{n+1} from Corollary A.1.2. It follows from the construction that these values remain the same on further stages. Indeed λ changes value only after initialization and μ changes value only when γ switches to outcome 0. We will prove that $\lambda\hat{\ }\mu \subset A$ and that $\lambda\hat{\ }\mu \in W$ or for every extension $\tau \supseteq \lambda\hat{\ }\mu$ we have $\tau \notin W$ and so the requirement G^W is satisfied.

By Lemma A.1.1 the value of the set E^γ_t does not change on γ-true stages $t > t^*_{n+1}$ and we will refer to it as E^γ. Finally γ has always its true outcome on true stages $t > t^*_{n+1}$.

If $\lambda\hat{\ }\mu(a) = 1$ then a is restrained in A by γ and by Lemmas A.1.2 and A.1.3 $a \in A$. If $\lambda\hat{\ }\mu(a) = 0$ and $a < lh(\lambda)$ then $a \in E^\gamma \subset \overline{A}$ so $A(a) = 0$. If $\lambda\hat{\ }\mu(a) = 0$ and $a \geq lh(\lambda)$ then a is extracted on every γ-true stage $t \geq t^*_{n+1}$ and $A(a) = 0$. Therefore $\lambda\hat{\ }\mu \subset A$.

If $\gamma\hat{\ }0 \subset f$ then this outcome was visited after we saw that $\lambda\hat{\ }\mu \in W_{t^*_{n+1}} \subset W$. If $\gamma\hat{\ }1 \subset f$ then $\mu = \varnothing$ and for all extensions $\tau \supseteq \lambda$ we have $\tau \notin W$. Indeed if there were an extension of λ, $\tau \in W$, then it would appear in the approximation of W on some finite stage and on the next γ-true stage we would have outcome 0 contradicting the choice of t^*_{n+1}.

This concludes the proof of the lemma and the theorem. □

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A.2 Cupping Δ_2^0 enumeration degrees to $\mathbf{0}'_e$

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Abstract: In this paper we prove that every nonzero Δ_2^0 e-degree is cuppable to $\mathbf{0}'_e$ by a 1-generic Δ_2^0 e-degree (so low and nontotal) and that every nonzero ω-c.e. e-degree is cuppable to $\mathbf{0}'_e$ by an incomplete 3-c.e. e-degree.

Introduction

Intuitively, we say that a set A is *enumeration reducible* to a set B, denoted as $A \leq_e B$, if there is an effective procedure to enumerate A, given any enumeration of B. More formally, $A \leq_e B$ if there is a computably enumerable set W such that

$$A = \{x : (\exists u)[\langle x, u\rangle \in W \ \&\ D_u \subseteq B]\}$$

where D_u is the finite set with canonical index u.

Let $\equiv_e$ denote the equivalence relation generated by $\leq_e$ and let $[A]_e$ be the equivalence class of A — the *enumeration degree* (*e-degree*) of A. The degree structure $\langle \mathcal{D}_e, \leq\rangle$ is defined by setting $\mathcal{D}_e = \{[A]_e : A \subseteq \omega\}$ and setting $[A]_e \leq [B]_e$ if and only if $A \leq_e B$. The operation of least upper bound is given by $[A]_e \vee [B]_e = [A \oplus B]_e$ where $A \oplus B = \{2x : x \in A\} \cup \{2x + 1 : x \in B\}$. The structure $\mathcal{D}_e$ is an upper semilattice with least element $\mathbf{0}_e$, the collection of computably enumerable sets. Gutteridge [9] proved that $\mathcal{D}_e$ does not have minimal degrees (see Cooper [1]).

An important substructure of $\mathcal{D}_e$ is given by the Σ_2^0 e-degrees i.e. the e-degrees of Σ_2^0 sets. Cooper [2] proved that Σ_2^0 e-degrees are the e-degrees below $\mathbf{0}'_e$, the e-degree of $\overline{K}$. An e-degree is Δ_2^0 if it contains a Δ_2^0 set, a set A with a computable approximation f such that for every element x, $f(x, 0) = 0$ and $\lim_s f(x, s)$ exists and equals to $A(x)$. Cooper and Copestake [5] proved that below $\mathbf{0}'_e$ there are e-degrees that are not Δ_2^0. These e-degrees are called *properly* Σ_2^0 *e*-degrees.

In this paper we are mainly concerned with the cupping property of Δ_2^0 e-degrees. An e-degree $\mathbf{a}$ is cuppable if there is an incomplete e-degree $\mathbf{c}$ such that $\mathbf{a} \cup \mathbf{c} = \mathbf{0}'_e$. In [6], Cooper, Sorbi and Yi proved that all nonzero Δ_2^0 e-degrees are cuppable and that there are noncuppable Σ_2^0 e-degrees.

Theorem A.2.1. *(Cooper, Sorbi and Yi* [6]*) Given a nonzero Δ_2^0 e-degree $\mathbf{a}$, there is a total Δ_2^0 e-degree $\mathbf{c}$ such that $\mathbf{a} \cup \mathbf{c} = \mathbf{0}'_e$, where an e-degree is total if it contains the graph of a total function. Meanwhile, noncuppable e-degrees exist.*

In this paper we first prove that each nonzero Δ_2^0 e-degree $\mathbf{a}$ is cuppable to $\mathbf{0}'_e$ by a non-total Δ_2^0 e-degree.

Theorem A.2.2. *Given a nonzero Δ_2^0 e-degree* **a**, *there is a 1-generic Δ_2^0 e-degree* **b** *such that* $\mathbf{a} \cup \mathbf{b} = \mathbf{0}'_e$. *Since 1-generic e-degrees are quasi-minimal and 1-generic Δ_2^0 e-degrees are low,* **b** *is nontotal and low.*

Here a set A is *1-generic* if for every computably enumerable set S of $\{0,1\}$-valued strings there is some initial segment σ of A such that either S contains σ or S contains no extension of σ. An enumeration degree is 1-generic if it contains a 1-generic set. Obviously, no nonzero e-degree below a 1-generic e-degree contains a total function and hence 1-generic e-degrees are quasi-minimal. Copestake proved that a 1-generic e-degree is low if and only if it is Δ_2^0 (see [7]).

Our second result is concerned with cupping ω-c.e. e-degrees to $\mathbf{0}'_e$. A set A is n-c.e. if there is an effective function f such that for each x, $f(x,0) = 0$, $|\{s+1 \mid f(x,s) \neq f(x,s+1)\}| \leq n$ and $A(x) = \lim_s f(x,s)$. A is is ω-c.e. if there are two computable functions $f(x,s), g(x)$ such that for all x, $f(x,0) = 0$, $|\{s+1 \mid f(x,s) \neq f(x,s+1)\}| \leq g(x)$ and $\lim_s f(x,s) \downarrow = A(x)$.

An enumeration degree is n-c.e. (ω-c.e.) if it contains an n-c.e. (ω-c.e.) set. It's easy to see that the 2-c.e. e-degrees are all total and coincide with the Π_1 e-degrees, see [3]. Cooper also proved the existence of a 3-c.e. nontotal e-degree. As the construction presented in [6] actually proves that any nonzero n-c.e. e-degree can be cupped to $\mathbf{0}'_e$ by an $(n+1)$-c.e. e-degree, we will prove that any nonzero ω-c.e. e-degree is cuppable to $\mathbf{0}'_e$ by a 3-c.e. e-degree.

Theorem A.2.3. *Given a nonzero ω-c.e. e-degree* **a**, *there is a 3-c.e. e-degree* **b** *such that* $\mathbf{a} \cup \mathbf{b} = \mathbf{0}'_e$.

This is the strongest possible result. We explain it as follows. Consider the standard embedding ι of $\mathcal{D}_T$ to $\mathcal{D}_e$ given by: $\iota(deg_T(A)) = deg_e(\chi_A)$ where χ_A denotes the graph of the characteristic function of A. It is well-known that ι is an order-preserving mapping and that the Π_1 enumeration degrees are exactly the images of the Turing c.e degrees under ι. Consider a noncuppable c.e. degree **a**. $\iota(\mathbf{a})$ is Π_1, hence ω-c.e., and $\iota(\mathbf{a})$ is not cuppable by any Π_1 e-degree, as ι preserves the least upper bounds. Therefore, no 2-c.e. e-degree cups $\iota(\mathbf{a})$ to $\mathbf{0}'_e$.

We use standard notation, see [4] and [10].

Basic ideas of Cooper-Sorbi-Yi's cupping

In this section we describe the basic ideas of Cooper-Sorbi-Yi's construction given in [6]. Let $\{A_s\}_{s<\omega}$ be a Δ_2^0 approximation of the given Δ_2^0 set A which is assumed to be not computably enumerable. We will construct two Δ_2^0 sets B and E (auxiliary) and an enumeration operator Γ such that the following requirements are satisfied:

$$S : \Gamma^{A,B} = \overline{K}$$

$$N_\Phi : E \neq \Phi^B$$

The first requirement is the global cupping requirement and it guarantees that the least upper bound of the degrees of A and B is $\mathbf{0}'_e$. Here $\Gamma^{A,B}$ denotes an enumeration operation relative to the enumerations of A and B.

The second group of requirements N_Φ, where Φ ranges over all enumeration operators, guarantees that the degree of B is not complete. Indeed, we have a witness — the degree of E is not below that of B.

To satisfy the global requirement S we will construct by stages an enumeration operator Γ such that $\overline{K} = \Gamma^{A,B}$. That is, at stage s we find all $x < s$ such that $x \in \overline{K}_s$ but $x \notin \Gamma^{A,B}[s]$, the approximation of $\Gamma^{A,B}$ at stage s, we define two markers a_x (bound of the A-part) and b_x (bound of the B-part and $b_x \in B$) and enumerate x into $\Gamma^{A,B}$ via the axiom $\langle x, A_s \upharpoonright a_x+1, B_s \upharpoonright b_x+1\rangle$. If x leaves $\overline{K}$ later, we can make this axiom invalid by extracting b_x from B or by a change (from 1 to 0) of A on $A_s \upharpoonright a_x + 1$. Intuitively we must use A-changes in the definition of Γ since otherwise B would be complete, contradicting the N-requirements. Since A is not in our control, if A does not provide such changes then we have to extract b_x out of B. We call this process the rectification of Γ at x.

Note that after stage s, at stage $t > s$ say, if $x \in \overline{K}_t$ but $A_t \upharpoonright a_x+1 \nsubseteq A_t$ or $B_s \upharpoonright b_x+1 \nsubseteq B_t$ then we need to put x into $\Gamma^{A,B}$ by enumerating a new axiom into Γ. If this happens infinitely often then x is not in $\Gamma^{A,B}$ and we cannot ensure that $\Gamma^{A,B}(x) = \overline{K}(x)$. To avoid this at stage t, when we re-enumerate x into $\Gamma^{A,B}$, we keep a_x the same as before, but let b_x be a bigger number. We put $b_x[t]$ into B and extract $b_x[s]$ from B (we want only one valid axiom enumerating x into $\Gamma^{A,B}$). Assuming that the G-strategies also do not change a_x after a certain stage, as A is Δ_2^0 there can be only finitely many changes in $A \upharpoonright a_x$ and hence we will eventually stop enumerating axioms for x in Γ.

Now we consider how to satisfy a N_Φ-requirement. We use variant of the Friedberg-Muchnik strategy. Namely, we select x as a witness, enumerate it into E and wait for $x \in \Phi^B$. If x never enters Φ^B then N_Φ is satisfied. Otherwise we will extract x from E, preserving $B \upharpoonright \phi(x)$ where $\phi(x)$ denotes the use function of the computation $\Phi^B(x) = 1$.

The need to preserve $B \upharpoonright \phi(x)$ conflicts with the need to rectify Γ. To avoid this before choosing x the N_Φ-strategy will first choose a (big) number k as its threshold and try to achieve $b_n > \phi(x)$ for all $n \geq k$. For elements $n < k$, S will be allowed to rectify Γ at its will. Whenever $\overline{K}$ changes below $k+1$ we reset this N_Φ-strategy by cancelling all associated parameters except for this k. Since k is fixed such a *resetting* process can happen at most $k+1$ many times, so we can assume that after a stage large enough this N_Φ-strategy will never be reset anymore.

If k enters K, the threshold is moved automatically to the next number in $\overline{K}$. Since $\overline{K}$ is infinite, eventually, the threshold will stop changing its value. This threshold will be the real threshold of the corresponding N_Φ-strategy.

In order to be able to preserve some initial segment of B for the diagonalization, N_Φ will first try to move all markers b_n for elements $n \geq k$ above the restraint. A useful A-change will facilitate this. In the event that no such useful change appears we will be able to argue that A is c.e. contrary to hypothesis. To do this we will have an extra parameter U, aimed to construct a c.e. set approximating A.

The N_Φ-strategy works as follows at stage s:

Setup: Define a threshold k to be a big number. Choose a witness $x > k$ and enumerate it in E.
K**-Check:** If a marker b_n for an element $n \leq k$ has been extracted from B during Γ-rectification then restart the attack.
Attack:

1. If $x \in \Phi^B$ go to step 2. Otherwise return to step 1 at the next stage.

2. Approximate A by $A_s \upharpoonright a_k$ at stage s. Extract $b_k[s]$ from B. Cancel all markers a_n and b_n for $n \geq k$. Define a_k new, bigger than any element seen so far in the construction. Go to step 3.

3. Initialize all strategies of lower priority. If a previous approximation of A defined at stage $t < s$ is not true then enumerate $b_k[t]$ back in B, extract x from E and go to step 4, otherwise go back to step 1.

4. While the observed change in A is still apparent, do nothing. Otherwise enumerate x back in E and extract $b_k[t]$ from B, go back to step 3.

If after a large enough stage the strategy waits at 1 or 4 forever then the N_Φ-requirement is obviously satisfied. In the latter case $\Phi^B(x) = 1 \neq 0 = E(x)$ and the construction of Γ will never change the enumeration of $\Phi^B(x) = 1$ since all γ-markers are lifted to bigger values by the changes of A below $a_k[s]+1$. This strategy will not go from 1 or 4 back to 3 infinitely often and hence the N_Φ-requirement is satisfied. Otherwise as A is Δ^0_2 it would pass through 2 infinitely often. Let $t_1 < t_2 < \cdots < t_n < \cdots$ be the stages at which this strategy passes through 2. Then for each i, $A_{t_i} \upharpoonright a_k[t_i] + 1 \subset A$. By this property we argue that A is computably enumerable as follows: for each x, x is in A if and only if x is in A_{t_i} for some i, or

$$x \in A \quad \Leftrightarrow \quad \exists i (x \in A_{t_i}).$$

This contradicts our assumption on A.

Cupping by 1-generic degrees

In this section we give a proof of Theorem A.2.2. That is, given an non-c.e. Δ^0_2 set A, we will construct a Δ^0_2 1-generic B satisfying the following requirements:

$$S : \Gamma^{A,B} = \overline{K};$$

$$G_i : (\exists \lambda \subset B)[\lambda \in W_i \vee (\forall \mu \supseteq \lambda)[\mu \notin W_i]].$$

If all requirements G_i together with the global requirement S are satisfied then B will have the intended properties. It is well known that the degree of a 1-generic set can not be complete.

Definition A.2.1. *The tree of outcomes will be a perfect binary tree T. Each node $\alpha \in T$ of length i will be labelled by the requirement G_i. We will say that α is a G_i-strategy.*

At stage 0 $B = \emptyset$, $\Gamma = \emptyset$, $U_\alpha = \emptyset$ for all α and all thresholds and witnesses will be undefined.

At stage s we start by rectifying Γ and then construct a path through the tree δ_s of length s visiting all nodes $\alpha \subset \delta_s$ and performing actions as stated in the construction.

The Γ-rectification module for satisfying the global S requirement is as follows:

Γ-rectification module.

Scan all elements $n < s$ and perform the following actions for the elements n such that $\Gamma^{A,B}(n) \neq \overline{K}(n)$:

- $n \in \overline{K}$.

  1. If $a_n \uparrow$, define $a_n = a_{n-1} + 1$(if n= 0, define $a_n = 1$). Note that this is the only case when the Γ-module changes the value of a_n. Once defined a_n can only be redefined due to a G-strategy. The idea is that eventually G-strategies will stop cancelling a_n, so that we can approximate $A \upharpoonright a_n$ correctly and obtain a true axiom for n.
  2. If $b_n \downarrow$ then extract it from B and cancel all markers $b_{n'}$ for $n' > n$.
  3. Define b_n to be big, i.e a number greater than any number mentioned in the construction so far, and enumerate it in B.
  4. Enumerate in Γ the axiom $\langle n, A \upharpoonright a_n + 1, \{b_m | m \leq n\}\rangle$.

- $n \notin \overline{K}$

  Then find all valid axioms in Γ for n – $\langle n, A \upharpoonright a+1, M_n\rangle$ and extract the greatest element of M_n from B.

Construction of δ_s.

We will define $\delta_s(n)$ for all $n < s$ by induction on n. Suppose we have already defined $\delta_s \upharpoonright i = \alpha$ working on requirement G_W. We will perform the actions assigned to α and choose its outcome $o \in \{0, 1\}$. Then $\delta_s(i) = o$.

α will be equipped with a threshold k and a witness λ, a finite binary string. When α is visited for the first time after initialization it starts from *Setup*. At further stages it always performs *Check* first. If the *Check* does not empty U_α then it continues with the *Attack* module from where it was directed to at the previous α-true stage. Otherwise it continues with the *Setup* to define λ again and then proceeds to step 1 of *Attack*.

Setup: If a threshold has not been defined or is cancelled then define k to be big – bigger than any element appeared so far in the construction. If a witness has not yet been defined choose a binary string λ of length $b_k + 1$ so that $\lambda = B \upharpoonright b_k + 1$.

Check: If a marker b_n for an element $n \leq k$ has been extracted from B during Γ-rectification at a stage t such that $s- < t \leq s$ where $s-$ is the previous α-true stage then initialize the subtree below α, empty U.

If $k \notin \overline{K}$ then define k to be the least $k' > k$ such that $k' \in \overline{K}$. I nitialize the subtree below α, empty U.

If b_k has changed since the last α-true stage and $\lambda \nsubseteq B$ then define λ to be $B \upharpoonright b_k$. Do not empty U.

Attack:

1. Check if there is a finite binary string $\mu \supseteq \lambda$ in W . If not then the outcome is $o = 1$. Return to step 1 at the next stage. If there is such a μ then remember the least one and go to step 2.
2. Enumerate in the guess list U a new entry $\langle A_s \upharpoonright a_k, \mu, b_k \rangle$. Extract b_k from B. Let $\hat{\mu}$ be the string μ but with position $b_k = 0$. For all elements $n > |\lambda|$ such that $\hat{\mu}(n)$ is defined let $B(n) = \hat{\mu}(n)$. Cancel all markers a_n and b_n for $n \geq k$. Define a_k to be bigger. Note that $\hat{\mu} \subset B$ and at the next stage Check will define a new value of λ to be $B \upharpoonright b_k + 1$ so that $\lambda \supseteq \hat{\mu}$. Go to step 3.
3. Initialize all strategies below α. Scan the guess list U for errors. The entries in the guess list will be of the following form $\langle U_t, \mu_t, b_t \rangle$ where U_t is a guess of A and b_t is the marker that was extracted from B when this guess was made at stage t. Note that to make $\mu_t \subset B$ we only need to enumerate b_t in B. If there is an error in the guess list, i.e. some $U_t \nsubseteq A_s$, then enumerate b_t in B and go to step 4 with current guess $G = \langle U_t, \mu_t, b_t \rangle$ where t is the least index of an error in U. If all elements are scanned and no errors are found go back to step 1.
4. If the current guess $G = \langle U_t, \mu, b_t \rangle$ has the property $U_t \nsubseteq A_s$ then let the outcome be $o = 0$. Come back to step 4 at the next stage. Otherwise extract b_t from B. If the Γ-rectification module has extracted a marker m for an axiom that includes b_t in its $B-$part since the last stage on which this strategy was visited then enumerate m back in B. Go back to step 3.

The Proof.

Define the true path $f \subset T$ to be the leftmost path through the tree that is visited infinitely many times, i.e. $\forall n \exists^\infty t (f \upharpoonright n \subseteq \delta_t)$ and $\forall n \exists t_n \forall t > t_n (\delta_t \not<_L f \upharpoonright n)$.

Lemma A.2.1. *For each strategy $f \upharpoonright n$ the following is true:*

1. *There is a stage $t_1(n) > t_n$ such that at all $f \upharpoonright n$-true stages $t > t_1(n)$ Check does not empty U.*
2. *There is a stage $t_2(n) > t_1(n)$ such that at all $f \upharpoonright n$-true stages $t > t_2(n)$ the Attack module never passes through step 3 and hence the strategies below $f \upharpoonright n$ are not initialized anymore, B is not modified by $f \upharpoonright n$, and the markers a_n for any elements n are not moved by $f \upharpoonright n$*

Proof. Suppose the two conditions are true for $m < n$. Let $f \upharpoonright n = \alpha$. Let t_0 be an α-true stage bigger than $t_2(m)$ for all $m < n$ and t_n.

Then after stage t_0 α will not be initialized anymore.

After stage t_0 all elements $n < k$ have permanent markers a_n. Indeed none of the strategies above α modify them anymore according to the induction hypothesis, strategies to the left are not accessible anymore and strategies to the right are initialized on stage t_0, hence the next time they are accessed they will have new thresholds greater than k.

The threshold k will stop shifting its value as $\overline{K}$ is infinite and we will eventually find the true threshold $k \in \overline{K}$.

As A is Δ_2^0, eventually all $A \upharpoonright a_n$ for element $n < k$ will have their final value and so will $\overline{K} \upharpoonright k$. Hence there is a stage $t_1(n) > t_0$ after which no markers b_n for elements $n \leq k$ will be extracted from B by the Γ-rectification and the *Check* module at α will never empty U again.

To prove the second clause suppose that the module passes through step 3 infinitely many times and consider the set $V = \bigcup L(U)$ where $L(U)$ denotes the left part of entries in the guess list U, that is the actual guesses at the approximation of A. By assumption A is not c.e. hence $A \neq V$.

If $V \nsubseteq A$ then there is a least stage t' and element p such that $p \in U_{t'} \backslash A$ and all U_t for $t < t'$ are subsets of A. Let $t_p > t_2$ be a stage such that the Δ_2^0 approximation of A settles down on p, i.e. for all $t > t_p$, $A_t(p) = A(p) = 0$. Then when we pass through step 3 after stage t_p we will spot this error, go to step 4 and never again return to step 3.

If $V \subset A$, let p be the least element such that $p \in A \backslash V$. Every guess in U is eventually correct and returns to step 1. To access step 3 again we pass through step 2, i.e. we pass through step 2 infinitely often. As a result a_k grows unboundedly and will eventually reach a value greater than p. As on all but finitely many stages t, $p \in A_t$, p will enter V.

□

Corollary A.2.1. *Every G_i-requirement is satisfied.*

Proof. Consider the G_i-strategy $\alpha = f \upharpoonright i$. Choose a stage $t_3 > t_2(i)$ from Lemma A.2.1, after which the Attack module is stuck at step 1 or step 4, we have a permanent value for a_k and $A \upharpoonright a_k$ remains unchanged. Then so will the marker b_k and we will never modify λ again and $\lambda \subseteq B_t$ at all $t > t_3$.

If the module is stuck at step 1 we have found a string λ such that $\lambda \subset B$ and no string $\mu \supset \lambda$ is in the set W_i.

If the module is stuck at step 4 we have found a string μ from the guess $G = \langle U_t, \mu, b_t \rangle$ which is in W_i. It follows from the construction that $\mu \subset B$. The current markers b_n, for $n \geq k$ at stage t were cancelled and $b_k[t] = b_t$ was extracted from B. Any axiom defined after stage t has b-marker greater than $|\mu|$. Hence the Γ-rectifying procedure will not extract any element below the restraint $B \upharpoonright |\mu|$ from B. It does not extract markers of elements $n < k$. If $n \geq k$ and $n \in \overline{K}$ then its current marker is greater than $|\mu|$. If $n > k$ and $n \notin \overline{K}$ then any axiom defined before stage t is invalid, because its b-marker is already extracted from B at a previous stage $t_0 < t$ or else it has an A-component that contains as a subset $U_t \nsubseteq A$. □

Lemma A.2.2. *The S-requirement is satisfied.*

Proof. At each stage s we make sure that Γ is rectified. For elements $n < s$, we have $\Gamma^{A,B}(n)[s] = \overline{K}(n)[s]$. This is enough to prove that $n \notin \overline{K} \Rightarrow n \notin \Gamma^{A,B}$. Indeed if we assume that $n \in \Gamma^{A,B}$ then there is an axiom $\langle n, A_n, M_n\rangle \in \Gamma$ and $A_n \subseteq A$, $M_n \subset B$. Hence this axiom is valid on all but finitely many stages. But according to our construction we will ensure $M_n \nsubseteq B$ on infinitely many stages, a contradiction.

To prove the other direction, $n \in \overline{K} \Rightarrow n \in \Gamma^{A,B}$, we have to establish that the N-strategies will stop modifying the markers a_n and b_n eventually. Indeed the markers can be modified only by N-strategies with thresholds $k < n$. The way we choose each threshold guarantees that there will be only finitely many nodes on the tree with this property. The nodes to the left of the true path will eventually not be accessible anymore and the nodes to the right will be cancelled and will choose new thresholds, bigger then n. Lemma A.2.1 proves that every node along the true path will eventually stop moving a_n and b_n by property 2.

Suppose the markers are not modified after stage t_1. After stage t_1, a_n has a constant value. As A is Δ^0_2 there will be a stage $t_2 > t_1$ such that for all $t > t_2$ $A \upharpoonright a_n[t] = A \upharpoonright a_n$. At stage $t_2 + 1$ we rectify Γ. If $n \in \Gamma^{A,B}$ then there is an axiom $\langle n, A_n, M_n\rangle$ in Γ such that $A_n \subset A \upharpoonright a_n$ and at all further stages this axiom will remain valid, so the Γ-rectifying procedure will not modify it again. Otherwise it will extract a b-marker for the last time and enumerate an axiom $\langle n, A \upharpoonright a_n, M'_n\rangle$ that will be valid at all further stages. In both cases we have found an axiom for n that is valid on all but finitely many stages, hence $n \in \Gamma^{A,B}$. □

Lemma A.2.3. *B is Δ^0_2.*

Proof. We need to show that for each n, n can be put in and moved out from B at most finitely times. To see this fix n and consider the G_i-strategy along the true path that has a threshold $k_i > n$. As we have already established in Corollary A.2.1 there is a stage $t_3 > t_2(i)$, after which we will never modify λ_i again and $\lambda \subseteq B_t$ on all $t > t_3$. As $n < |\lambda_i|$ then $B_t(n)$ will remain constant on all stages $t > t_3$. This means that $B(n)$ changes at most t_3 many times. □

Cupping the ω-c.e. degrees

In this section we give a proof of Theorem A.2.3. Suppose we are given an ω-c.e. set A with bounding function g. We will modify the construction of the set B so that it will turn out to be $3 - c.e.$. The requirements are:

$$S : \Gamma^{A,B} = \overline{K}$$

$$N_\Phi : E \neq \Phi^B$$

The structure of the axioms enumerated in Γ will be more complex. Again we will have an a-marker a_n for each element n, but instead of just one marker b_n we will have a set of b-markers B_n of size $g_n + 1$ where $g_n = \sum_{x<a_n} g(x)$ together with a counter c_n that will tell us which element we should extract if we need to. Every time $A \upharpoonright a_n$ changes we will extract from B a

different element – the c_n-th element $b_n \in B_n$ and then add 1 to c_n to ensure that each element in B will be extracted only once. If we need to restore a computation due to the N-strategies we will enumerate the extracted marker back in B, hence B is $3-c.e.$. Note that if a restored computation has to be destroyed again, we will need to extract a different marker from B. This could destroy further computations. That is why will always try to restore the last computation $\Phi^B(x)$.

Γ-rectification module.

Scan all elements $n < s$ and perform the following actions for the elements n such that $\Gamma^{A,B}(n) \neq \overline{K}(n)$:

- $n \in \overline{K}$.

  1. If $a_n \uparrow$ then define $a_n = a_{n-1} + 1$(if n = 0, define $a_n = 1$).
  2. If $B_n \downarrow$. Extract the c_n-th member of B_n. Move the counter c_n to the next position $c_n + 1$. Cancel all $B_{n'}$ for $n' > n$.
  3. If $B_n \uparrow$ then define a set of new markers B_n of size $g_n + 1$ where $g_n = \sum_{x<a_n} g(x)$ and a new counter $c_n = 1$ and enumerate B_n in B.
  4. Enumerate in Γ the axiom $\langle n, A_s \upharpoonright a_n + 1, \bigcup\{B_{n'}(c_{n'}) | n' \leq n\}\rangle$ where $B_{n'}(c_{n'})$ is the set of all elements in $B_{n'}$ with positions greater than or equal to $c_{n'}$.

- $n \notin \overline{K}$

  Then find all valid axioms in Γ for n – $\langle n, A_t \upharpoonright a + 1, M_n\rangle$ where $M_n = \bigcup\{B_{n'} | n' \leq n\}$ and extract the least member of B_n that has not yet been extracted from B. Increment the counter c_n that corresponds to the set of markers B_n.

Construction of δ_s.

Setup: If a threshold has not been defined or is cancelled then define k to be big, bigger than any element appeared so far in the construction. If a witness has not yet been defined choose $x > k$ and enumerate it in E.

Check: If a marker from B_n for an element $n < k$ has been extracted from B during Γ-rectification at a stage t, $s- < t \leq s$ where $s-$ is the previous α-true stage, then initialize the subtree below α, empty U.

If $k \notin \overline{K}$ then shift it to the next possible value and redefine x to be bigger. Again initialize the subtree below α and empty U.

Attack:

1. Check if $x \in \Phi^B$. If not then the outcome is $o = 1$, return to step 1 at the next stage. If $x \in \Phi^B$ go to step 2.

2. Initialize all strategies below α. Scan the guess list U for errors. If there is an error then take the last entry in the guess list, say the one with index t: $\langle U_t, B_t, c_t\rangle \in U$ and $U_t \nsubseteq A_s$. Enumerate the $(c_t - 1)$-th member of B_t back in B. Extract x from E and go to step 4 with current guess $G = \langle U_t, B_t, c_t\rangle$. If all elements are scanned and no errors are found go to step 3.

3. Enumerate in the guess list U a new entry $\langle A_s \upharpoonright a_k, B_k, c_k\rangle$. Extract the c_k-th member of B_k from B and move c_k to the next position $c_k + 1$. Cancel all markers a_n and B_n for $n \geq k$. Define a_k new, bigger than any element seen so far in the construction. Go to back to step 1.

   Note that this ensures that our guesses at the approximation of A are monotone. Hence if there is an error in the approximation, this error will be apparent in the last guess. This allows us to always use the computation corresponding to the last guess. We will always be able to restore it.

4. If the current guess $G = \langle U_t, B_t, c_t\rangle$ has the property $U_t \nsubseteq A_s$ then let the outcome be $o = 0$. Come back to step 4 at the next stage. Otherwise enumerate x back in E and extract the c_t-th member of B_t from B and move the value of the counter to $c_t + 1$. If at this stage during the Γ-rectification procedure a different marker m for an axiom that contains B_t was extracted then enumerate m back in B. Go back to step 1.

The Proof.

The construction ensures that for any n, at any stage t, at most one axiom in Γ defines $\Gamma^{A,B}(n)$. Generally, we extract a number from B_n to drive n out of $\Gamma^{A,B}$. When an N-strategy α acts at step 3 of the Attack module, at stage s say, α needs to preserve $\Phi^B(x_\alpha)$. All lower priority strategies are initialized and an element b_1 in B_{k_α} is extracted from B to prevent the S-strategy from changing B on $\phi(x_\alpha)$. Note that all axioms for elements $n \geq k_\alpha$ contain B_{k_α}. So at stage s, when we extract b_1 from B, n is driven out of $\Gamma^{A,B}$. As in the remainder of the construction, at any stage, we will have either that A has changed below a_n or B has changed on B_n, these axioms will never be active again. As the Γ-module acts first, it may still extract a marker m from an axiom for $n > k_\alpha$ if $A \upharpoonright a_n$ has changed back and thereby injure $B \upharpoonright \phi(x_\alpha)$. But when α is visited it will correct this by enumerating m back in B and extracting a further element $b_2 \in B_{k_\alpha}$ from B to keep Γ true. This makes our N-strategies and the S-strategy consistent. We comment here that such a feature is also true in the proof of Theorem A.2.2, but there we do not worry about this as we are constructing a Δ^0_2 set. In the proof of Theorem A.2.3, this becomes quite crucial, as we are constructing B as a 3-c.e. set, and we have less freedom to extract numbers out from B.

The construction ensures that B is a $3 - c.e.$ set. First we prove that the counter c_n never exceeds the size of its corresponding set B_n and therefore we will always have an available marker to extract from B if it is necessary.

Lemma A.2.4. *For every set of markers B_n and corresponding counter c_n at all stages of the construction $c_n < |B_n|$ and the c_n-th member of B_n is in B.*

Proof. For each set of markers B_n only one node along the true path can enumerate its elements back into B. Indeed if B_n enters the guess list U_t at some node α on the tree then at stage t, B_n is the current set of markers for n and n is the threshold for α. When α enumerates B_n in its U_t, it cancels the current markers for the element n. Hence B_n does not belong to any $U^{\beta}_{t'}$ for $t' \leq t$ and any node β or else B_n will not be current and B_n will not enter $U^{\beta}_{t''}$ at any stage $t'' \geq t$ and any node β as it is not current anymore.

We ensure that n being a threshold is in $\overline{K}$, hence after stage t the Γ-rectification procedure will not modify $B \upharpoonright B_n$. Before stage t while the markers were current the counter c_n was moved only when the Γ-rectification procedure observed a change in $A \upharpoonright a_n$, i.e some element that was in $A \upharpoonright a_n$ at the previous stage is not there anymore. After stage t α will move the marker c_n once at entry in U_t and then only when it observes a change in $A \upharpoonright a_n$, i.e $U_n = A \upharpoonright a_n[t]$ was a subset of A at a previous step but is not currently. Altogether c_n will be moved at most $g_n + 1 < |B_n|$ times.

Otherwise B_n belongs to an axiom which contains the set B_k for a particular threshold k and $n \notin \overline{K}$. Then again its members are enumerated back in B only in reaction to a change in $A \upharpoonright a_n$. □

We will now prove that Lemma A.2.1 is valid for this construction as well. Note that this construction is a bit different, therefore we will need a new proof. The true path f is defined in the same way.

Lemma A.2.5. *For each strategy $f \upharpoonright n$ the following is true:*

1. *There is a stage $t_1(n) > t_n$ such that at all $f \upharpoonright n$-true stages $t > t_1(n)$ Check does not empty U.*
2. *There is a stage $t_2(n) > t_1(n)$ such that at all $f \upharpoonright n$-true stages $t > t_2(n)$ the Attack module never passes through step 2 and hence the strategies below α are not initialized anymore, B is not modified by $f \upharpoonright n$, and the markers a_n for any elements n are not moved by $f \upharpoonright n$*

Proof. Suppose the two conditions are true for $m < n$. Let $f \upharpoonright n = \alpha$. Let t_0 be an α-true stage bigger than $t_2(m)$ for all $m < n$ and t_n.

Then after stage t_0 α will not be initialized anymore. The proof of the the existence of stage $t_1(n)$ satisfying the first property is the same as in Lemma 1.

To prove the second clause suppose that the module passes through step 2 infinitely many times and consider the set $V = \bigcup L(U)$ where $L(U)$ denotes the left part of entries in the guess list U. By assumption A is not c.e. hence $A \neq V$.

If $V \nsubseteq A$ then there is element p such that $p \in V \backslash A$. Let $t_p > t_2$ be a stage such that the approximation of A settles down on p, i.e. for all $t > t_p$, $A_t(p) = A(p) = 0$. Then when we pass through step 2 after stage t_p we will spot this error, go to step 4 and never again return to step 1.

If $V \subset A$, let p be the least element such that $p \in A \backslash V$. Every guess in U is eventually correct and allows us to move to step 3, i.e. we pass through step 3 infinitely often. As a result a_k grows unboundedly and will eventually reach a value greater than p. As on all but finitely many stages t, $p \in A_t$, p will enter V. $\square$

Corollary A.2.2. *Every N_i-requirement is satisfied.*

Proof. Let $\alpha \subset f$ be an N_i-strategy. As a corollary of Lemma A.2.5 there is a stage $t_3 > t_2(i)$ after which the *Attack* module is stuck at step 1, and hence $x \notin \Phi^B$, but $x \in E$. Or else the module is stuck at step 4, in which case $x \in \Phi^B$ and $x \notin E$. Indeed step 4 was accessed with $G = \langle U_t, B_t, c_t \rangle$, belonging to the last entry in the guess list $\langle U_t, B_t, c_t \rangle$. At stage t we had $x \in \Phi^B[t]$. The current markers b_n , for $n \geq k$ were cancelled and $b_k[t]$ was extracted from B. Hence the Γ-rectifying procedure will not extract any element below the restraint $B \upharpoonright \phi(x)$ from B. It does not extract markers of elements $n < k$. If $n \geq k$ and $n \in \overline{K}$ then its current marker is greater than $\phi(x)$. If $n > k$ and $n \notin \overline{K}$ then any axiom defined before stage t is invalid, because one of its b-markers is extracted from B at a previous stage or else it has an A-component $U_t \not\subseteq A$. Any axiom defined after stage t has b-markers greater than $\phi(x)$.

After stage t, if α modifies B it will be in the set of markers B_t, and when step 4 is accessed we have $B_t \subset B$. $\square$

Lemma A.2.2 is now valid for Theorem A.2.3 as well, hence all requirements are satisfied and this concludes the proof of Theorem A.2.3.

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A.3 Total Degrees and Nonsplitting Properties of Σ_2^0 Enumeration Degrees

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This paper continues the project, initiated in [3], of describing general conditions under which relative splittings are derivable in the local structure of the enumeration degrees.

The main results below include a proof that any high total e-degree below $\mathbf{0}_e'$ is splittable over any low e-degree below it, and a construction of a Π_1^0 e-degree unsplittable over a Δ_2 e-degree below it.

In [3] it was shown that using semirecursive sets one can construct minimal pairs of e-degrees by both effective and uniform ways, following which new results concerning the local distribution of total e-degrees and of the degrees of semirecursive sets enabled one to proceed, via the natural embedding of the Turing degrees in the enumeration degrees, to results concerning embeddings of the diamond lattice in the e-degrees. A particularly striking application of these techniques was a relatively simple derivation of a strong generalisation of the Ahmad Diamond Theorem.

This paper extends the known constraints on further progress in this direction, such as the result of Ahmad and Lachlan [2] showing the existence of a nonsplitting Δ_2^0 e-degree $> \mathbf{0}_e$, and the recent result of Soskova [13] showing that $\mathbf{0}_e'$ is unsplittable in the Σ_2^0 e-degrees above some Σ_2^0 e-degree $< \mathbf{0}_e'$. This work also relates to results (e.g. Cooper and Copestake [8]) limiting the local distribution of total e-degrees.

For further background concerning enumeration reducibility and its degree structure, the reader is referred to Cooper [6], Sorbi [12] or Cooper 7, chapter 11.

Theorem 1. *If $\mathbf{a} < \mathbf{h} \leq \mathbf{0}'$, $\mathbf{a}$ is low and $\mathbf{h}$ is total and high then there is a low total e-degree $\mathbf{b}$ such that $\mathbf{a} \leq \mathbf{b} < \mathbf{h}$.*

Corollary 2. *Let $\mathbf{a} < \mathbf{h} \leq \mathbf{0}'$, $\mathbf{h}$ be a high total e-degree, $\mathbf{a}$ be a low e-degree. Then there are Δ_2^0 e-degrees $\mathbf{b}_0 < \mathbf{h}$ and $\mathbf{b}_1 < \mathbf{h}$ such that $\mathbf{a} = \mathbf{b}_0 \cap \mathbf{b}_1$ and $\mathbf{h} = \mathbf{b}_0 \cup \mathbf{b}_1$.*

Proof. Immediately follows from Theorem 1, and Theorem 6 of [3]. □

Proof of Theorem 1. Assume A has low e-degree, $H \oplus \overline{H}$ has high e-degree (i.e., H has high Turing degree) and $A \leq_e H \oplus \overline{H}$.

We want to construct an H-computable increasing sequence of initial segments $\{\sigma_s\}_{s\in\omega}$ such that the set $B = \cup_s \sigma_s$ satisfies the requirements

$$P_n : n \in A \iff (\exists y)[\langle n, y\rangle \in B] \text{ and}$$
$$R_n : (\exists \sigma \subset B)[n \in W_n^\sigma \vee (\forall \tau \supset \sigma)[\tau \in S^A \implies n \notin W_n^\tau]]$$

for each $n \in \omega$, where

$$S^A = \{\tau : (\forall x)(\forall y)[\tau(\langle x, y\rangle) \downarrow = 1 \implies x \in A]\}.$$

Note that P_n-requirements guarantee that $A \leq_e B$, and hence $A \leq_e B \oplus \overline{B}$. To prove that the R_n-requirements provide $B' \equiv_T \emptyset'$, first note that $S^A \equiv_e A$, which has low e-degree, and

$$X = \{\langle \sigma, n\rangle : (\exists \tau \supset \sigma)[\tau \in S^A \;\&\; n \in W_n^\tau]\} \leq_e S^A.$$

Then $X \in \Delta^0_2$ and

$$n \notin B' \iff (\exists \sigma \subset B)[\langle \sigma, n\rangle \notin X],$$

so that B' is co-c.e. in $B \oplus \emptyset' \equiv_T \emptyset'$. Thus $B' \leq_T \emptyset'$ by Post's Theorem.

Since the set B will be computable in H, the set

$$Q = \{n : (\forall \sigma \subset B)(\exists \tau \supset \sigma)[\tau \in S^A \;\&\; n \in W_n^\tau]\}$$

will be computable in $(H \oplus \emptyset')' \equiv_T H'$ – indeed, we have $n \in Q \iff (\forall \sigma \subset B)[\langle \sigma, n\rangle \in X]$, so that Q is co-c.e. in $H \oplus \emptyset'$. Now to construct the desired set B we can apply the Recursion Theorem and fix an H-computable function g such that $Q(x) = \lim_s g(x, s)$.

Let $\{A_s\}_{s\in\omega}$ and $\{S^A_s\}_{s\in\omega}$ be respective H-computable enumerations of A and S^A.

Construction.

Stage $s = 0$. $\sigma_0 = \lambda$.

Stage $s + 1 = 2\langle n, z\rangle$ (to satisfy P_n). Given σ_s define $l = |\sigma_s|$.

If $n \notin A_s$, then let $\sigma_{s+1} = \sigma_s\widehat{\ }0$.

If $n \in A_s$, then choose the least $k \geq l$ such that $k = \langle n, y\rangle$ for some $y \in \omega$ and define $\sigma_{s+1} = \sigma_s\widehat{\ }0^{k-l}\widehat{\ }1$ (so that $\sigma_{s+1}(k) = 1$).

Stage $s+1 = 2\langle n, z\rangle+1$ (to satisfy R_n). H-computably find the least stage $t \geq s$ such that either $g(n, t) = 0$, or $n \in W^\tau_{n,t}$ for some τ satisfying $\tau \in S^A_t$ and $\tau \supset \sigma_s$. (Such stage t exists since if $\lim_s g(n, s) = 1$ then $n \in Q$, and hence there exists some $\tau \supset \sigma_s$ such that $n \in W_n^\tau$ and $\tau \in S^A$.)

If $g(n, t) = 0$ then define $\sigma_{s+1} = \sigma_s\widehat{\ }0$.

Otherwise, choose the first $\tau \supset \sigma_s$ such that $\tau \in S^A_t$ and $n \in W^\tau_{n,t}$. Define $\sigma_{s+1} = \tau$.

This completes the description of the construction.

Let $B = \cup_s \sigma_s$. Clearly $B \leq_T H$ since each σ_s is obtained effectively in H. Each P_n-requirement is satisfied by the even stages of the construction since $\sigma_s \in S^A$ for any $s \in \omega$.

To prove that each R_n-requirement is met suppose that

$$(\forall \sigma \subset B)(\exists \tau \supseteq \sigma)[\tau \in S^A \;\&\; n \in W_n^\tau]$$

for some n. This means that $n \in Q$. Choose any odd stage $s = 2\langle n, z\rangle + 1$ such that $g(n, t) = 1$

for all $t \geq s$. Then by the construction $n \in W_n^{\sigma_s}$.

Hence $A \leq_e B \oplus \overline{B} \leq_e H \oplus \overline{H}$, and $\deg_e(B \oplus \overline{B})$ is low. $\square$

Theorem 3. *There is a* Π_1^0 *e-degree* $\mathbf{a}$ *and a* 3*-c.e. e-degree* $\mathbf{b} < \mathbf{a}$ *such that* $\mathbf{a}$ *is not splittable over* $\mathbf{b}$.

Proof. We construct a Π_1^0 set A and 3-c.e. set B satisfying both the global requirement:

$G : B = \Omega(A)$,

and the requirements

$$R_{\Xi,\Psi,\Theta} : A = \Xi(\Psi(A) \oplus \Theta(A)) \Longrightarrow (\exists \text{ e-operator } \Gamma)A = \Gamma(\Psi(A) \oplus B) \vee$$
$$(\exists \text{ e-operator } \Lambda)A = \Lambda(\Theta(A) \oplus B)$$

for each triple of e-operators Ξ, Ψ, Θ, and

$$N_\Phi : A \neq \Phi(B)$$

for each e-operator Φ.

In fact A will be constructed as a 2-c.e. set. Note that the e-degrees of Π_1 sets coincide with the e-degrees of 2-c.e. sets. Hence this will still produce the desired enumeration degrees.

Basic Strategies

Suppose we have an effective listing of all requirements $R_1, R_2, \dots$ and $N_1, N_2, \dots$ The requirements will then be arranged by priority in the following way: $G < R_1 < N_1 < R_2 < N_2 < \dots$

To satisfy the requirement G we will make sure that every time we enumerate an element into the set B, we enumerate a corresponding axiom into the set Ω; and every time we extract an element from B, we make the corresponding axiom invalid by extracting elements from A. More precisely every element y that enters B will have a corresponding marker m in A and an axiom $\langle y, \{m\}\rangle$ in Ω. If y is extracted from B then we extract m from A. If y is later re-enumerated into B – this can happen since B is 3-c.e. – then we will just enumerate the axiom $\langle y, \emptyset\rangle$ into Ω.

To satisfy the requirements R_i we will initially try to construct an operator Γ using information from both of sets B and $\Psi(A)$. Again, enumeration of elements into A is always accompanied by enumeration of axioms into Γ, and extraction of elements from A can be rectified via B-extractions.

The N-strategies follow a variant of the Friedberg -Muchnik strategy while at the same time respecting the Γ-rectification, so we will call them (N_Φ, Γ)-strategies. They choose a follower x, enumerate it in A, then wait until $x \in \Phi(B)$. If this happens - they extract the element x from A while restraining $B \upharpoonright \varphi(x)$ in B. The need to rectify Γ after the extraction of the follower x from A can be in conflict with the restraint on B. To resolve this conflict we try to obtain a change in the set $\Psi(A)$ which would enable us to rectify Γ without any extraction from the set B. To do this we monitor the length of agreement

$$l_{\Xi,\Psi,\Theta}(s) = \max\{y : (\forall y < x)[y \in A[s] \iff y \in \Xi(\Psi(A) \oplus \Theta(A))[s]]\}.$$

We only proceed with actions directed at a particular follower once it is below the length of agreement. This ensures that the extraction of x from A will have one of the following consequences

1. The length of agreement will never return so long as at least one of the axioms that ensure $x \in \Xi(\Psi(A) \oplus \Theta(A))$ remains valid.
2. There is a useful change in the set $\Psi(A)$.
3. There is a useful change in the set $\Theta(A)$.

We will initially assume that it is the case that the third consequence is true and commence a backup strategy (N_Φ, Λ) which is devoted to building an enumeration operator Λ with information from A and $\Theta(A)$. This is a new copy of the N-strategy working with the same follower. It will try to make use of this change in $\Theta(A)$ to satisfy the requirement. Only when we are provided with evidence that our assumption is wrong will we return to the initial strategy (N_Φ, Γ).

Basic module for an N_Φ-strategy below one $R_{\Xi,\Psi,\Theta}$-strategy

We will first consider the simple case involving just two requirements. Assume we have N_Φ, which we refer to as the N-requirement, below $R_{\Xi,\Psi,\Theta}$, which we refer to as the R-requirement.

At the root we have the R-strategy denoted by (R, Γ). It will have two outcomes $e <_L gw$. The R-strategy will monitor all elements $x \notin A$. In the case in which there is an element $x \notin A$ such that $x \in \Gamma(\Psi(A) \oplus B)$ the operator Γ cannot be rectified. The (R, Γ)-strategy will then have outcome gw, and we will be able to argue that $x \in \Xi(\Psi(A) \oplus \Theta(A))$, which indicates a global win for the R-requirement. Strategies working below this outcome will follow a simple Friedberg-Muchnik strategy and preserve the difference at x by using followers of big enough value. In case there is no such x the operator, Γ can be rectified and the (R, Γ)-strategy will have outcome e.

Below e we will try to meet N satisfying $A = \Gamma(\Psi(A) \oplus B)$. The (N, Γ)-strategy will have four outcomes: three finitary outcomes, f, w and l, and one infinitary outcome λ. The outcomes are arranged in the following way: $\lambda <_L f <_L w <_L l$. Outcome l indicates that at that node the R-requirement is globally satisfied since the follower x enumerated in A is not in $\Xi(\Psi(A) \oplus \Theta(A))$. Outcome w indicates that Γ is correct on x and the N-requirement is satisfied as $x \in A - \Phi(B)$. Outcome f is only accessible once a follower x has been returned. It will indicate that Γ is again correct on x and the N-requirement is satisfied via $x \in \Phi(B) - A$.

Below outcome λ strategies will be devoted to constructing an operator Λ with $A = \Lambda(\Theta(A) \oplus B)$ where they will receive their followers from (N, Γ). Again we have a controlling strategy (R, Λ) with only one outcome e which makes sure that the operator Λ can be rectified at all times. In case it sees an element $x \notin A$ for which the axiom enumerated in Λ is valid, it will send x back to (N, Γ). We will be able to argue that x has provided evidence of a useful change in $\Psi(A)$.

Below (R, Λ)'s only outcome e we try to meet N by (N, Λ) with $A = \Lambda_\Phi(\Theta(A) \oplus B)$. The strategy below the outcome λ acts only when the (N, Γ)-strategy *sends* its follower x. It performs similar actions with regard to (N, Γ) and has two outcome $f <_L w$ both indicating that the N-requirement is satisfied and the operator Λ remains intact.

The R strategy:

1. Scan all followers $x \notin A$ defined up to the current stage.
2. If $x \in \Gamma(\Psi(A) \oplus B)$, then let the outcome be $o = gw$.
3. If all followers are scanned and none has produced an outcome $o = gw$, then let the outcome be $o = e$.

The (N, Γ) strategy:

At stage s the strategy will start its work at the step of the module indicated at the previous stage.

Setup 1) Choose a new follower x as a fresh number (bigger than any previously set up restraint). Enumerate it into A_s.

2) If there are finite sets $G(x), H(x), L(x)$ with $x \in \Xi(G(x) \oplus H(x))$, $G(x) \subset \Psi(L(x))$, $H(x) \subset \Theta(L(x))$ and $L(x) \subset A$ then restrain A on $\max(L(x))$ and go to *Setup 3*. Otherwise let the outcome be $o = l$ and return to *Setup*2) at the next stage.

3) Define $x's$ B-marker $y(x)$, along with its corresponding A-marker $m(x)$, as fresh numbers bigger than any previously set restraint on A or B. Enumerate $y(x)$ in B_s and $m(x)$ in A_s. Define a new axiom $\langle y(x), \{m(x)\}\rangle$ for Ω_s.

Enumerate each $\langle z, G_x \oplus B \upharpoonright y(x)\rangle$ into Γ where z is either x, or $m(x)$, or a follower $z \in A$ from a previous cycle of the strategy. Note that we enumerate axioms for previous followers as well. So at this point the operator Γ is rectified. Let the outcome be $o = w$. Go to $Wait$ at the next stage.

Wait If $x \in \Phi(B_s)$ then go to *Attack*. Otherwise let the outcome be $o = w$ and return to $Wait$ at the next stage.

Attack 1) Check if any previously sent follower has been returned. If so go to *Result*. Otherwise go to *Attack*2.

2)Let $v(x) = \max(\varphi(x), y(x))$ and restrain B on $v(x)$. Extract $y(x)$ from B_s and $m(x)$ from A_s, noting that x is still in $\Xi(\Psi(A) \oplus \Theta(A))$ as the marker $m(x)$ is chosen as a fresh number after $G(x)$ and $H(x)$ are already defined.

Send x. Let the outcome be $o = \lambda$. At the next stage start from *Setup*1, choosing a new current follower. The strategy working below outcome λ will believe B only below a right boundary $R_s = y(x)$. Note that the next follower will choose its B-marker of greater value. So if the outcome λ is visited infinitely often then the right boundary R will grow unboundedly.

Result Let the returned follower be x. Put $y(x)$ into B_s and $\langle y(x), \emptyset\rangle$ into Ω_s. For each follower z of this strategy such that $z \in A$ put the axiom $\langle z, \emptyset\rangle$ into Γ_s.

1) For the returned follower we know that $x \notin A_s$ and $H(x) \subset \Theta(A_s)$. The outcome λ will not be accessible anymore so we can preserve $H(x) \subseteq \Theta(A_t)$ at further stages t. Also if $G(x) \subseteq \Psi(A_s)$ then the (R, Γ)-strategy would have outcome gw preserving the difference and satisfying R globally. The (N, Γ)-strategy would not be accessible any longer. Otherwise $G(x) \nsubseteq A$ and the outcome is $o = f$. Return at *Result*1 at the next stage.

The (R, Λ)-strategy below outcome λ :

1. Scan all followers $x \notin A$.
2. If $x \in \Lambda(\Theta(A) \oplus B)$ then return x. End this stage.
3. If all followers are scanned and none have been returned then let the outcome be e.

The (N, Λ)-strategy below outcome λ :

Setup 1) Let $x \in A$ be a new integer which was sent by the (N, Γ)-strategy. Now x becomes the *follower* of the (N, Λ)-strategy. Go to *Setup*2.

2) Put $\langle x, H_x \oplus B \upharpoonright v(x)\rangle$ into Λ. Go to $Wait$.

Wait If $x \in \Phi(B)$ with use $\varphi(x) < R_s$ then go to *Attack*. Otherwise the outcome is $o = w$, return to $Wait$ at the next stage.

Attack Extract x from A. Go to result.

Result Let the outcome be $o = f$. Return to *Result* at the next stage.

The (N, FM)-strategy below outcome l or gw :

Setup Choose a new follower x bigger than any previously set restraint on A and enumerate it into A. Go to $Wait$.

Wait If $x \in \Phi(B)$ go to *Attack*. Otherwise the outcome is $o = w$, return to $Wait$ at the next stage.

Attack Extract x from A and go to *Result*.

Result Let the outcome be $o = f$. Return to *Result* at the next stage.

Now the (N, FM) strategy below outcome l will also be changing A. To keep Γ and Λ rectified, every time we initialise the (N, FM)-strategy and cancel its follower x, if $x \in A$ we will add the axiom $\langle x, \emptyset \rangle$ in Γ and Λ.

If the (R, Γ)-strategy has outcome gw on stage s for the first time, then the (N, FM)-strategy working below will will be initialised on the previous stage and will choose its follower x anew, respecting the restraint on A that (N, Γ) has set up. So (R, Γ) will have outcome gw on all further stages and B will not be modified any longer. The (N, FM)-strategy will be able to satisfy its requirement.

Suppose that (R, Γ)-strategy never has outcome gw. We will analyse all possible outcomes of the N-strategies and see that in each case the requirements are satisfied.

Consider first the possible outcomes of the strategy (N, Γ). If one of the cycles stops at *Setup*2, i.e. on all stages $t > s$ the strategy has outcome l, then the true outcome will be $(o = l)$. The length of agreement $l_{\Xi,\Psi,\Theta}(s) = \max\{y : (\forall y < x)[y \in A[s] \iff y \in \Xi(\Psi(A) \oplus \Theta(A))[s]]\}$ is bounded and hence the requirement R is trivially satisfied.

The set B is not modified after stage s and the simple strategy (N, FM), active on all stages $t \geq s$ succeeds to satisfy the requirement N.

Suppose now that no cycle of the (N, Γ)-strategy stops at *Setup*2. In this case the (N, FM)-strategy may be activated infinitely many times and will be initialised every time the $(N\Gamma)$-strategy moves on to $Wait$. The current follower x of the (N, FM)-strategy will be cancelled and if it is not yet extracted from A the corresponding axiom $\langle x, \emptyset \rangle$ will be enumerated in Γ and Λ. This ensures that both operators will be correct at x for all cancelled followers x of the strategy (N, FM).

We first consider the case when the (N, Γ)-strategy during its work sends only finitely many integers. Then some cycle with a follower x stops either at $Wait$ or reaches *Result*. If the cycle stops at $Wait$ then the outcome is $o = w$ and $x \in A - \Phi(B)$, hence the N-requirement is satisfied. On the other hand for all followers z we have $z \in A \iff z \in \Gamma(\Psi(A) \oplus B)$ and $m(z) \in A \iff m(z) \in \Gamma(\Psi(A) \oplus B)$ since $y(z) \in B \iff z = x$. Hence Γ is correct at all followers z.

If the cycle reaches *Result* then we have $y(x) \in B$ and hence $x \in \Phi(B) - A$, so N is satisfied. Also $H_x \subseteq \Theta(A)$ via some finite set $P_x \subset A$. If $G_x \subseteq \Psi(A)$ then this will be apparent at some finite stage s, i.e. on stage s we will see a finite set $Q_x \subset A$ such that $G_x \subseteq \Psi(Q_x)$. Then from stage s on the (R, Γ)-strategy will have outcome $o = gw$, contradicting our assumption. So $G_x \nsubseteq \Psi(A)$ giving $x \notin \Gamma(\Psi(A) \oplus B)$. Since again $y(z) \in B \iff z = x$ we have $z \in A \iff z \in \Gamma(\Psi(A) \oplus B)$ and $m(z) \in A \iff m(z) \in \Gamma(\Psi(A) \oplus B)$ for any follower z. Hence the operator Γ remains correct at all further stages.

Suppose now that the (N, Γ)-strategy during its work sends infinitely many integers. In particular, no x is returned to (N, Γ). Then the true outcome is $o = \lambda$ and we will see that the (N, Λ)-strategy is successful.

If the (N, Λ)- strategy stops at $Wait$ then $x \in A - \Phi(B)$. Indeed if we assume that $x \in \Phi(B)$ then there is some finite $M_x \subset B$ such that $x \in \Phi(M_x)$. The right boundary R grows unboundedly, so eventually there will be a stage s with $R_s > \max(M_x)$ and the strategy will move on to *Attack*.

The second case is if the strategy reaches *Result*. Then $x \in \Phi(B) - A$ because at some stage s we found a set $M_x \subset B_s$ with $\max M_x < R$ such that $x \in \Phi(M_x)$. The strategy (N, Γ) will not extract any more markers from B after stage s that are below the right boundary R_s, hence $x \in \Phi(B)$.

At this stage of the construction we can only prove that Λ will be correct at the follower x and all cancelled followers of the strategy (N_Φ, FM). To prove that the operator is correct at the rest of the followers enumerated in A by the (N, Γ)-strategy we will need to consider how all N-strategies will work together.

Basic module for many N_Φ-strategies under one $R_{\Xi,\Psi,\Theta}$-strategy

We will try to meet all requirements $N_{\Phi_1}, N_{\Phi_2}, \ldots$. Each requirement N_{Φ_j} will be denoted by N_j and met by one of the following strategies:

1. (N_j, Γ) with outcomes λ, f, w and l;
2. (N_j, FM) with outcomes f and w and situated in the subtree of the strategy (N_i, Γ) with outcome l, where $i \leq j$.
3. (N_j, Λ) with outcomes f and w and situated in the subtree of the strategy (N_i, Γ) with outcome λ where where $i \leq j$.

We now need to be more careful as more strategies will enumerate and extract markers from A and B. We will have to ensure that the operator constructed on the true path is correct and manages to satisfy the R-requirement.

The first rule that we will implement in order to achieve this follows the idea of cancelling followers of the (N, FM)-strategy from the previous section. Namely, whenever we initialise a strategy (N_j, S) on an node α in the tree of strategies whose follower x is in A we will enumerate an axiom $\langle x, \emptyset \rangle$ into all operators Γ and Λ that are constructed on nodes $\beta < \alpha$. If $m(x)$ is in A we will also enumerate an axiom $\langle m(x), \emptyset \rangle$ into these operators.

Secondly we will be more careful when enumerating axioms in the corresponding operators. Instead of just using the sets $G(x)$ and $H(x)$, we will use the information from all axioms defined up until now. More precisely we will modify the modules of the strategies from the previous section in the following way:

The (N_j, Γ)**-strategy** is the same as the as the (N_Φ, Γ)-strategy with the exception of step *Setup*3, which is now as follows:

*Setup*3) Enumerate all $\langle z, G_x \oplus B \upharpoonright y_x \cup U \rangle$ into Γ where z is either x, or m_x, or a follower $z \in A$ from a previous cycle of the strategy and U is the union of all sets D such that $\langle v, D \rangle$ is a valid axiom in Γ, where $v \in A$ is a follower of the strategy (N_i, Γ) with $i < j$.

The (N_{Φ_j}, Λ)**-strategy** is the same as the (N_Φ, Λ)-strategy with the exception of *Setup*2), which is now as follows:

*Setup*2) Enumerate $\langle x, (H_x \oplus B \upharpoonright v(x)) \cup U \rangle$ into Λ where U is the union of all finite sets D such that $\langle v, D \rangle \in \Lambda$ for some follower $v \in A$ of an (N_k, Λ)-strategy with $k < j$.

The main idea behind the added sets U in the axioms is that a strategy α working below another strategy β where α and β construct the same operator O believes that $\beta's$ work is final and the axioms enumerated in O by β will remain true. In the case that β changes its mind and invalidates one of these axioms α will be initialised as β will have an outcome to the left of α. If $\alpha's$ followers are still in A then an axioms for them will be enumerated in the operator as stated in above. But if $\alpha's$ follower is not in A, then we need to ensure that there isn't a valid axiom in O for it. α will not be able to monitor this follower any longer, so the job is going to be transferred to β automatically via the set U which includes an axiom for $\beta's$ follower, which β observes and makes sure is invalid.

Two R-requirements

Now we need to consider the case when there are two R-requirements. Corresponding to them there are nodes on the tree: an (R_1, Γ_1)-strategy and an (R_2, Γ_2)-strategy along each path, scanning for an appropriate global win for the R-requirements. Below outcome gw for an R_i-strategy the N-requirements simply ignore the requirement R_i and act as in the previous section.

There now more possibilities for an N-strategy working below outcomes e of both (R_i, Γ_i)-strategies depending on how it believes the R_i-requirements will be satisfied.

The main strategy will be again the one that deals with operators Γ_1 and Γ_2. It will try to obtain the necessary changes in the sets $\Psi_1(A)$ and $\Psi_2(A)$ using backup strategies that try to satisfy the R requirements in a different manner. The requirement R_1 is of higher priority. The method for satisfying the lower priority requirement R_2 will be decided after we have established the method for satisfying R_1 unless we have already evidence that the R_2-requirement is trivially satisfied. The N-strategy starts off assuming that the requirements will be satisfied via operators Γ_1 and Γ_2. It will be denoted by (N, Γ_1, Γ_2). Its outcomes are $\lambda_2 <_L f <_L w <_L l_2 <_L l_1$. Outcomes w and f will represent the fact that the strategy has succeeded in satisfying its requirement while keeping both operators rectified.

Outcome l_1 will represent a global win for R_1. The price we pay for it is that the operator Γ_2 will not be rectified. Below this outcome there will be a backup (N, FM_1, Γ_2')-strategy. It will construct a new operator Γ_2' and meet the requirement N. Its outcomes are $\lambda_2 <_L f <_L w <_L l_2$ and it acts just as the (N, Γ)-strategy from the previous section.

Outcome l_2 will represent a global win for R_2. Below it we have a strategy (N, Γ_1, FM_2) which continues to construct the same operator Γ_1 as the (N, Γ_1, Γ_2)-strategy. Strategies below will simply treat R_2 as satisfied - that is, this requirement will be invisible to them.

Below outcome λ_2 is the (R_2, Λ_2)-strategy followed by a backup strategy (N, Γ_1, Λ_2). It continues to construct the same operator for the first strategy Γ_1 but switches the method for the second strategy to Λ_2. Its outcomes are $\lambda_1 <_L f <_L w$.

Below outcome λ_1 is the (R_1, Λ_1)-strategy a backup strategy that changes the method for satisfying the requirement R_1. As a consequence the method for R_2 must be decided again. The strategy is $(N, \Lambda_2, \Gamma_2'')$ with outcomes $\lambda_2 <_L f <_L <_w <_L l_2$. The method for satisfying R_1

cannot be switched anymore. The method for R_2 can be further switched via (N, Λ_1, FM_2) below l_2 and to $(N, \Lambda_1, \Lambda_2'')$ below outcome λ_2.

In this way all possible combinations of methods for satisfying the two R-requirements are distributed through the tree.

The modules for each of the described strategies above follow the basic steps as outlined in the previous section. The (N, Γ_1, Γ_2) strategy chooses a follower x. It tries to define the parameters for R_1 - $H_1(x)$, $G_1(x)$, $y_1(x)$ and $m_1(x)$ and rectifies Γ_1. Then it focuses on the second requirement R_2. Once R_2's parameters are defined a new element $m_2(x)$ will be enumerated in A. The new point is that this new change in A must be reflected in the definition of Γ_1. So an axiom $\langle m_2(x), G_1(x) \oplus \{y_2(x)\}\rangle$ is enumerated in Γ_1. If $m_2(x)$ is extracted from A then we will extract $y_2(x)$ from B and this axiom will not be valid. We will enumerate $y_2(x)$ back in B only if x has been returned in which case $G_1(x) \nsubseteq \Psi_1(A)$.

The axioms enumerated in Γ_2 will have to include additionally $m_1(x)$ and all $m_1(z)$ for previously defined followers of this strategy from previous cycles, that are still in A.

Once we have established that $x \in \Phi(B)$, we start the attack by sending the follower x with defined $v(x) = \max(\varphi(x), y_1(x), y_2(x))$ to (N, Γ_1, Λ_2). This strategy will need to get further permission from Γ_1. An axiom $\langle z, H_2(x) \oplus B \restriction v(x)\rangle$ will be enumerated for each z which is a follower from a previous cycle, x or $m_1(x)$. This strategy also starts an attack by sending x to $(N, \Lambda_1, \Gamma_2'')$ and extracting $y_1(x)$ and $m_1(x)$ from A once it has observed that $x \in \Phi(B)$. Note that this will make the axiom for x in Λ_2 invalid.

The $(N, \Lambda_1, \Gamma_2'')$-strategy now must define parameters $G_2''(x)$ and $H_2''(x)$, markers $y_2''(x)$ and $m_2''(x)$. And then it will initiate the last attack sending x to $(N, \Lambda_1, \Lambda_2'')$.

Once the follower is extracted from A it can climb back up these strategies. (R_2, Λ_2'') will send it back to $(N, \Lambda_1, \Gamma_2'')$ in case $H_2''(x) \subset \Theta_2(A)$.

(R_1, Λ_1) will send the follower x back to (N, Γ_1, Λ_2) in case $H_1(x) \subset \Theta_1(A)$.

Then (R, Λ_2) will send it back to $(N, \Gamma_1\Gamma_2)$ in case $H_2(x) \subset \Theta_2(A)$.

When the $(N, \Gamma_1\Gamma_2)$-strategy re-receives x it will have proof that $H_1(x) \subseteq \Theta_1(A)$, so that $G_1(x) \nsubseteq \Psi_1(A)$ and Γ_1 is rectified and $H_2(x) \subset \Theta_2(A)$, so $G_2(x) \nsubseteq \Psi_2(A)$ and Γ_2 is rectified.

Considering two requirements we can justify the need for the (R_i, Λ_i)-strategies. Suppose $\alpha\hat{\ }l_2 \subset \beta$ and β is sharing the same method Λ_1 as α. If a follower x of β is extracted from A we must ensure that the axioms for x defined in the operator Λ_1 are invalid. It could be the case that α moves on to outcome w and initialises β. The follower x will not be observed any longer. But as $\Theta_1(A)$ is not in our control it is possible that $H_1(x) \subset \Theta_1(A)$ and this is revealed at a later stage after x has been cancelled. If x is not sent back, then Λ_1 will not be correct. This is why we need the (R_1, Λ_1) strategy which observes all followers. It will return x even after x is cancelled.

The (R, Γ_1) strategy plays a similar role. Suppose that $\alpha\hat{\ }l_2 \subset \beta$. Now β is sharing the same method Γ_1 as α. If a follower x of β is extracted from A we must ensure that the axioms for x defined in the operator Γ_1 are invalid. If α moves on to outcome w thereby initialising β we lose control on x and it could happen that $G_1(x) \subset \Psi_1(A)$ at a later stage. We will be able to argue that if the axiom for x in Γ_1 is valid, then $H_1(x) \subset \Theta_1(A)$ and (R, Γ_1) will have outcome gw at all further stages.

In [4] we combine the ideas from the above description to obtain the construction that meets all requirements. □

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