Minimal covers in the Weihrauch degrees

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Abstract

In this paper, we study the existence of minimal covers and strong minimal covers in the Weihrauch degrees. We characterize when a problem f is a minimal cover or strong minimal cover of a problem h. We show that strong minimal covers only exist in the cone below id and that the Weihrauch lattice above id is dense. From this, we conclude that the degree of id is first-order definable in the Weihrauch degrees and that the first-order theory of the Weihrauch degrees is computably isomorphic to third-order arithmetic.

1 Introduction

In a partial order $(\mathcal{D}, \leq)$, an element $\mathbf{a}$ is a *minimal cover* of $\mathbf{b}$ if $\mathbf{b} < \mathbf{a}$ and there is no $\mathbf{c}$ such that $\mathbf{b} < \mathbf{c} < \mathbf{a}$. In other words, the interval between $\mathbf{a}$ and $\mathbf{b}$ is empty. We say that $\mathbf{a}$ is a *strong minimal cover* of $\mathbf{b}$ if $\mathbf{b} < \mathbf{a}$ and for all $\mathbf{c}$ if $\mathbf{c} < \mathbf{a}$ then $\mathbf{c} \leq \mathbf{b}$.

Understanding the properties and the distribution of minimal covers and strong minimal covers can provide deep insights into the structure of a partial order, and indeed there is an extensive literature on the construction of minimal covers and strong minimal covers in the Turing degrees. In contrast, despite its growing popularity, the structure of Weihrauch degrees is vastly unexplored, as most of the efforts up to this date have concentrated on the classification of the Weihrauch degrees of specific problems. In particular, very little is known on the existence of (strong) minimal covers in the Weihrauch degrees. In this paper, we fill this gap by providing complete characterizations of minimal covers and strong minimal covers in the Weihrauch degrees. We will see that this analysis is then able to answer a number of other questions.

1.1 Background

Weihrauch reducibility [13, 6, 1] classifies partial multi-valued functions according to their uniform computational strength, and it is often used to characterize the computability-theoretical complexity of $\forall\exists$-statements. We briefly recall the main notions we need in this paper, and we refer the reader to [1] for a more thorough presentation on Weihrauch reducibility.

If f and g are partial multi-valued functions on Baire space (denoted by $f, g :\subseteq \mathbb{N}^\mathbb{N} \rightrightarrows \mathbb{N}^\mathbb{N}$), we say that f is *Weihrauch reducible* to g, and write $f \leq_\mathrm{W} g$, if there are two computable functionals $\Phi :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ and $\Psi :\subseteq \mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ such that

$$(\forall p \in \mathrm{dom}(f))[\ \Phi(p) \in \mathrm{dom}(g) \land (\forall q \in g(\Phi(p)))\ \Psi(p, q) \in f(p)\ ].$$

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We use $(\mathcal{W}, \le_{\mathrm{W}})$ to denote the structure of degrees induced by Weihrauch reducibility. In the computable analysis literature, Weihrauch reducibility is often defined in a more general context, where f and g are partial multi-valued functions between represented spaces. However, it is well-known that every Weihrauch degree contains a representative with domain and codomain $\mathbb{N}^\mathbb{N}$ (see, e.g., [1, Lemma 11.3.8]). In other words, in order to study the structure of the Weihrauch degrees, there is no loss of generality in restricting our attention to computational problems on Baire space. In what follows, with a small abuse of notation we will identify a natural number n with the infinite string constantly equal to n.

The Weihrauch degrees are known to form a distributive lattice where join and meet are induced, respectively, by the following operators:

- $f \sqcup g$ is the problem with domain $\{0\} \times \mathrm{dom}(f) \cup \{1\} \times \mathrm{dom}(g)$ defined as
$$(f \sqcup g)(i, x) := \begin{cases} f(x) & \text{if } i = 0, \\ g(x) & \text{if } i = 1. \end{cases}$$
- $f \sqcap g$ is the problem with domain $\mathrm{dom}(f) \times \mathrm{dom}(g)$ defined as
$$(f \sqcap g)(x, z) := \{0\} \times f(x) \cup \{1\} \times g(z).$$

The degree of the empty function is a natural bottom element in the Weihrauch degrees. The existence of a top element is equivalent to the failure of a (relatively weak) form of choice. In particular, under ZFC, there is no top element in $(\mathcal{W}, \le_{\mathrm{W}})$.

The statements and proofs of our main theorems exploit the interplay between Weihrauch and Medvedev reducibility. For the sake of completeness, we recall some basic facts on Medvedev reducibility and refer the reader to [8, 12] for more details. Given $A, B \subseteq \mathbb{N}^\mathbb{N}$, we say that A is *Medvedev reducible* to B, and write $A \le_{\mathrm{M}} B$, if there is a computable functional $\Phi :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ such that $\Phi(B) \subseteq A$. We write $(\mathcal{M}, \le_{\mathrm{M}})$ for the degree structure induced by Medvedev reducibility. It is well-known that the Medvedev degrees form a distributive lattice with a top element (the degree of $\emptyset$) and a bottom element (the degree of $\mathbb{N}^\mathbb{N}$).

There is a close connection between Weihrauch and Medvedev reducibility. Indeed, we can rephrase the definition of Weihrauch reducibility as follows: $f \le_{\mathrm{W}} g$ iff there are two computable functionals $\Phi :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ and $\Psi :\subseteq \mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ such that Φ witnesses $\mathrm{dom}(g) \le_{\mathrm{M}} \mathrm{dom}(f)$ and, for every $p \in \mathrm{dom}(f)$, $\Psi(p, \cdot)$ witnesses $f(p) \le_{\mathrm{M}} g(\Phi(p))$. This suggests two possible embeddings of the Medvedev degrees in the Weihrauch degrees [7]. For our purposes, we explicitly mention the following one (see [3, Section 5] for a discussion of the other): For every $A \subseteq \mathbb{N}^\mathbb{N}$, let $d_A \colon A \to \{0^\mathbb{N}\}$ be the constant function that maps every element of A to the constantly 0 string. The map $d := A \mapsto d_A$ induces a lattice embedding of $\mathcal{M}^{\mathrm{op}} = (\mathcal{M}, \ge_{\mathrm{M}})$ in $(\mathcal{W}, \le_{\mathrm{W}})$ [7, Lemma 5.6].

A simple inspection reveals that the range of the embedding d is exactly the set of uniformly computable degrees. To see this, let $\mathrm{id} \colon \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ be the identity function on Baire space. It is immediate that a multi-valued function f is uniformly computable iff $f \le_{\mathrm{W}} \mathrm{id}$. In fact, writing id_X for the restriction of id to X, every problem $f \le_{\mathrm{W}} \mathrm{id}$ is Weihrauch-equivalent to $\mathrm{id}_{\mathrm{dom}(f)}$. In other words, the uniformly computable problems are precisely those equivalent to one of the form id_X for some $X \subseteq \mathbb{N}^\mathbb{N}$, which is, in turn, equivalent to d_X. So the lower cone of id is isomorphic to $\mathcal{M}^{\mathrm{op}}$, hence far from trivial. In particular, this implies that the Medvedev degrees are first-order definable in $(\mathcal{W}, \le_{\mathrm{W}}, \mathbf{1})$, where $\mathbf{1}$ is the Weihrauch degree of id. The question of whether $\mathbf{1}$ is first-order definable in $(\mathcal{W}, \le_{\mathrm{W}})$ was raised by Pauly during the conference "Computability and Complexity in Analysis 2020" and the Oberwolfach meeting 2117 [2], see also [10]. Our results answer this question affirmatively.

The empty intervals in the Medvedev degrees have been fully characterized in the literature. For every $p \in \mathbb{N}^\mathbb{N}$, let $\{p\}^+ := \{(e)^\frown q \,:\, \Phi_e(q) = p \text{ and } q \not\le_{\mathrm{T}} p\}$.

Theorem 1.1 (Dyment [4, Cor. 2.5]). *For every $A <_{\mathrm{M}} B$, B is a minimal cover of A iff*

$$(\exists p \in A)[\ A \equiv_{\mathrm{M}} B \wedge \{p\} \text{ and } B \wedge \{p\}^{+} \equiv_{\mathrm{M}} B\ ],$$

where $P \wedge Q := (0)^\frown P \cup (1)^\frown Q$ is the meet in the Medvedev degrees.

The set $\{p\}^+$ is the immediate successor of $\{p\}$ in the Medvedev degrees. In fact, the strong minimal covers in $\mathcal{M}^{\mathrm{op}}$ are precisely those of the form $(\{p\}^+, \{p\})$. This implies that the property of being a degree of solvability (i.e., being Medvedev equivalent to a singleton) is first-order definable in $(\mathcal{M}, \leq_{\mathrm{M}})$ (Dyment [4, Cor. 2.1]).

In particular, the fact that the lower cone of id (in the Weihrauch degrees) is isomorphic to $\mathcal{M}^{\mathrm{op}}$ immediately yields:

Corollary 1.2. *For every $p \in \mathbb{N}^{\mathbb{N}}$, $\mathrm{id}_{\{p\}}$ is a strong minimal cover of $\mathrm{id}_{\{p\}^+}$.*

Since $\mathrm{NR} := \{q : q \not\leq_{\mathrm{T}} 0\} \equiv_{\mathrm{M}} \{0^{\mathbb{N}}\}^+$, we also obtain:

Corollary 1.3. id *is a strong minimal cover of* $\mathrm{id}_{\mathrm{NR}}$.

1.2 Our main theorems

Unlike the Medvedev degrees, there are no results describing the structure of minimal covers and strong minimal covers in the Weihrauch degrees. Recently, Dzhafarov, Lerman, Patey, and Solomon [5] showed that no Weihrauch degree can be minimal. This result can be obtained as a corollary of our first main theorem:

Theorem 1.4. *Let f and h be partial multi-valued functions on Baire space. The following are equivalent:*

(1) *f is a minimal cover of h in the Weihrauch degrees.*

(2) *$f \equiv_{\mathrm{W}} h \sqcup \mathrm{id}_{\{p\}}$ for some p with $\mathrm{dom}(h) \not\leq_{\mathrm{M}} \{p\}$ and $\mathrm{dom}(h) \leq_{\mathrm{M}} \{p\}^+$.*

The second main theorem provides a similar characterization for strong minimal covers:

Theorem 1.5. *Let f and h be partial multi-valued functions on Baire space. The following are equivalent:*

(1) *f is a strong minimal cover of h in the Weihrauch degrees.*

(2) *There is $p \in \mathbb{N}^{\mathbb{N}}$ such that $f \equiv_{\mathrm{W}} \mathrm{id}_{\{p\}}$ and $h \equiv_{\mathrm{W}} \mathrm{id}_{\{p\}^+}$.*

A multi-valued function is called *pointed* if it has a computable point in its domain. A computational problem f is pointed iff $\mathrm{id} \leq_{\mathrm{W}} f$. In particular, the cone above id is exactly the cone of pointed degrees. Using Theorem 1.4, we can further characterize the non-pointed degrees.

Corollary 1.6. *Let g be a multi-valued function. The following are equivalent:*

(1) $\mathrm{id} \not\leq_{\mathrm{W}} g$.

(2) *There are f, h such that $g \leq_{\mathrm{W}} h <_{\mathrm{W}} f$ and f is a minimal cover of h.*

Proof. The direction (2) $\Rightarrow$ (1) is straightforward as Theorem 1.4 implies that the bottom of a minimal cover cannot have a computable point in its domain. To show that (1) $\Rightarrow$ (2), observe that if $\mathrm{id} \not\leq_{\mathrm{W}} g$, then $g \leq_{\mathrm{W}} g \sqcup \mathrm{id}_{\mathrm{NR}} <_{\mathrm{W}} g \sqcup \mathrm{id}$. In particular, since $g \sqcup \mathrm{id}_{\mathrm{NR}} \sqcup \mathrm{id}_{\{0^{\mathbb{N}}\}} \equiv_{\mathrm{W}} g \sqcup \mathrm{id}$, by Theorem 1.4, the interval $g \sqcup \mathrm{id}_{\mathrm{NR}} <_{\mathrm{W}} g \sqcup \mathrm{id}$ is empty. □

Corollary 1.7. *The pointed Weihrauch degrees are dense.*

Our results provide two different first-order definitions of the degree of id, thus the property of being uniformly computable is lattice-theoretic, answering the above-mentioned question by Pauly.

Theorem 1.8. *The Weihrauch degree of* id *is first-order definable in* $(\mathcal{W}, \leq_W)$. *In particular, it is both:*

(1) *the greatest degree that is a strong minimal cover, and*

(2) *the least degree such that the cone of Weihrauch degrees above it is dense.*

Proof. The first definition follows from Theorem 1.5 and the fact that id is a strong minimal cover (Corollary 1.3). The second is immediate from Corollary 1.6. □

Finally, the definability of id implies that the first-order theory of the Weihrauch degrees is computably isomorphic to the third-order theory of arithmetic, and therefore it is "as complicated as possible".

Theorem 1.9. *The first-order theory of the Weihrauch degrees, the first-order theory of the Weihrauch degrees below* id*, and the third-order theory of true arithmetic are pairwise recursively isomorphic.*

Proof. It is routine to check that Weihrauch reducibility between two multi-valued functions f and g can be defined using a Π^1_2 formula (with free third-order variables f and g). This immediately implies that $\mathrm{Th}(\mathcal{W}(\leq \mathrm{id})) \leq_1 \mathrm{Th}_3(\mathbb{N})$ and $\mathrm{Th}(\mathcal{W}) \leq_1 \mathrm{Th}_3(\mathbb{N})$. The fact that the degree of id is first-order definable in $(\mathcal{W}, \leq_W)$ (Theorem 1.8) yields $\mathrm{Th}(\mathcal{W}(\leq \mathrm{id})) \leq_1 \mathrm{Th}(\mathcal{W})$. Since, as mentioned, the lower cone of id is isomorphic to $\mathcal{M}^{\mathrm{op}}$, the statement follows from the fact that $\mathrm{Th}(\mathcal{M}) \equiv_1 \mathrm{Th}_3(\mathbb{N})$ ([11, Thm. 3.13] and independently [9, Thm. 2]). □

2 Proof of the main theorems

Before proving the main theorems, we need some preliminary results. The following lemma is a step towards proving the first main theorem. It implicitly shows that if f is not a minimal cover of h, then there is a uniform way to construct a problem g such that $h <_W g <_W f$.

Lemma 2.1. *Let* $h \leq_W f$. *If* f *is a minimal cover of* h *then there is* g *with* $|\mathrm{dom}(g)| = 1$ *such that* $f \equiv_W h \sqcup g$.

Proof. We first outline the proof strategy: we construct a partial single-valued function $\xi :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}$ in stages, so that ξ is defined on at most s points by stage s. Let F_ξ be the problem defined as $F_\xi(p, \xi(p)) := f(p)$. The function ξ "scrambles" the domain of f: it is immediate that, for every choice of the function ξ, $F_\xi \leq_W f$. The converse reduction trivially holds when ξ is computable, but it does not hold in general. The construction attempts to build a function ξ so that

- $F_\xi \not\leq_W h$
- $h \sqcup F_\xi <_W f$.

Since this would contradict our assumptions, we argue that the construction must fail. The failure of our construction will result in the desired function g.

The construction of ξ proceeds as follows: for every stage s, we define a partial function ξ_s. We start the construction by letting $\xi_0 := \emptyset$. For the sake of readability, let us write $F_s := F_{\xi_s}$.

At stage $s + 1 = 2\langle e, i\rangle$, we extend ξ_s so that $F_\xi \not\leq_W h$ via Φ_e, Φ_i. Since ξ_s has finite domain and codomain $\mathbb{N}$, it has a total computable extension $\hat{\xi} :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}$. As observed, $F_{\hat{\xi}} \equiv_W f$, hence in particular $F_{\hat{\xi}} \not\leq_W h$ via Φ_e, Φ_i. By the definition of Weihrauch reducibility, there is some $p_0 \in \mathrm{dom}(f)$ such that either $\Phi_e(p_0, \hat{\xi}(p_0)) \notin \mathrm{dom}(h)$ or, for some $q \in h(\Phi_e(p_0, \hat{\xi}(p_0)))$, $\Phi_i((p_0, \hat{\xi}(p_0)), q) \notin F_{\hat{\xi}}(p_0, \hat{\xi}(p_0)) = f(p_0)$. Fix some p_0 as above. Defining $\xi_{s+1} := \xi_s \cup \{(p_0, \hat{\xi}(p_0))\}$ ensures that there is no extension ξ' of ξ_{s+1} such that $F_{\xi'} \le_{\mathrm{W}} h$ via the functionals Φ_e, Φ_i.

At stage $s+1 = 2\langle e, i\rangle + 1$, we try to extend ξ_s so that $f \not\le_{\mathrm{W}} h \sqcup F_\xi$ via Φ_e, Φ_i. The construction stops if Φ_e, Φ_i witness $f \le_{\mathrm{W}} h \sqcup F_s$, as the same pair of functionals would witness the reduction $f \le_{\mathrm{W}} h \sqcup F_{\xi'}$ for any extension ξ' of ξ_s. In this case, we simply define $\xi := \xi_s$. Assume therefore that Φ_e, Φ_i do not witness $f \le_{\mathrm{W}} h \sqcup F_s$, and let p_1 be such that either $\Phi_e(p_1) \notin \mathrm{dom}(h \sqcup F_s)$ or, for some $q \in (h \sqcup F_s)(\Phi_e(p_1))$, $\Phi_i(p_1, q) \notin f(p_1)$. If

- $\Phi_e(p_1)\uparrow$, or
- $\Phi_e(p_1) = (0, r)$ for some $r \notin \mathrm{dom}(h)$, or
- $\Phi_e(p_1) = (1, (p, k))$ for some p such that $p \notin \mathrm{dom}(f)$ or $p \in \mathrm{dom}(\xi_s)$ with $\xi_s(p) \neq k$, or
- $\Phi_e(p_1) \in \mathrm{dom}(h \sqcup F_s)$ and for some $q \in (h \sqcup F_s)(\Phi_e(p_1))$, $\Phi_i(p_1, q) \notin f(p_1)$,

then there is no extension ξ' of ξ_s such that $f \le_{\mathrm{W}} h \sqcup F_{\xi'}$ via the functionals Φ_e, Φ_i, hence we can just define $\xi_{s+1} := \xi_s$. The remaining case is that $\Phi_e(p_1) = (1, (p, k))$ for some $p \in \mathrm{dom}(f) \setminus \mathrm{dom}(\xi_s)$. We define $\xi_{s+1} := \xi \cup \{(p_1, k+1)\}$. Again, this ensures that there is no extension ξ' of ξ_{s+1} such that Φ_e, Φ_i witness that $f \le_{\mathrm{W}} h \sqcup F_{\xi'}$.

Observe that, if for every $s+1 = 2\langle e, i\rangle + 1$, $f \not\le_{\mathrm{W}} h \sqcup F_s$, then, in the limit, we obtain a function ξ such that $h <_{\mathrm{W}} h \sqcup F_\xi <_{\mathrm{W}} f$, against the assumption that f is a minimal cover of h. This implies that, for some s as above, $f \le_{\mathrm{W}} h \sqcup F_s = h \sqcup F_\xi$. Moreover, ξ, and hence F_ξ, has finite domain, which in turn implies that $f \equiv_{\mathrm{W}} h \sqcup F_\xi$.

Let $(p_i)_{i<n}$ be an enumeration of $\mathrm{dom}(\xi)$, and let c_i be the function with domain $\{p_i\}$ defined as $c_i(p_i) := f(p_i)$. Since $\mathrm{dom}(\xi)$ is finite, we immediately have $F_\xi \le_{\mathrm{W}} \bigsqcup_{i<n} c_i$. If, for all $i < n$, $c_i \le_{\mathrm{W}} h$, we would have $f \le_{\mathrm{W}} h \sqcup F_\xi \le_{\mathrm{W}} h$, which is a contradiction. Hence, for some $i < n$, $h <_{\mathrm{W}} h \sqcup c_i \le_{\mathrm{W}} h \sqcup F_\xi \equiv_{\mathrm{W}} f$. The fact that f is a minimal cover of h implies that $f \equiv_{\mathrm{W}} h \sqcup c_i$. Letting $g := c_i$ concludes the proof. □

Corollary 2.2. *If f is a strong minimal cover of h, then there is $g \equiv_{\mathrm{W}} f$ such that $|\mathrm{dom}(g)| = 1$.*

Proof. By Lemma 2.1, there is g with $|\mathrm{dom}(g)| = 1$ such that $f \equiv_{\mathrm{W}} h \sqcup g$. Since $g \le_{\mathrm{W}} f$, the fact that f is a strong minimal cover of h, implies that $g \equiv_{\mathrm{W}} f$ or $g \le_{\mathrm{W}} h$. The latter is readily seen to yield a contradiction, as $g \le_{\mathrm{W}} h \Rightarrow h <_{\mathrm{W}} f \equiv_{\mathrm{W}} h \sqcup g \equiv_{\mathrm{W}} h$, hence $g \equiv_{\mathrm{W}} f$. □

For every set A, let χ_A denote the characteristic function of A.

Lemma 2.3. *If $f \not\le_{\mathrm{W}} h$, then there are at most countably many $D \subseteq \mathbb{N}$ such that $f \sqcap \chi_D \le_{\mathrm{W}} h$.*

Proof. We argue that each pair of potential reduction witnesses Φ, Ψ for $f \sqcap \chi_D \le_{\mathrm{W}} h$ can work for at most one choice of D. To see this, assume that $f \sqcap \chi_D \le_{\mathrm{W}} h$ via Φ, Ψ. Notice that if there is $n_0 \in \mathbb{N}$ such that

$$(\forall x \in \mathrm{dom}(f))(\forall y \in h(\Phi(x, n_0)))(\exists z \in f(x))\ \Psi((x, n_0), y) = (0, z),$$

then we would have $f \le_{\mathrm{W}} h$, against the hypothesis.

This implies that, for every $D \subseteq \mathbb{N}$, if Φ, Ψ witness the reduction $f \sqcap \chi_D \le_{\mathrm{W}} h$ then for every $n \in \mathbb{N}$ there exists some $x \in \mathrm{dom}(f)$ and some $y \in h(\Phi(x, n))$ such that $\Psi((x, n), y) = (1, \chi_D(n))$. In other words, membership of n in D is determined by the pair Φ, Ψ, and hence the same pair cannot witness the reduction $f \sqcap \chi_E \le_{\mathrm{W}} h$ for any set $E \neq D$. □

In fact, if $f \not\le_{\mathrm{W}} h$ then there are exactly countably many $D \subseteq \mathbb{N}$ such that $f \sqcap \chi_D \le_{\mathrm{W}} h$ if and only if $\mathrm{dom}(h) \le_{\mathrm{M}} \mathrm{dom}(f)$. (Otherwise, it is never the case that $f \sqcap \chi_D \le_{\mathrm{W}} h$.)

Lemma 2.4. *If $f \not\le_{\mathrm{W}} \mathrm{id}$ and f has singleton domain, then for all h such that $f \not\le_{\mathrm{W}} h$ there is $g <_{\mathrm{W}} f$ such that $g \not\le_{\mathrm{W}} h$. It follows that $h <_{\mathrm{W}} h \sqcup g <_{\mathrm{W}} h \sqcup f$.*

Proof. Fix f as above with $\mathrm{dom}(f) = \{x\}$ and let h be such that $f \not\le_{\mathrm{W}} h$. By Lemma 2.3, there is $D \subseteq \mathbb{N}$ such that $f \sqcap \chi_D \not\le_{\mathrm{W}} h$. Let $g := f \sqcap \chi_D$. Note that $g <_{\mathrm{W}} f$ because every instance of χ_D (and hence of g) has computable solutions, while our assumptions that $f \not\le_{\mathrm{W}} \mathrm{id}$ and $|\mathrm{dom}(f)| = 1$ ensure that f does not have computable solutions.

To prove the last part of the statement, observe that the reductions $h \le_{\mathrm{W}} h \sqcup g \le_{\mathrm{W}} h \sqcup f$ are immediate as $\sqcup$ is the join in the Weihrauch lattice, and $h \sqcup g \not\le_{\mathrm{W}} h$ follows immediately from $g \not\le_{\mathrm{W}} h$. A reduction $h \sqcup f \le_{\mathrm{W}} h \sqcup g$ would, in particular, yield $f \le_{\mathrm{W}} h \sqcup g$. This is a contradiction as f is join-irreducible (since $|\mathrm{dom}(f)| = 1$) and $f \not\le_{\mathrm{W}} h$ and $f \not\le_{\mathrm{W}} g$. This concludes the proof. □

We are now able to prove the first main theorem, which we state again for the sake of readability.

Theorem 1.4. Let f, h be partial multi-valued functions on Baire space. The following are equivalent:

(1) f is a minimal cover of h in the Weihrauch degrees,

(2) $f \equiv_{\mathrm{W}} h \sqcup \mathrm{id}_{\{p\}}$ for some p with $\mathrm{dom}(h) \not\le_{\mathrm{M}} \{p\}$ and $\mathrm{dom}(h) \le_{\mathrm{M}} \{p\}^+$.

Proof. (1) ⇒ (2): By Lemma 2.1, there is g with $\mathrm{dom}(g) = \{p\}$ such that $f \equiv_{\mathrm{W}} h \sqcup g$. In particular $g \not\le_{\mathrm{W}} h$. If $g \not\le_{\mathrm{W}} \mathrm{id}_{\{p\}}$ then g satisfies the hypotheses of Lemma 2.4, hence there is G such that $h <_{\mathrm{W}} h \sqcup G <_{\mathrm{W}} h \sqcup g \equiv_{\mathrm{W}} f$, contradicting the fact that f is a minimal cover of h. Therefore $g \equiv_{\mathrm{W}} \mathrm{id}_{\{p\}}$. The fact that $\mathrm{dom}(h) \not\le_{\mathrm{M}} \{p\}$ is immediate as

$$\mathrm{dom}(h) \le_{\mathrm{M}} \{p\} \Rightarrow \mathrm{id}_{\{p\}} \le_{\mathrm{W}} h \Rightarrow h \equiv_{\mathrm{W}} h \sqcup \mathrm{id}_{\{p\}} \equiv_{\mathrm{W}} f,$$

contradicting $h <_{\mathrm{W}} f$. Observe that $\mathrm{dom}(h) \not\le_{\mathrm{M}} \{p\}^+$ would lead to a contradiction with the fact that f is a minimal cover of h. Indeed, we would obtain

$$h <_{\mathrm{W}} h \sqcup \mathrm{id}_{\{p\}^+} <_{\mathrm{W}} h \sqcup \mathrm{id}_{\{p\}} \equiv_{\mathrm{W}} f,$$

where the first two reductions are strict, respectively, since $\mathrm{dom}(h) \not\le_{\mathrm{M}} \{p\}^+$ and $\mathrm{dom}(h) \cup \{p\}^+ \not\le_{\mathrm{M}} \{p\}$ (as $\{p\}^+ \not\le_{\mathrm{M}} \{p\}$).

(2) ⇒ (1): Assume towards a contradiction that there is g such that $h <_{\mathrm{W}} g <_{\mathrm{W}} h \sqcup \mathrm{id}_{\{p\}}$. The forward functional Φ of the reduction $g \le_{\mathrm{W}} h \sqcup \mathrm{id}_{\{p\}}$ lets us define two restrictions g_0, g_1 of g by letting $\mathrm{dom}(g_i) := \{p \in \mathrm{dom}(g) : \Phi(g)(0) = i\}$. In particular, we obtain $g \equiv_{\mathrm{W}} g_0 \sqcup g_1$, $g_0 \le_{\mathrm{W}} h$ (and hence $\mathrm{dom}(h) \le_{\mathrm{M}} \mathrm{dom}(g_0)$), and $g_1 \le_{\mathrm{W}} \mathrm{id}_{\{p\}}$. The latter reduction implies $g_1 \equiv_{\mathrm{W}} \mathrm{id}_A$ for some $A \ge_{\mathrm{M}} \{p\}$, hence we obtain $h <_{\mathrm{W}} g_0 \sqcup \mathrm{id}_A <_{\mathrm{W}} h \sqcup \mathrm{id}_{\{p\}}$.

Note that $h <_{\mathrm{W}} g_0 \sqcup \mathrm{id}_A$ implies that $\mathrm{id}_A \not\le_{\mathrm{W}} h$, i.e., $\mathrm{dom}(h) \not\le_{\mathrm{M}} A$. This, in turn, implies that $A \equiv_{\mathrm{M}} \{p\}$, as $\{p\} <_{\mathrm{M}} A$ would mean that $A \ge_{\mathrm{M}} \{p\}^+ \ge_{\mathrm{M}} \mathrm{dom}(h)$, contradicting $\mathrm{dom}(h) \not\le_{\mathrm{M}} A$. In other words, we obtain $h <_{\mathrm{W}} g_0 \sqcup \mathrm{id}_{\{p\}} <_{\mathrm{W}} h \sqcup \mathrm{id}_{\{p\}}$. This is a contradiction, as $h \le_{\mathrm{W}} g_0 \sqcup \mathrm{id}_{\{p\}}$ implies that $h \sqcup \mathrm{id}_{\{p\}} \le_{\mathrm{W}} g_0 \sqcup \mathrm{id}_{\{p\}}$. □

Observe that, since $\emptyset$ is the top of the Medvedev lattice and the bottom of the Weihrauch lattice, the following is immediate.

Corollary 2.5 ([5])**.** *There are no minimal degrees in $\mathcal{W}$.*

Theorem 1.4 allows us to show also that there is a close connection between antichains in the Turing degrees and minimal covers in the Weihrauch degrees: for every family of pairwise Turing incomparable sets $\{p_\alpha\}_{\alpha<\kappa}$ with $\kappa < 2^{\aleph_0}$, there is a multi-valued function h whose minimal covers are exactly those of the form $h \sqcup \mathrm{id}_{\{p_\alpha\}}$.

Corollary 2.6. *For every cardinal $\kappa \le 2^{\aleph_0}$, there is a problem h with exactly κ minimal covers.*

Proof. The case for $\kappa = 0$ follows from the fact that there are no minimal degrees in the Weihrauch lattice. It can also be proved using the fact that the cone above id is dense (Corollary 1.7).

Let $\{p_\alpha \in \mathbb{N}^\mathbb{N} : \alpha < \kappa\}$ be pairwise Turing incomparable and let h be any problem with

$$\operatorname{dom}(h) := \bigcup_{\alpha<\kappa} \{q \in \mathbb{N}^\mathbb{N} : p_\alpha <_\mathrm{T} q\}.$$

By Theorem 1.4, for every $\alpha < \kappa$, $h \sqcup \operatorname{id}_{\{p_\alpha\}}$ is a minimal cover of h. Observe also that $h \sqcup \operatorname{id}_{\{p_\alpha\}} \le_\mathrm{W} h \sqcup \operatorname{id}_{\{p_\beta\}}$ implies $p_\beta \le_\mathrm{T} p_\alpha$ (as $\operatorname{dom}(h) \not\le_\mathrm{M} \{p_\alpha\}$). This shows that h has at least κ minimal covers.

Theorem 1.4 implies that every h has at most $2^{\aleph_0}$ minimal covers. To conclude the proof, assume that $k < 2^{\aleph_0}$ and fix $p \in \mathbb{N}^\mathbb{N}$ such that, for every $\alpha < \kappa$, $p \not\equiv_\mathrm{T} p_\alpha$. We show that $h \sqcup \operatorname{id}_{\{p\}}$ is not a minimal cover of h. If there is $\alpha < \kappa$ such that $p_\alpha \le_\mathrm{T} p$ then $\operatorname{dom}(h) \le_\mathrm{M} \{p\}$ and hence $h \sqcup \operatorname{id}_{\{p\}} \le_\mathrm{W} h$. If for every $\alpha < \kappa$, $p_\alpha \not\le_\mathrm{T} p$, then there is some $q >_\mathrm{T} p$ such that, for every α, $p_\alpha \not\le_\mathrm{T} q$. This follows from the fact that in the Turing degrees p has $2^{\aleph_0}$ minimal covers and for every $\alpha < \kappa$ there is at most one m that is a minimal cover of p such that $p_\alpha \le_\mathrm{T} m$. Indeed, if $p_\alpha \le_\mathrm{T} m$ and m is a minimal cover of p then $p <_\mathrm{T} p \oplus p_\alpha \le_\mathrm{T} m$ implies $m \equiv_\mathrm{T} p \oplus p_\alpha$. So $\kappa < 2^{\aleph_0}$ implies the existence of the desired q as some minimal cover of p. It follows that $\operatorname{dom}(h) \not\le_\mathrm{M} \{p\}^+$ and so by Theorem 1.4 we have that $h \sqcup \operatorname{id}_{\{p\}}$ is not a minimal cover of h. □

With a similar argument, we can also show the following:

Corollary 2.7. *For every h, if there is $p \in \mathbb{N}^\mathbb{N}$ such that $\operatorname{dom}(h) \equiv_\mathrm{M} \{p\}^+$, then h has a unique minimal cover.*

We finish with our characterization of strong minimal covers.

Theorem 1.5. Let f, h be partial multi-valued functions on Baire space. The following are equivalent:

(1) f is a strong minimal cover of h in the Weihrauch degrees,

(2) There is $p \in \mathbb{N}^\mathbb{N}$ such that $f \equiv_\mathrm{W} \operatorname{id}_{\{p\}}$ and $h \equiv_\mathrm{W} \operatorname{id}_{\{p\}^+}$.

Proof. To see that (2) implies (1), we just recall that $\{p\}$ is a strong minimal cover of $\{p\}^+$ in $\mathcal{M}^{\mathrm{op}}$, which is isomorphic to the lower cone of id.

We proceed to argue that (1) implies (2). As a strong minimal cover requires an empty interval, by Theorem 1.4, we can restrict ourselves to the case $f \equiv_\mathrm{W} h \sqcup \operatorname{id}_{\{p\}}$ where $\operatorname{dom}(h) \not\le_\mathrm{M} \{p\}$ and $\operatorname{dom}(h) \le_\mathrm{M} \{p\}^+$. As the top element of a strong minimal cover has to be join-irreducible, we find that $f \le_\mathrm{W} \operatorname{id}_{\{p\}}$ (as $f \not\le_\mathrm{W} h$), and therefore $f \equiv_\mathrm{W} \operatorname{id}_{\{p\}}$. This, in turn, implies that $h \equiv_\mathrm{W} \operatorname{id}_A$ for some $A \subseteq \mathbb{N}^\mathbb{N}$ such that $A \not\le_\mathrm{M} \{p\}$ and $A \le_\mathrm{M} \{p\}^+$. Since $\{p\} \le_\mathrm{M} A$ (as $h \le_\mathrm{W} f$), this yields $A \equiv_\mathrm{M} \{p\}^+$ as claimed. □

In particular, this shows that every Weihrauch degree has a most one strong minimal cover.

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