The theory of the enumeration degrees, definability, and automorphisms

Mariya I. Soskova
Department of Mathematics
University of Wisconsin
Madison, WI 53706, USA
msoskova@math.wisc.edu

1 Introduction

Enumeration reducibility captures a natural relationship between sets of natural numbers in which positive information about the first set is used to produce positive information about the second set. It was introduced independently several times in works by Friedberg and Rogers [10], Myhill [39], and Selman [45], who were searching for a natural way to extend the notion of relative Turing computability from total functions to partial functions. Informally, $A \subseteq \omega$ is enumeration reducible to $B \subseteq \omega$ if there is a uniform way to compute an enumeration of A from an enumeration of B. The formal definition that we give below is the one by Friedberg and Rogers [10].

Definition 1.1. $A \subseteq \omega$ is *enumeration reducible* to a set $B \subseteq \omega$ ($A \leq_e B$) if there is a c.e. set W such that

$$A = \{n \colon (\exists e)[\langle n, e\rangle \in W \text{ and } D_e \subseteq B]\},$$

where D_e is the eth finite set in a canonical enumeration.

By identifying sets that are reducible to each other we obtain an algebraic representation of this reducibility as a partial order: the structure of the enumeration degrees $\mathcal{D}_e$. The degree structure $\mathcal{D}_e$ is an upper semi-lattice with least upper bound induced by the effective join operation $A \oplus B = \{2n \mid n \in A\} \cup \{2n+1 \mid n \in B\}$ and a least element $\mathbf{0}_e$, the degree of all c.e. sets.

Motivation for the interest in the enumeration degrees comes from its nontrivial connections to the study of the Turing degrees. In Turing reducibility, $\leq_T$,

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we use membership information, both positive and negative, from a given oracle set B to obtain the same type of membership information about a reduced set A. Enumeration reducibility, restricts us to both using and producing only positive information. There is a further relation that sits between Turing and enumeration reducibility. The relation *relative computable enumerability* (c.e. in) uses positive and negative information about an oracle set B to produce only positive information about a set A. We can express the positive and negative information about a set A in a positive way by replacing it with $A \oplus \overline{A}$. This gives the following relationship between the three reducibilities:

Proposition 1.2. *$A \leq_T B \Leftrightarrow A \oplus \overline{A}$ is B-c.e. $\Leftrightarrow A \oplus \overline{A} \leq_e B \oplus \overline{B}$.*

Myhill [39] used this relationship to define a natural embedding of the Turing degrees into the enumeration degrees. He proved that the embedding $\iota\colon \mathcal{D}_T \to \mathcal{D}_e$, defined by

$$\iota(d_T(A)) = d_e(A \oplus \overline{A}),$$

preserves the order and the least upper bound. Thus we have a copy of the Turing degrees sitting inside the enumeration degrees. Medvedev [35] observed that sufficiently generic sets have enumeration degrees outside of the range of this embedding and so the enumeration degrees properly extend the Turing degrees.

In this article we outline some of the more recent results in the study of the enumeration degrees. We will focus on three aspects of the structure of the enumeration degrees:

I. The first order theory of $\mathcal{D}_e$ and its fragments;

II. First order definability;

III. Automorphisms and automorphism bases.

We will outline the current state of the art and discuss open problems on each topic that we believe mark important goals for the advancement of knowledge in the field. We will see that enumeration reducibility and the enumeration degrees have nontrivial interactions with other parts of mathematics, most prominently with topology.

The structure of this article is motivated by a theorem that was proved by Slaman and Woodin [53] for the Turing degrees and extended by Soskova [54] to the enumeration degrees. We state it here and elaborate on it throughout the paper:

Theorem 1.3 (Slaman and Woodin, Soskova)**.** *The following are equivalent:*

1. *$\mathcal{D}_e$ is* biinterpretable *with second order arithmetic.*

2. *The definable relations in $\mathcal{D}_e$ are exactly the ones induced by degree invariant definable relations in second order arithmetic.*

3. *$\mathcal{D}_e$ is a rigid structure.*

Thus, the three aspects we consider are related and provide different outlooks on one main problem.

Question 1.4. Are there nontrivial automorphisms of the enumeration degrees?

2 The first order theory of $\mathcal{D}_e$ and its fragments

We start our explorations with the theory of the enumeration degrees $Th(\mathcal{D}_e)$, the set of sentences in the language of partial orders that are true in the structure. Our first observation is that enumeration reducibility is a relation that can be defined in second order arithmetic Z_2. Assuming some basic facts from classical recursion theory, it is straightforward to check that enumeration reducibility is a Σ^0_3 relation. Therefore, the theory $Th(\mathcal{D}_e)$ can be effectively interpreted in second order arithmetic. This does not really help us with understanding the complexity of the set, except for providing the evident upper bound $Th(\mathcal{D}_e) \leq_T Th(Z_2)$. It does, however, set the stage for our investigations and hints to the connections between $\mathcal{D}_e$ and Z_2 outlined in Theorem 1.3. We will first consider fragments of the theory that we obtain by restricting the quantifier complexity of the sentences.

2.1 The existential theory of $\mathcal{D}_e$

The simplest fragment of the theory of $\mathcal{D}_e$ is the existential theory, the set of existential sentences true in the structure. We denote this set by $\exists\text{-}Th(\mathcal{D}_e)$. An existential statement has the form

$$(\exists \mathbf{x}_1)\dots(\exists \mathbf{x}_n)[\varphi(\mathbf{x}_1,\dots \mathbf{x}_n)],$$

where φ is either obviously false because it contradicts the axioms of partial orderes or else it is a disjunction of quantifier free formulas that partially describe a finite partial order. In order for an existential sentence to be true in $\mathcal{D}_e$, it first must comply with the axioms of partial order. This is something that can be effectively checked. It follows that in order to decide whether an existential sentence is true in $\mathcal{D}_e$ we must understand which finite partial orders can be embedded in $\mathcal{D}_e$. The answer is simple: all partial orders can be embedded. This result can be traced back to Sacks [43], who showed that every countable partial order can be embedded in $\mathcal{D}_T$ (in fact, he showed it for $\mathcal{D}_T(\leq_T \mathbf{0}')$, the initial interval of the Turing degrees bounded by $\mathbf{0}'$), in combination with the embedding of $\mathcal{D}_T$ in $\mathcal{D}_e$. In the enumeration degrees this result was extended first by Lagemann [26], who showed that every countable partial order can be embedded below any nonzero Δ^0_2 enumeration degree, then by Bianchini [4] who found such embeddings in any nonempty interval of Σ^0_2 enumeration degrees, then by Soskov and Soskova [55], who replaced Σ^0_2 with *good*. A good enumeration degree is a degree that contains a set with a *good approximation*. The good approximations were introduced by Lachlan and Shore [24]. They use them to show density for the n-c.e.a. degrees: a hierarchy of enumeration degrees based on the relation c.e. in. Most recently this series of results have been extended by Slaman and Sorbi [48].

Theorem 2.1 (Slaman, Sorbi)**.** *Every countable partial order can be embedded below any nonzero enumeration degree.*

Note that the statement of the theorem above, reveals an important structural property of the enumeration degrees, initially proved by Gutteridge [16]: the enumeration degrees are downwards dense. We will see that this statement will play a trick on us when we consider more complex fragments of the theory. It also provides an example of a structural difference between $\mathcal{D}_e$ and $\mathcal{D}_T$, where minimal degrees exist.

In any case, we now know that the existential theory of the enumeration degrees is decidable. We move on to the next quantifier complexity level, where the situation is less clear.

2.2 The two quantifier theory of $\mathcal{D}_e$

We only have partial understanding of the two quantifier theory of the enumeration degrees $\forall\exists$-$Th(\mathcal{D}_e)$. In order to describe this, let us again consider the problem in more detail. A two quantifier statement has the form:

$$(\forall \mathbf{x}_1)\dots(\forall \mathbf{x}_k)(\exists \mathbf{y}_1)\dots(\exists \mathbf{y}_n)[\varphi(\mathbf{x}_1,\dots \mathbf{x}_k,\mathbf{y}_1,\dots \mathbf{y}_n)],$$

where φ is once again a disjunction of conjunctions. Let $\psi_1,\dots\psi_r$ be formulas that describe the complete quantifier free type of the variables $\mathbf{x}_1,\dots \mathbf{x}_k$. The statement above is equivalent to the following:

$$\bigwedge_{i<r}(\forall \mathbf{x}_1)\dots(\forall \mathbf{x}_k)[\psi_i(\mathbf{x}_1,\dots \mathbf{x}_k)\to(\exists \mathbf{y}_1)\dots(\exists \mathbf{y}_n)\varphi(\mathbf{x}_1,\dots \mathbf{x}_k,\mathbf{y}_1,\dots \mathbf{y}_n)].$$

And so it is sufficient to decide statements of the form:

$$(\forall \mathbf{x}_1)\dots(\forall \mathbf{x}_k)[\psi(\mathbf{x}_1,\dots \mathbf{x}_k)\to\bigvee_{i<t}(\exists \mathbf{y}_1)\dots(\exists \mathbf{y}_n)\varphi_i(\mathbf{x}_1,\dots \mathbf{x}_k,\mathbf{y}_1,\dots \mathbf{y}_n),$$

where ψ describes the quantifier free type of the variables $\mathbf{x}_1, \dots \mathbf{x}_k$ and each φ_i describes one possible quantifier free type of the variables $\mathbf{x}_1, \dots, \mathbf{x}_k, \mathbf{y}_1, \dots, \mathbf{y}_n$ that is consistent with the type described by ψ. We can restate this question in a structural way as follows:

Problem 2.2. *If P is a finite partial order and Q_1, $\dots Q_t$ are finite extensions of P, decide whether every embedding of P into the enumeration degrees can be extended to an embedding of one of the Q_i.*

The simpler problem when $t = 1$ is the *the extension of embeddings problem.*

For $\mathcal{D}_T$ the problem above is decidable. Lerman [30] showed that every finite lattice P can be embedded as an initial segment of $\mathcal{D}_T$. Thus, if P is a lattice then this embedding of P can be extended to an embedding of Q only if no new element in $Q \setminus P$ is below any element of P. In addition, Q must respect least upper bounds: i.e. if $x \in Q \setminus P$ is above two old elements $u, v \in P$ then x must be above $u \vee v$. If P is not a lattice then points in $Q \setminus P$ can also take the place of least upper bounds that need to be added just because we are embedding in an upper semi-lattice. Shore [46] and Lerman [31] then proved that these are the only obstacles and so the decision problem is computable. The algorithm does not even use the possibility of selecting different possible extensions in different situations: the decision problem is reduced to its simplest case, it is equivalent to the extension of embeddings problem.

In $\mathcal{D}_e$ the situation is very interesting for the following reasons. As we mentioned earlier, Gutteridge [16] showed that the enumeration degrees are downwards dense and so no finite lattice can be embedded as an initial segment. Cooper [8] proved, however, that the enumeration degrees are not dense and Slaman and Calhoun [6] extended Cooper's results by showing that there are empty intervals in the Π^0_2-enumeration degrees. Kent, Lewis-Pye, and Sorbi [21] showed that there are strong minimal covers in the enumeration degrees:

Definition 2.3. The degree $\mathbf{b}$ is a *strong minimal* cover of $\mathbf{a}$ if $\mathbf{a} < \mathbf{b}$ and every degree $\mathbf{x} < \mathbf{b}$ is also bounded by $\mathbf{a}$.

Consider the two-element lattice P consisting of two elements $u < v$. The embedding of P to degrees $\mathbf{a} < \mathbf{b}$ such that $\mathbf{b}$ is a strong minimal cover of $\mathbf{a}$ extends to an embedding of Q only if new elements $x \in Q \setminus P$ that are strictly below v are also below u. The embedding of P to degrees $\mathbf{0}_e < \mathbf{b}$, on the other hand, extends to an embedding of Q only if new elements $x \in Q \setminus P$ are above u. Using Theorem 2.1 and a fairly standard forcing construction we can conclude that these are the only obstacles. Thus for this lattice P we can decide the problem of extending to one of many Q_i's: every embedding of P extends to an embedding

$Q_1, \dots Q_n$, if and only if there is one Q_i that places elements strictly below v also below u and there is another Q_j that places new elements above u. The decision procedure is already slightly more complicated than that for the same lattice in $\mathcal{D}_T$ and is not equivalent to the extension of embeddings problem.

Towards a possible decision procedure for the more general problem Lempp, Slaman, and Soskova [28] prove the following

Theorem 2.4 (Lempp, Slaman, Soskova)**.** *Every finite distributive lattice can be embedded as an interval* $[\mathbf{a}, \mathbf{b}]$*, so that if* $\mathbf{x} \leq \mathbf{b}$ *then* $\mathbf{x} \in [\mathbf{a}, \mathbf{b}]$ *or* $\mathbf{x} < \mathbf{a}$*.*

Note that in the theorem above the range of our embedding is the whole interval $[\mathbf{a}, \mathbf{b}]$, and so this is an extension of the existence of strong minimal covers in the enumeration degrees. We will say that this is a *strong interval lattice embedding*. This turns out to be sufficient to decide the extension of embeddings problem:

Theorem 2.5 (Lempp, Slaman, Soskova)**.** *The extension of embeddings problem for* $\mathcal{D}_e$ *is decidable.*

Some very important questions remain open. First of all, we do not know, whether we can remove the distributivity restriction in the theorem above.

Question 2.6. Does every finite lattice have a strong interval embedding in $\mathcal{D}_e$?

Even if we had a positive answer to the question above, we still do not know, whether we can decide the two quantifier theory of $\mathcal{D}_e$. Before we can answer that question, we need to understand more about the structure of $\mathcal{D}_e$. One particularly difficult structural questions concerns the existence of a strong minimal pair.

Definition 2.7. A pair of degrees $\mathbf{a}$ and $\mathbf{b}$ form a *strong minimal pair* if

- $\mathbf{a}$ and $\mathbf{b}$ are incomparable degrees with only $\mathbf{0}_e$ as their common lower bound.
- if $\mathbf{x} \leq \mathbf{b}$ then $\mathbf{a} \vee \mathbf{x} \geq \mathbf{b}$, and, similarly, if $\mathbf{x} \leq \mathbf{a}$ then $\mathbf{b} \vee \mathbf{x} \geq \mathbf{b}$.

Question 2.8. Are there strong minimal pairs in $\mathcal{D}_e$?

2.3 The three quantifier theory of $\mathcal{D}_e$

Finite distributive lattices are already fairly complicated structures. Nies [40] showed that their $\forall\exists\forall$-theory in the language of partial orders is hereditarily undecidable. He also gave a way to transfer this undecidability to structures in which we can define finite distributive lattices.

Definition 2.9. Let $\mathcal{C}$ be a class of structures in a finite relational language $L = \{R_1, \dots R_n\}$. We say that $\mathcal{C}$ is Σ_k-*elementary definable* in $\mathcal{D}_e$ if there are Σ_k formulas φ_U, φ_{R_i}, and $\varphi_{\neg R_i}$ for $i \leq n$ such that for every $C \in \mathcal{C}$ there are parameters $\vec{\mathbf{p}} \in \mathcal{D}_e$ that make the structure with universe $U = \{\mathbf{x} \mid \mathcal{D}_e \models \varphi_U(\mathbf{x}, \vec{\mathbf{p}})\}$ and relations R_i defined as $\{\vec{\mathbf{x}} \mid \mathcal{D}_e \models \varphi_{R_i}(\vec{\mathbf{x}}, \vec{\mathbf{p}})\} = \{\vec{\mathbf{x}} \mid \mathcal{D}_e \models \neg\varphi_{\neg R_i}(\vec{\mathbf{x}}, \vec{\mathbf{p}})\}$ isomorphic to C.

Theorem 2.4 implies that the class of finite distributive lattices is Σ_1-elementary definable in the partial order $\mathcal{D}_e$ with two parameters: $\varphi_U(\mathbf{x}, \mathbf{a}, \mathbf{b})$ is the formula $\mathbf{a} \leq \mathbf{x}$ & $\mathbf{x} \leq \mathbf{b}$ and $=, \neq, \leq$ and $\nleq$ are interpreted via $=, \neq, \leq$ and $\nleq$. We next apply the Nies Transfer Lemma [40]:

Lemma 2.10 (Nies Transfer Lemma). *Let $r \geq 2$ and $k \geq 1$. If a class of models $\mathcal{C}$ is Σ_k-elementarily definable in $\mathcal{D}_e$ and the $r+1$-quantifier fragment of $\mathcal{C}$ is hereditarily undecidable then the $k+r$-quantifier fragment of $\mathcal{D}_e$ is hereditarily undecidable.*

We can now conclude:

Corollary 2.11. *The three quantifier theory of $\mathcal{D}_e$ is (hereditarily) undecidable.*

This makes the question of the decidability of the $\forall\exists$-theory of $\mathcal{D}_e$ all the more interesting, as it would exactly give us the quantifier complexity where decidability breaks down.

2.4 The full theory of $\mathcal{D}_e$

Let us now turn to the full theory of the enumeration degrees. It follows from what we have said so far that this theory is not decidable. But how complicated is it? Well, first of all, as we already discussed, enumeration reducibility is arithmetically definable in Z_2 and so Z_2 can interpret $\mathcal{D}_e$. This sets an upper bound to the complexity of $Th(D_e)$, namely we see that it is 1-reducible to $Th(Z_2)$. Slaman and Woodin [52] prove that the reverse is true as well.

Theorem 2.12 (Slaman, Woodin). *The first order theory of the enumeration degrees is computably isomorphic to the theory of second order arithmetic.*

In other words, they show that there is an algorithm that allows us to translate a sentence φ in the language of second order arithmetic to a sentence ψ in the language of partial orders so that $Z_2 \models \varphi$ if and only if $\mathcal{D}_e \models \psi$. The main tool that they use is their *Coding Theorem*:

Theorem 2.13 (Coding Theorem)**.** *There is a uniform way to define every countable relation on $\mathcal{D}_e$ using parameters. In other words, for every n there is a formula φ_n such that for every countable relation $R \subseteq \mathcal{D}_e^n$ there are parameters $\vec{\mathbf{p}}$ such that $R(\mathbf{a}_1, \dots \mathbf{a}_n)$ is true if and only if $\mathcal{D}_e \models \varphi_n(\mathbf{a}_1, \dots \mathbf{a}_n, \vec{\mathbf{p}})$.*

The Coding Theorem lets us pick out in a definable way a tuple of parameters $\vec{\mathbf{p}}$ that codes unary relations $N_{\vec{\mathbf{p}}}$ and $C_{\vec{\mathbf{p}}}$, 3-ary relations $R_{+,\vec{\mathbf{p}}}$ and $R_{*,\vec{\mathbf{p}}}$, such that the structure $M_{\vec{\mathbf{p}}} = (N_{\vec{\mathbf{p}}}; +_{\vec{\mathbf{p}}}, *_{\vec{\mathbf{p}}}, C_{\vec{\mathbf{p}}})$, where $\mathbf{a}_1 +_{\vec{\mathbf{p}}} \mathbf{a}_2 = \mathbf{a}_3$ if and only if $R_{+,\vec{\mathbf{p}}}(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$ and, similarly, $\mathbf{a}_1 *_{\vec{\mathbf{p}}} \mathbf{a}_2 = \mathbf{a}_3$ if and only if $R_{*,\vec{\mathbf{p}}}(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$, is isomorphic to true arithmetic with a predicate for a set C. We say that $\vec{\mathbf{p}}$ codes the model $(\mathbb{N}, +, *, C)$.

The *biinterpretability conjecture* suggests that the relationship between Z_2 and $\mathcal{D}_e$ is even stronger:

Conjecture 2.14 (Biinterpretability conjecture)**.** *The relation $Bi(\vec{\mathbf{p}}, \mathbf{c})$, true when $\vec{\mathbf{p}}$ codes a model $(\mathbb{N}, +, *, C)$ and $\deg_e(C) = \mathbf{c}$, is first order definable in $\mathcal{D}_e$.*

Slaman and Woodin formulate this conjecture for a number of degree structures, including $\mathcal{D}_T$, $\mathcal{D}_e$, the arithmetical degrees $\mathcal{D}_a$, the hyperarithmetical degrees $\mathcal{D}_h$. In their fundamental work [53] on the analysis on the automorphism group of $\mathcal{D}_T$ they prove that the conjecture is true for the hyperarithmetical degrees. For $\mathcal{D}_T$ they are only able to show that it is true modulo the use of a single parameter. Soskova [54], extends their work and shows that, as anticipated, the same is true for $\mathcal{D}_e$.

Theorem 2.15 (Slaman and Woodin, Soskova)**.** *There is a single parameter $\mathbf{g}$ and a formula φ such that $Bi(\vec{\mathbf{p}}, \mathbf{c})$ is true if and only if $\mathcal{D}_e \models \varphi(\mathbf{g}, \vec{\mathbf{p}}, \mathbf{c})$.*

Using the equivalence proved in Theorem 1.3 we can infer something about our next theme: first order definability. However, in this case as well, we need to use a parameter.

Corollary 2.16. *Every relation on $\mathcal{D}_e$ that is induced by a degree invariant definable relation in second order arithmetic can be defined in $\mathcal{D}_e$ using a single parameter $\mathbf{g}$.*

3 First order definability in $\mathcal{D}_e$

One of the most celebrated first order definability results in the structure of the Turing degrees is the first order definability of the Turing jump. Recall, that the Turing jump is an operator on $\mathcal{D}_T$ that maps a degree $\mathbf{a}$ to $\mathbf{a}' > \mathbf{a}$ and is defined by relativizing the halting problem to an arbitrary set. Slaman and Woodin's [53] analysis of the automorphism group of $\mathcal{D}_T$ allows them to prove that for every automorphism on the Turing degrees π and every degree $\mathbf{x}$ we have that $\pi(\mathbf{x})'' = \pi(\mathbf{x}'')$. In other words, the double jump operator is preserved by automorphisms. They also show that if a relation is invariant under all automorphisms then it must be definable without parameters. As a result they get the first order definability of the double jump operator. Shore and Slaman [47] then build on top of that result to prove that the jump operator is also definable. The proof uses the Kumabe-Slaman forcing method. As you can probably guess, the first order definition of the jump operator that comes out of this elaborate proof is not intuitive and has fairly high quantifier complexity. We will see in this section that definability in the enumeration degrees is quite different.

3.1 The enumeration jump and the total enumeration degrees

Before we can illustrate how definability differs in the enumeration degrees, we need to isolate interesting relations on $\mathcal{D}_e$ whose first order definability would be informative. And what better way to start than with an enumeration degree analog of the jump operator. Recall that the halting set K^A relative to a set A is the uniform join of all c.e. in A sets. When we try to adapt the definition to the world of enumeration degrees we naturally consider the set $K_A = \bigoplus_{e<\omega} \Gamma_e(A)$—the uniform join of the sets that are enumeration reducible to A. Unfortunately, this does not give rise to a very interesting operator, because $K_A \equiv_e A$. In the proof that $A \not\leq_T K^A$ we actually use the fact that $\overline{K^A}$ is not c.e. in A. This idea gives rise to the following definition of the enumeration jump operator introduced by Cooper [7].

Definition 3.1. The *enumeration jump* of a set A is the set $A' = K_A \oplus \overline{K}_A$. The jump of a degree is $\deg_e(A)' = \deg_e(A')$.

The enumeration jump operator has many of the properties that we expect from a jump operator: for instance for all $\mathbf{a}$ we have that $\mathbf{a} < \mathbf{a}'$ and $\mathbf{a} \leq \mathbf{b}$ implies $\mathbf{a}' \leq \mathbf{b}'$. It also agrees with the Turing jump under the standard embedding ι. So naturally we may wonder: Is the enumeration jump operator first order definable?

Another, possibly more important, class of enumeration degrees is the class of all total enumeration degrees.

Definition 3.2. A set A is *total* if $\overline{A} \leq_e A$. An enumeration degree is *total* if it contains a total set.

To understand where the name *total* note that the graph of a total function is total and that every total degree contains the graph of a total function. An equivalent way of defining total degrees is as the enumeration degrees of sets of the form $A \oplus \overline{A}$. In other words, the total enumeration degrees are exactly the degrees that are images of Turing degrees under the standard embedding ι. It was Rogers [42] who asked first whether the total degrees are first order definable in $\mathcal{D}_e$. In fact, Rogers [42] had a list of questions among which were whether $\mathcal{D}_e$ and $\mathcal{D}_T$ are rigid, whether the Turing jump is first order definable and whether definability is equivalent to invariance under automorphisms.

We will see that in $\mathcal{D}_e$ both the enumeration jump and the total enumeration degrees have natural, simple first order definitions. At the heart of these definition is a notion introduced by Jockusch [18] in his thesis.

Definition 3.3. A set A is semi-computable if and only if there is a total computable selector function $s_A \colon \omega^2 \to \omega$—a function such that $\forall x, y \in \omega$ we have that $s_A(x, y) \in \{x, y\}$ and whenever $\{x, y\} \cap A \neq \emptyset$ we have that $s_A(x, y) \in A$.

Jockusch [18] characterized semi-computable sets as left cuts in computable linear orderings on ω. One direction of this characterization is straightforward: if $\leq_L$ is a computable linear ordering then the function $s(x, y)$ that compares its inputs and outputs the one that is smaller with respect to $\leq_L$ witnesses that all left cuts in that linear ordering are semi-computable. Jockusch also proved that semi-computable sets are far from computable. In fact, every Turing degree contains a semi-computable set that is neither c.e. nor co-c.e. Translated through the embedding ι into enumeration degree theoretic terms this shows that every total enumeration degrees is the nontrivial join of the enumeration degrees of a semi-computable set and its complement. Here by nontrivial, we mean that neither of these two degrees is $\mathbf{0}_e$. Arslanov, Cooper and Kalimullin [3] realized that the enumeration degrees of a semi-computable set and its complement satisfy an unusual structural property:

Definition 3.4. A pair of enumeration degrees $\{\mathbf{a}, \mathbf{b}\}$ is a *robust minimal pair* if and only if:

$$(\forall \mathbf{x})[(\mathbf{a} \vee \mathbf{x}) \wedge (\mathbf{b} \vee \mathbf{x}) = \mathbf{x}.]$$

Note that $\mathbf{0}$ forms a robust minimal pair with any other degree. We will call this a *trivial* robust minimal pair. The reason that we called the property above unusual is once again rooted in intuition coming from the Turing degrees. Posner and Robinson [41] prove that if $D \geq_T \emptyset'$ and $\{A_i\}_{i<\omega}$ is a sequence of uniformly D-computable incomputable sets then there is a set G such that

$$(\forall i)(A_i \oplus G \equiv_T G' \equiv_T D)$$

As a consequence we get that for any pair of nonzero Turing degrees $\{\mathbf{a}, \mathbf{b}\}$ there is a Turing degree $\mathbf{g}$ such that $\mathbf{a} \vee \mathbf{g} = \mathbf{b} \vee \mathbf{g} = \mathbf{g}'$. As $\mathbf{g} < \mathbf{g}'$, the degree $\mathbf{g}$ witnesses that the pair $\{\mathbf{a}, \mathbf{b}\}$ is not a robust minimal pair. So there are no nontrivial robust minimal pairs in the Turing degrees.

It is alluring to hope that the robust minimal pairs define semi-computable pairs, as that would give a fairly simple definition of the nonzero total enumeration degrees: joins of nontrivial robust minimal pairs. Kalimullin [19] showed that this is, unfortunately, not the case. He gave a combinatorial characterization of the pairs of sets whose degrees form robust minimal pairs:

Definition 3.5. Sets A and B form a *Kalimullin pair* ($\mathcal{K}$-pair) relative to a set U if and only if there is a set $W \leq_e U$ such that $A \times B \subseteq W$ and $\overline{A} \times \overline{B} \subseteq \overline{W}$.

Theorem 3.6 (Kalimullin). *A pair of sets A and B are a $\mathcal{K}$-pair relative to a set U if and only if their enumeration degrees $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{u}$ satisfy*

$$(\forall \mathbf{x} \geq \mathbf{u})[(\mathbf{a} \vee \mathbf{u} \vee \mathbf{x}) \wedge (\mathbf{b} \vee \mathbf{u} \vee \mathbf{x}) = \mathbf{x}$$

We will say that $\mathbf{a}$ and $\mathbf{b}$ are a *robust minimal pair relative to* $\mathbf{u}$ if they satisfy the formula from the definition above. And so, the robust minimal pairs are exactly the degrees of $\mathcal{K}$-pairs relative to any c.e. set. We call such pairs simply $\mathcal{K}$-pairs. It is not difficult to show that this class is much larger than the class of semi-computable pairs. Nevertheless, Kalimullin [19] was able to show that they are extremely useful for definability results in $\mathcal{D}_e$. He proved that the enumeration jump can be characterized via robust minimal pairs.

Theorem 3.7 (Kalimullin). *The jump of an enumeration degree $\mathbf{u}$ is the greatest degree that can be represented as $\mathbf{a} \vee \mathbf{b} \vee \mathbf{c}$ where $\{\mathbf{a}, \mathbf{b}\}$, $\{\mathbf{b}, \mathbf{c}\}$, and $\{\mathbf{a}, \mathbf{c}\}$ all form robust minimal pairs relative to $\mathbf{u}$.*

Ganchev and Soskova [14] gave an alternative definition of the enumeration jump which only relies on unrelativized robust minimal pairs:

Theorem 3.8 (Ganchev, Soskova). *The jump of an enumeration degree $\mathbf{u}$ is the greatest degree that can be represented as $\mathbf{a} \vee \mathbf{b}$ for a nontrivial robust minimal pair $\{\mathbf{a}, \mathbf{b}\}$ such that $\mathbf{a} \leq \mathbf{u}$.*

It took several more years to arrive at the correct approach to the first order definability of the total enumeration degrees. Ganchev and Soskova [14] realized that semi-computable pairs satisfy a stronger structural feature: they are *maximal* robust minimal pairs.

Definition 3.9. A robust minimal pair $\{\mathbf{a}, \mathbf{b}\}$ is *maximal* if and only if whenever $\{\mathbf{c}, \mathbf{d}\}$ is a robust minimal pair such that $\mathbf{a} \leq \mathbf{c}$ and $\mathbf{b} \leq \mathbf{d}$, we have that $\mathbf{a} = \mathbf{c}$ and $\mathbf{b} = \mathbf{d}$.

In other words, these are robust minimal pairs such that neither side can be further lifted to form a higher robust minimal pair. The final piece of the puzzle was to show that for every $\mathcal{K}$-pair $\{A, B\}$ there is a semi-computable set C such that $A \leq_e C$ and $B \leq_e \overline{C}$. Ganchev and Soskova [14] showed that this holds for $\mathcal{K}$-pairs bounded by $\mathbf{0}_e'$. The full result was then obtained by Cai, Ganchev, Lempp, Miller and Soskova [5]:

Theorem 3.10 (Cai, Ganchev, Lempp, Miller and Soskova)**.** *A nonzero enumeration degree is total if and only if it can be represented as* $\mathbf{a} \vee \mathbf{b}$*, for a* maximal *robust minimal pair.*

The first order definability of the total enumeration degrees clarified a lot of the parallels that we were observing between $\mathcal{D}_T$ and $\mathcal{D}_e$. For example, Theorem 2.12, Theorem 2.13, and Theorem 2.15 are now a direct consequences of the corresponding facts true of the Turing degrees.

This definition of the total enumeration degrees lead to an additional surprising consequence: the first order definability of the image of the relation "c.e. in". Recall that for Turing degrees $\mathbf{a}$ and $\mathbf{b}$, we say that $\mathbf{a}$ is c.e. in $\mathbf{b}$ if and only if there is some set $A \in \mathbf{a}$ which is c.e. in some set (or equivalently all sets) $B \in \mathbf{b}$.

Theorem 3.11 (Cai, Ganchev, Lempp, Miller and Soskova [5])**.** *The relation*

$$\{(\mathbf{a}, \mathbf{b}) \in \mathcal{D}_e^2 \mid \mathbf{a}, \mathbf{b} \textit{ are total} \ \& \ \iota^{-1}(\mathbf{a}) \textit{ is c.e. in } \iota^{-1}(\mathbf{b})\}$$

is first order definable in $\mathcal{D}_e$*.*

So far we have only looked at relations and classes that are closely related to our study and understanding of the Turing degrees. We next take a look at a class that arises differently: from the study of effective mathematics.

3.2 The continuous degrees

Computable analysis allows us to lift computability theoretic notions from sets of natural numbers to more complex mathematical objects, such as real numbers, continuous functions, elements of the Hilbert cube. All of these are examples of points in computable metric spaces, a notion introduced by Lacombe [25].

Definition 3.12. A *computable metric space* is a metric space $\mathcal{M}$ together with a countable dense sequence $Q^{\mathcal{M}} = \{q_n^{\mathcal{M}}\}_{n \in \omega}$ on which the metric is computable, i.e. there is a computable function that maps a pair of indices i, j and a precision $\varepsilon \in \mathbb{Q}^+$ to a rational that is within ε of $d_{\mathcal{M}}(q_i, q_j)$.

The canonical example of a computable metric space is $\mathbb{R}$ with $Q^{\mathbb{R}}$ some computable listing of the rational numbers $\mathbb{Q}$. But many other second countable metric spaces (metric spaces with a countable base) can be supplied with a listing of a dense sequence to make them computable: for example, Cantor space 2^ω and Baire space ω^ω with the usual metric, the continuous functions on the unit interval $\mathbf{C}[0,1]$ with the uniform metric, the Hilbert cube $[0,1]^\omega$ with metric $d(\alpha,\beta) = \sum_{n\in\omega} \frac{|\alpha(n)-\beta(n)|}{2^n}$.

To every member x of a computable metric space we associate a set of *names*—discrete objects that give us a way to approximate x with arbitrary precision using the distinguished dense sequence:

Definition 3.13. $\lambda\colon \mathbb{Q}^+ \to \omega$ is a *name* of a point x in a computable metric space $\mathcal{M}$ if for all rationals $\varepsilon > 0$ we have $d_{\mathcal{M}}(x, q^{\mathcal{M}}_{\lambda(\varepsilon)}) < \varepsilon$.

We think of the set of names for x as carrying the algorithmic content of x. In particular, we can define a computable function on computable metric spaces $f\colon \mathcal{M} \to \mathcal{N}$ as (represented by) a computable functional Ψ that takes names of a point $x \in \mathcal{M}$ to names of $f(x) \in \mathcal{N}$. We can also talk about the *Turing degree* of a point: the least Turing degree of a name for that point. It is fairly easy to see that every real number r has a Turing degree, the Turing degree of its Dedekind cut $\{q \in \mathbb{Q} \mid q < r\} \oplus \{q \in \mathbb{Q} \mid q > r\}$. Pour El and Lempp asked whether this is also true for continuous functions on the real numbers. To answer their question, Miller [36] introduced a way to compare the computable strength of points in arbitrary computable metric spaces.

Definition 3.14. If x and y are members of (possibly different) computable metric spaces, then $x \leq_r y$ if there is a uniform way to compute a name for x from a name for y.

This reducibility induces a degree structure, which Miller [36] called the *continuous degrees*. His reason for the choice of name comes from the following characterization:

Theorem 3.15 (Miller)**.** *Every continuous degree contains a point from* $[0,1]^\omega$ *and a point from* $\mathbf{C}[0,1]$.

In other words, we can think of the continuous functions on the unit interval and of the Hilbert cube as universal spaces. Using the universality of $[0,1]^\omega$ Miller [36] was able to show that the continuous degrees embed into the enumeration degrees. For $\alpha \in [0,1]^\omega$, let

$$C_\alpha = \bigoplus_{i\in\omega} \{q \in \mathbb{Q} \mid q < \alpha(i)\} \ \oplus\ \{q \in \mathbb{Q} \mid q > \alpha(i)\}.$$

Enumerating C_α is exactly as hard as computing a name for α. So $\alpha \mapsto C_\alpha$ induces the aforementioned embedding. Each element of 2^ω, ω^ω, and $\mathbb{R}$ is mapped onto the total degree of its least Turing degree name (i.e., the image of its Turing degree). Lempp and Pour El's question can be restated in terms of this embedding as: is there a continuous degree that is non-total. Miller [36] answered this question:

Theorem 3.16 (Miller). *There is a nontotal continuous degree.*

It is worth pointing out that every known proof of this result uses nontrivial topological facts: Miller [36] used a variant of Brouwer's fixed point theorem for multivalued functions on an infinite dimensional space. Day and Miller [9] gave an alternative proof that relies on neutral measures. Levin [33] used Sperner's lemma to construct such measures. More recently, Kihara and Pauly [23], and independently Hoyrup (unpublished) used results from topological dimension theory—that $[0,1]^\omega$ is strongly infinite dimensional and therefore not the countable union of finite dimensional spaces.

The continuous degrees therefore constitute an interesting class of enumeration degrees. They properly extend the Turing degrees. Miller [36] proved that no continuous degree can be quasiminimal, so they are a proper subclass of the enumeration degrees. Are they first order definable? Surprisingly, the answer turns out to be: yes and they have a very natural first order definition. Andrews, Igusa, Miller and Soskova [2] use an effective version of Urysohn's metrization theorem due to Schröder [44] to show the following:

Theorem 3.17 (Andrews, Igusa, Miller, and Soskova). *An enumeration degree* $\mathbf{a}$ *is continuous if and only if it is* almost total*: if* $\mathbf{x} \not\leq \mathbf{a}$ *and* $\mathbf{x}$ *is total then* $\mathbf{a} \vee \mathbf{x}$ *is total.*

It follows from the definability of the total enumeration degrees that the continuous degrees are first order definable. In this case as well, the definability of the continuous degrees has a pleasing further consequence. Recall, that a Turing degree $\mathbf{a}$ is *PA above* a Turing degree $\mathbf{b}$ if $\mathbf{a}$ computes a path in every infinite $\mathbf{b}$-computable tree. Using the embedding ι we can transfer this relation to total degrees. Miller [36] proved that nontotal continuous degrees can be used to characterize this relation.

Theorem 3.18. *For total degrees* $\mathbf{a}$ *is* PA above $\mathbf{b}$ *if and only if there is a nontotal continuous degree* $\mathbf{c}$ *such that* $\mathbf{b} < \mathbf{c} < \mathbf{a}$.

The definability of the non-total continuous degrees now yields:

Corollary 3.19. *The image of the relation "PA above" is first order definable in* $\mathcal{D}_e$.

Ganchev, Kalimullin, Miller, and Soskova [11] give an alternative first order definition of the continuous degrees that relies only on $\mathcal{K}$-pairs and avoids invoking the definability of the total degrees. They show that an enumeration degree is continuous if and only if it is not half of any nontrivial relativized $\mathcal{K}$-pair. This gives a structural dichotomy in the enumeration degrees:

Theorem 3.20 (Ganchev, Kalimullin, Miller, and Soskova)**.** *For every enumeration degree* $\mathbf{a}$*, exactly one of the following two properties holds:*

1. *The degree* $\mathbf{x}$ *is continuous, so for every total enumeration degree* $\mathbf{x} \not\leq \mathbf{a}$*,* $\mathbf{a} \vee \mathbf{x}$ *is total.*

2. *There is a total enumeration degree* $\mathbf{x} \not\leq \mathbf{a}$ *such that* $\mathbf{a} \vee \mathbf{x}$ *is a strong quasiminimal cover of* $\mathbf{x}$.

3.3 The skip operator and the cototal enumeration degrees

Before we can explore first order definability in the enumeration degrees further, we must accumulate a collection of classes and relations on the enumeration degrees and understand their interactions with classes that we have already explored. Andrews, Ganchev, Kuyper, Lempp, Miller, A. Soskova and M. Soskova [1] initiate the study of a natural operator on the enumeration degrees: *the skip operator*, and the related class of the cototal degrees. Recall that by K_A we denote the uniform join of all set that are enumeration reducible to A.

Definition 3.21. The *skip* of a set A is the set $A^\Diamond = \overline{K}_A = \bigoplus_{e<\omega} \overline{\Gamma_e(A)}$.

It is straightforward to check that $A \leq_e B$ implies $\overline{K}_A \leq_1 \overline{K}_B$, and so the skip operator on sets induces an operator on degrees: $\deg_e(A)^\Diamond = \deg_e(A^\Diamond)$. There are several ways in which it can be argued that the skip operator is the more natural analog of the Turing jump operator for the structure $\mathcal{D}_e$. Andrews, et al. [1] prove that:

1. $A \leq_e B$ if and only if $A^\Diamond \leq_1 B^\Diamond$, but there are sets A and B such that $A' \leq_1 B'$ but $A \not\leq_e B$.

2. If $\mathbf{b} \geq_e \mathbf{0}_e'$ then there is a degree $\mathbf{a}$ such that $\mathbf{a}^\Diamond = \mathbf{b}$. In fact, $\mathbf{a}$ can be chosen to be quasiminimal.

3. On total enumeration degrees the skip and the jump coincide.

On the other hand, there are ways in which the skip behaves differently: even though the skip of a degree is never below that degree it can be, and most often is, to the side of it, so $\mathbf{a}$ and $\mathbf{a}^\Diamond$ are usually incomparable degrees. Andrews et al. [1] push this to the extreme with the following theorem:

Theorem 3.22 (Andrews, et al.)**.** *There are degrees* $\mathbf{a}$ *and* $\mathbf{b}$ *that form a* skip 2-cycle, *i.e.* $\mathbf{a}^\Diamond = \mathbf{b}$ *and* $\mathbf{b}^\Diamond = \mathbf{a}$*. Such degrees* $\mathbf{a}$ *and* $\mathbf{b}$ *must be above every hyperarithmetic degree.*

There is much more to investigate about the skip operator and its structural behavior, in particular, the authors leave open:

Question 3.23. Is the skip operator first order definable in $\mathcal{D}_e$?

The class of degrees on which the skip behaves just like the jump operator is the class of cototal enumeration degrees:

Definition 3.24. A set A is *cototal* if $A \leq_e \overline{A}$. An enumeration degree is *cototal* if and only if it contains a cototal set.

Clearly, every total degree is cototal: $A \oplus \overline{A} \equiv_1 \overline{A \oplus \overline{A}} = \overline{A} \oplus A$. Andrews et al. [1], also show that every Σ^0_2 enumeration degree and every continuous degree is cototal. On the other hand, sufficiently generic degrees are not cototal. Thus we have a proper superclass of the continuous degrees.

Motivation for the study of the cototal enumeration degrees came from symbolic dynamics. Jeandel and his group were studying the spectrum of a *minimal subshift.*

Definition 3.25. A set $S \subseteq 2^\omega$ is called a *subshift* if S is topologically closed and closed under the shift operator that maps $\alpha(0)\alpha(1)\alpha(2)\dots$ to $\alpha(1)\alpha(2)\dots$.

The subshift is *minimal* if it has no proper nonempty subset that is also a subshift.

The spectrum of a subshift S is the set of Turing degrees that compute a member of S.

Jeandel [17] had noticed that a Turing degree computes a member of a given minimal subshift S if and only if it can enumerate the set L_S, the *language of* S, consisting of all finite subwords of elements of S. Thus, the spectrum of a minimal subshift S is exactly characterized by the enumeration degree of L_S. He also noticed that the set L_S can be uniformly enumerated given any enumeration of its complement (the set of forbidden words in S), i.e. L_S is cototal. McCarthy [34] proved that every cototal degree contains the set L_S for some minimal subshift S and so we get a characterization of the cototal degrees.

Theorem 3.26 (Jeandel, McCarthy). *An enumeration degree is cototal if and only if it contains the language of a minimal subshift.*

It turns out that cototal degrees have numerous characterization arising in all kinds of areas of effective mathematics. The cototal degrees are:

1. The degrees on which the skip and the jump operator coincide.
2. The degrees of complements of maximal independent sets in computable graphs on ω.
3. The degrees of complements of maximal antichains in $\omega^{<\omega}$.
4. The enumeration degrees such that the set of Turing degrees above them is the spectrum of a structure and the the upward closure of an F_σ subset of ω^ω.
5. The degrees of sets with good approximations.
6. The degrees of points in computable G_δ topological spaces.

(1) and (2) are proved by Andrews et al. [1]. (3) is proved by McCarthy [34]. (4) was proved by Montalbán [38] and McCarthy [34]. (5) was proved by Miller and Soskova [37]. They used this characterization to prove that the cototal enumeration degrees are dense. (6) is proved by Kihara, Ng, and Pauly [22].

And so we come to a second open question related to definability:

Question 3.27. Are the cototal enumeration degrees first order definable in $\mathcal{D}_e$?

The last characterization of the cototal degrees is part of a more general program, initiated by Kihara and Pauly [23] and extended in Kihara, Ng, and Pauly [22], to transfer topological spaces and topological properties to the enumeration degrees.

Definition 3.28. A *represented space* is a pair of a second countable topological space X and listing of an open base $B^X = \{B_i\}_{i<\omega}$.

A *name* for a point $x \in X$ is an enumeration of the set $N_x = \{i \mid x \in B_i\}$.

For $x \in X$ and $y \in Y$, where X and Y are (possibly different) represented spaces, we say that $x \leq y$ if and only if every name for y uniformly computes a name for x.

Thus a represented space X gives rise to a class of enumeration degrees $\mathcal{D}_X \subseteq \mathcal{D}_e$. For example:

1. $\mathcal{D}_{S^\infty} = \mathcal{D}_e$, where S is the Sierpinski topology $\{\emptyset, \{1\}, \{0,1\}\}$.
2. $\mathcal{D}_{2^\omega} = \mathcal{D}_\mathbb{R}$ is the class of all total enumeration degrees.
3. $\mathcal{D}_{[0,1]^\omega}$ is the class of the continuous degrees.
4. $\mathcal{D}_{\mathbb{R}^<}$, where $\mathbb{R}^<$ denotes the reals equipped with the lower topology which is generated by $\{(q,\infty)\}_{q\in\mathbb{Q}}$, is the class of all semi-computable degrees.

Kihara, Ng, and Pauly [22] further investigate $\mathcal{D}_X$, where X is the ω-power of: the cofinite topology on ω, the telophase space, the double origin space, the quasi-Polish Roy space, the irregular lattice space. Thus we have many more classes whose first order definability in $\mathcal{D}_e$ can be pursued.

4 Automorphisms and automorphism bases

4.1 Global automorphisms

In this section we discuss the implications of definability for the automorphism group of the enumeration degrees.

Definition 4.1. Let $\mathfrak{A}$ be a structure with domain A. A set $B \subseteq A$ is an *automorphism base* for $\mathfrak{A}$ if any two automorphisms that agree on B coincide.

Equivalently, B is a base if the only automorphism that fixes all members of B is the identity.

Let us take a look at some highlights in Slaman and Woodin's [53] automorphism analysis:

Theorem 4.2 (Slaman, Woodin)**.** *The Turing degrees have at most countably many automorphisms.*

There is a single degree $\mathbf{g} \leq \mathbf{0}_T^{(5)}$ *that is an automorphism base for* $\mathcal{D}_T$*.*

Relations on $\mathcal{D}_T$ *induced by definable relations in* Z_2 *are first order definable in* $\mathcal{D}_T$ *with such a parameter* $\mathbf{g}$*.*

Relations on $\mathcal{D}_T$ *induced by definable relations in* Z_2 *that are furthermore invariant under automorphisms are first order definable in* $\mathcal{D}_T$ *(without parameters).*

We will extract from this theorem a lot of information about the automorphisms of the enumeration degrees using the definability of the total degrees and the following old result of Selman [45].

Theorem 4.3 (Selman)**.** $\mathbf{a} \leq \mathbf{b}$ *if and only if every total degree above* $\mathbf{b}$ *is also above* $\mathbf{a}$*.*

Thus, the total enumeration degrees are a *definable automorphism base* for $\mathcal{D}_e$. Definability implies that every automorphism of the enumeration degrees π induces an automorphism of the Turing degrees $\pi^*(\mathbf{a}) = \iota^{-1}(\pi(\iota(\mathbf{a})))$. The fact that the total degrees form an automorphism base tells us that this mapping is injective, and, in particular, a nontrivial automorphism of $\mathcal{D}_e$ gives rise to a nontrivial automorphism of $\mathcal{D}_T$. As promised we get the following:

Corollary 4.4. *$\mathcal{D}_e$ has at most countably many automorphisms. Furthermore, a single total degree below $\mathbf{0}_e^{(5)}$ is an automorphism base for $\mathcal{D}_e$.*

The most pressing open question here is therefore, whether the reverse relationship holds.

Question 4.5. Does every automorphism of $\mathcal{D}_T$ extend to an automorphism of $\mathcal{D}_e$?

A positive answer to the question above would give us that the two automorphism groups are isomorphic. It would also imply that automorphisms of the Turing degrees preserve the relations c.e. in and PA above, as they both have definable images in $\mathcal{D}_e$. By Theorem 4.2 this yields their first order definability in $\mathcal{D}_T$. If on the other hand we can rule out the existence of nontrivial automorphisms of $\mathcal{D}_T$ that preserve these relations then we would get that $\mathcal{D}_e$ is rigid. Our hope is that by proceeding in this fashion and uncovering more definable classes of total enumeration degrees we will put more and more restrictions on the possible extendable nontrivial automorphisms of $\mathcal{D}_T$ to eventually get rigidity.

4.2 Local and global structural interactions

The local structure of the enumeration degrees $\mathcal{D}_e(\leq \mathbf{0}'_e)$ consists of the initial interval bounded by $\mathbf{0}'_e$. Every degree in that interval consists entirely of Σ^0_2 sets. The local structure has been studied extensively as well. Cooper [7] proved that it is a dense structure and, as we mentioned earlier, Bianchini [4] extended this result to prove that every countable partial order can be embedded densely in $\mathcal{D}_e(\leq \mathbf{0}'_e)$. This gives the decidability of the the existential theory of $\mathcal{D}_e(\leq \mathbf{0}'_e)$. The two quantifier theory is much more difficult to analyze. Density prevents us from using the initial segment embedding method that Lerman and Shore [32] use for $\mathcal{D}_T(\leq \mathbf{0}'_T)$. Nevertheless, there are partial results: Lempp and Sorbi [29] show that every finite lattice can be embedded in $\mathcal{D}_e(\leq \mathbf{0}'_e)$ preserving least and greatest element and Lempp, Slaman, and Sorbi [27] prove that the extension of embeddings problem is decidable. Kent [20] showed that the three quantifier theory of $\mathcal{D}_e(\leq \mathbf{0}'_e)$ is not decidable. So the first open problem we have for $\mathcal{D}_e(\leq \mathbf{0}'_e)$ matches the one we have for $\mathcal{D}_e$:

Question 4.6. Is the two quantifier theory of $\mathcal{D}_e(\leq \mathbf{0}'_e)$ decidable?

The full theory was shown to be computably isomorphic to first order arithmetic by Ganchev and Soskova [13]. Their proof relies on a local version of the Coding Theorem that was already established by Slaman and Woodin [52] and the local definability of $\mathcal{K}$-pairs [12], which was not previously known. The local definability of $\mathcal{K}$-pairs unlocked a series of other first order definability results, proved in a series of papers by Ganchev and Soskova [12, 14, 15]:

Theorem 4.7 (Ganchev, Soskova)**.** *The following classes have first order definitions in $\mathcal{D}_e$:*

1. *The downwards properly Σ^0_2 degrees, degrees that bound no nonzero Δ^0_2 degree.*
2. *The upwards properly Σ^0_2 degrees, degrees that are not bounded by any incomplete Δ^0_2 degree.*
3. *The Δ^0_2 total enumeration degrees.*
4. *The low enumeration degrees, degrees with $\mathbf{a}' = \mathbf{0}'_e$.*
5. *All members of the jump hierarchy: the low_n and the high_n degrees for $n \geq 1$.*

The local structure $\mathcal{D}_e(\leq \mathbf{0}'_e)$ relates to first order arithmetic in a similar way as the global structure $\mathcal{D}_e$ relates to second order arithmetic. We can formulate a biinterpretability conjecture for the local structure as well. The Σ^0_2 sets form a countable class that can be naturally indexed. For example we can set $U_e = \Gamma_e(\emptyset')$, where $\{\Gamma_e\}_{e<\omega}$ list all enumeration operators.

Question 4.8. The biinterpretability conjecture for $\mathcal{D}_e(\leq \mathbf{0}'_e)$ is that there is a definable coded model of first order arithmetic $\mathcal{M} = (\mathbb{N}^{\mathcal{M}}, +^{\mathcal{M}}, *^{\mathcal{M}})$ and a definable function $\varphi : \mathbb{N}^{\mathcal{M}} \to \mathcal{D}_e(\leq \mathbf{0}'_e)$ such that $\varphi(e^{\mathcal{M}}) = \deg_e(U_e)$. Is it true?

The function φ above is called *an indexing* of the degrees in $\mathcal{D}_e(\leq \mathbf{0}'_e)$. Slaman and Soskova [50] prove that a similar biinterpretability conjecture for $\mathcal{D}_T(\leq \mathbf{0}'_T)$ is true modulo the use of finitely many parameters. Their approach transferred to the enumeration degrees does not lead to a similar result, so even biinterpretability for $\mathcal{D}_e(\leq \mathbf{0}'_e)$ with parameters remains open. It does, however, allow them to uncover an important relationship between local and global structure. The starting point in both approaches is the following theorem of Slaman and Woodin [51], a consequence of the local coding theorem:

Theorem 4.9 (Slaman, Woodin)**.** *There is an indexing of the c.e. Turing degrees that is definable from Δ^0_2 parameters in the local structure $\mathcal{D}_T(\leq \mathbf{0}'_T)$.*

Slaman and Soskova [49] start with the result above transferred to $\mathcal{D}_e(\leq \mathbf{0}'_e)$ via the standard embedding ι. They use the local definability of the total degrees, the low enumeration degrees, as well as several technical priority constructions, to prove that if a set of finitely many parameters defines an indexing of the image of the c.e. Turing degrees then the same set of parameters defines an indexing of the image of the Δ^0_2 Turing degrees. Next, using the definability of the jump operator and the image of the relation c.e. in, they show that every set of parameters that defines an indexing of the image of the Δ^0_2 Turing degrees also defines an indexing of the image of the degrees that are c.e. in and above some Δ^0_2 Turing degree. Next, using properties of sufficiently generic sets, they show that every set of parameters that defines an indexing as above, also defines an indexing of the image of all Turing degrees bounded by $\mathbf{0}''_T$. The last two steps can now be iterated any finite number of times to show that:

Theorem 4.10 (Slaman, Soskova)**.** *Any set of parameters that defines an indexing of the image of the c.e. Turing degrees also defines an indexing of the image of the Turing degrees bounded by $\mathbf{0}^{(n)}_T$ for every natural number n.*

This theorem combines well with what we know about the automorphism group of $\mathcal{D}_e$, in particular, the fact that there is a single total degree below $\mathbf{0}^{(5)}_e$ that forms an automorphism base for $\mathcal{D}_e$. If a set of parameters defines an indexing of the image of the Turing degrees below $\mathbf{0}^{(5)}_T$ and an automorphism fixes these parameters then it is not difficult to see that the automorphism must fix all elements in the range of the definable indexing. As this includes the degree that by itself is an automorphism base for all of $\mathcal{D}_e$, the automorphism must be the identity.

Theorem 4.11 (Slaman, Soskova)**.** *There is a finite set of Δ^0_2 total degrees that forms an automorphism base for the global structure $\mathcal{D}_e$.*

Similar arguments lead Slaman and Soskova [49] to the following consequence of their theorem:

Corollary 4.12 (Slaman, Soskova)**.** *If $\mathcal{D}_e$ has a nontrivial automorphism then so does:*

1. *The local structure $\mathcal{D}_e(\leq \mathbf{0}'_e)$.*
2. *The structure of the Δ^0_2 Turing degrees $\mathcal{D}_T(\leq \mathbf{0}'_T)$.*
3. *The structure of the c.e. Turing degrees.*

Naturally, we wonder:

Question 4.13. Do the automorphisms of any of these structures extend to automorphisms of $\mathcal{D}_e$?

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