Genericity and Nonbounding in the Enumeration degrees

Mariya Ivanova Soskova *

University of Leeds

Abstract. The structure of the semi lattice of enumeration degrees has been investigated from many aspects. One aspect is the bounding and nonbounding properties of generic degrees. Copestake proved that every 2-generic enumeration degree bounds a minimal pair and conjectured that there exists a 1-generic degree that does not bound a minimal pair. In this paper we verify this longstanding conjecture by constructing such a degree using an infinite injury priority argument.
Key words: Enumeration reducibility, Nonbounding degrees, Generic degrees, Minimal pairs

1 Introduction

In contrast to the Turing case where every 1-generic degree bounds a minimal pair as proved in [5] we construct a 1-generic set whose e-degree does not bound a minimal pair in the semi-lattice of the enumeration degrees.

In her paper [1] Copestake examines the n-generic degrees for every $n < \omega$. She proves that every 2-generic enumeration degree bounds a minimal pair and states that there is a 1-generic enumeration degree that does not bound a minimal pair. Her proof of the statement does not appear in the academic press. Later Cooper, Li, Sorbi and Yang show in [2] that every Δ_2^0 enumeration degree bounds a minimal pair and construct a Σ_2^0 enumeration degree that does not bound a minimal pair. In the same paper the authors state that their construction can be used to build a 1-generic degree that does not bound a minimal pair. Initially the goal of this paper was to build a 1-generic enumeration degree with the needed properties by following the construction from [2]. In the working process it turned out that significant modifications of the construction had to be made in order to get the desired 1-generic degree. The enumeration degree that is constructed is also Σ_2^0 and generalizes the result from [2].

2 Constructing a 1-generic degree that does not bound a minimal pair

Definition 1. *A set A is* enumeration reducible *to a set B if there is a c.e. set Φ such that:*

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$$n \in A \Leftrightarrow \exists u(\langle n, u\rangle \in \Phi \wedge D_u \subset B)$$

where D_u denotes the finite set with code u under the standard coding of finite sets. The c.e. set Φ is an enumeration operator *and its elements will be called* axioms*. We will write $A \leq_e B$ to denote that A is enumeration reducible to B and $A = \Phi(B)$ to denote the fact that A is enumeration reducible to B via the enumeration operator Φ.*

We will denote enumeration operators by capital Greek letters $\Phi, \Theta \ldots$.

As with Turing reducibility, enumeration reducibility gives rise to a degree structure. Note that all c.e. sets have degree 0, the least enumeration degree.

We will use lower case Greek letters (especially ρ, τ) for finite binary strings and let $\tau \subseteq \rho$ indicate that τ is an initial segment of ρ. When A is a set $\tau \subset A$ means that τ is an initial segment of $A's$ characteristic function χ_A considered as an infinite binary sequence.

Definition 2. *A set A is* 1-generic *if for every c.e. set X of finite binary strings*

$$\exists \tau \subset A(\tau \in X \vee \forall \rho \supseteq \tau(\rho \notin X)).$$

An enumeration degree is 1-generic if it contains a 1-generic set.

Definition 3. *A pair of enumeration degrees a and b form a* minimal pair *in the semi-lattice of the enumeration degrees if:*

1. *$a > 0$ and $b > 0$.*
2. *For every enumeration degree c ($c \leq a \wedge c \leq b \rightarrow c = 0$).*

Theorem 1. *There exists a 1-generic enumeration degree a that does not bound a minimal pair in the semi-lattice of the enumeration degrees.*

We will use the priority method with infinite injury to build a set A whose e-degree will have the intended properties. The construction involves a priority tree of strategies. For further definitions of both computability theoretic and tree notations and terminology we refer the reader to [3] and [4].

2.1 The Requirements

We will construct a set A satisfying the following requirements:

1. A is generic. Therefore for all c.e. sets W we have a requirement:

$$G^W : \exists \tau \subseteq A(\tau \in W \vee \forall \mu \supseteq \tau(\mu \notin W)),$$

where τ and μ are finite binary strings.
Let Req^G be the set of all G^W requirements.

2. A does not bound a minimal pair. For each pair of enumeration operators Θ_0 and Θ_1 we will have a requirement:

$$R^{\Theta_0\Theta_1} : \Theta_0(A) \text{ is c.e.} \vee \Theta_1(A) \text{ is c.e.} \vee$$
$$\vee \exists D(D \leq_e \Theta_0(A) \wedge D \leq_e \Theta_1(A) \wedge D \text{ is not c.e.}).$$

Let Req^R be the set of all $R^{\Theta_0\Theta_1}$ requirements.
Fix a requirement $R^{\Theta_0\Theta_1}$. Let $X = \Theta_0(A)$ and $Y = \Theta_1(A)$. This requirement is too complicated to be satisfied at once and we will break it up into subrequirements:

$$R^{\Theta_0\Theta_1} : (\exists \Phi_0)(\exists \Phi_1)(\forall \text{c.e. sets } W)[S^W]$$

where S^W is the subrequirement:

$$S^W : X \text{ is c.e.} \vee Y \text{ is c.e.} \vee [\Phi_0(X) = \Phi_1(Y) = D \wedge \exists d(W(d) \neq D(d))].$$

Let $Req^S_{R^{\Theta_0\Theta_1}}$ be the set of all S^W subrequirements of $R^{\Theta_0\Theta_1}$.

Let $Req = Req^G \cup Req^R \cup (\bigcup_{R \in Req_R} Req^S_R)$.

2.2 Priority Tree of Strategies

The set *Req* is linearly ordered with order type ω and requirements in earlier positions have higher priority. Each particular requirement can be satisfied in more than one way. We connect to each such way an outcome. The choice of the correct way to satisfy a certain requirement depends on the outcomes of higher priority requirements. Therefore we represent the set of all possible sequences of outcomes as a *tree of strategies*. Each node α on the tree is labelled by a requirement $P \in Req$ and the node α will be referred to as a P-strategy. The children of α correspond to each of α's possible outcomes. So, although each of those nodes will be labelled by the same requirement, each may have a different approach to satisfying its requirement depending on what it "believes" to be the outcome of α.

The set of all possible outcomes for each requirement will be linearly ordered ($<_L$, defined below) and the nodes of the tree of strategies will be ordered by the induced lexicographical ordering $\leq$. The construction is by stages; in each stage s we construct a set A_s approximating A and a string δ_s of length s in the tree of strategies. The initial segments $\delta \subseteq \delta_s$ are the nodes of the tree visited during stage s of the construction; they are the strategies that might act to satisfy their requirements. The intent is that there will be a true path, a leftmost path of nodes visited infinitely often, such that all nodes along the true path are able to satisfy their requirements. If the node β is visited on stage s, we say that s is a β-true stage.

Each node (say of length n) will build its own approximation A^n_s, so $A_s = A^s_s$; nodes will obey restrictions on A and $\overline{A}$ set by higher priority requirements. Ultimately A will be the set of all natural numbers a such that

$$(\exists t_a)(\forall t > t_a)[a \in A_t].$$

At the end of stage s we *initialize* all strategies $\delta > \delta_s$ by setting all parameters to their initial values and *cancelling* any witnesses.

We will proceed to describe what general actions the different types of strategies, corresponding to the different types of requirements, will make.

1. Let γ be a G^W-strategy. The actions that γ makes when visited are the following:
   (a) γ chooses a finite string λ_γ according to rules that ensure compatibility with strategies of higher priority.
   (b) Then it searches for a string μ such that $\lambda_\gamma\hat{}\mu \in W$. If it finds such a string then γ remembers the shortest one, μ_γ, and has outcome 0. If not then $\mu_\gamma = \varnothing$ and the outcome is 1. The order between the two outcomes is $0 <_L 1$. The strategy will be successful if $\lambda_\gamma\hat{}\mu_\gamma \subseteq A$. γ will restrain some elements out of and in A to ensure this.
2. Let α be a $R^{\Theta_0\Theta_1}$-strategy. It acts as a mother strategy to all its substrategies ensuring that they work correctly. We assume that on this level the two enumeration operators Φ_0 and Φ_1 are built. They are common to all substrategies of α. This strategy has only one outcome: 0.
3. Let β be a S^W-strategy. It is a substrategy of one fixed $R^{\Theta_0\Theta_1}$-strategy $\alpha \subset \beta$. The actions that β makes are the following:
   (a) First it tries to prove that the set X is c.e. by building a c.e. set U which should turn out to be equal to X. On each stage it adds elements to U and then looks if any errors have occurred in the set. While there are no errors the outcome is ∞_X.
   (b) If an error occurs then some element that was assumed to be in the set X has been extracted from X. The strategy can not fix the error by extracting the element from U because we want U to remain c.e. In this case β gives up on its desire to make X c.e. It finds the smallest error $k \in U \backslash X$ and forms a set E_k which is called an agitator set for k. The agitator contains an element a for every axiom for k in the current approximation of Θ_0, say $\langle k, D_k \rangle$, such that $a \in D_k$. So extracting the agitator set from A will make sure that each axiom for k in Θ_0 will not be valid for $\Theta_0(A) = X$, that is will make sure that $k \notin X$. On the other hand with some additional actions we will make sure that if the agitator is a subset of A then $k \in X$. And so the agitator will have the following property which we will refer to as the control property:

$$k \in X \Leftrightarrow E_k \subseteq A.$$

   The strategy now turns its attention to Y. It tries to prove that it is c.e. by constructing a similar set V_k that would turn out equal to be Y. It makes similar actions, checking at the same time if the agitator for k preserves its control property. Note that the agitator will lose this property if a new axiom for k is enumerated in Θ_0. While there are no errors in V_k the outcome is $\langle \infty_Y, k \rangle$.

(c) If an error is found in V_k, the strategy chooses the least $l \in V_k \backslash Y$ and forms an agitator F_l^k for l in a similar way. F_l^k now has the control property:

$$l \in Y \Leftrightarrow F_l^k \subseteq A.$$

Now β has some control over the sets X and Y, namely using the agitators it can determine whether or not $k \in X$ and $l \in Y$. It adds axioms $\langle d, \{k\}\rangle \in \Phi_0$ and $\langle d, \{l\}\rangle \in \Phi_1$ for some witness d, constructing a difference between D and W. If $d \in W$ the outcome is $\langle l, k\rangle$ and the agitators are kept out of A. If $d \notin W$ then the agitators are enumerated back in A, so $d \in D$ and the outcome is the symbol d_0.

The possible outcomes of a S^W-strategy are:

$$\infty_X <_L T_0 <_L T_1 <_L \cdots <_L T_k <_L \cdots <_L d_0$$

where T_k is the following group of outcomes:

$$\langle \infty_Y, k\rangle <_L \langle 0, k\rangle <_L \langle 1, k\rangle <_L \cdots <_L \langle l, k\rangle \ <_L \ldots$$

The priority tree of strategies is a computable function T with $\mathrm{Dom}(T) \subseteq \{0, 1, \infty_X, \langle \infty_Y, k\rangle, \langle l, k\rangle, d_0 | k, l \in \mathbb{N}\}^{<\omega}$ and $\mathrm{Range}(T) = Req$ for which the following properties hold:

1. If $\alpha \in \mathrm{Dom}(T)$ and $T(\alpha) \in Req^R$ then $\alpha\hat{\ }0 \in \mathrm{Dom}(T)$.
2. If $\gamma \in \mathrm{Dom}(T)$ and $T(\gamma) \in Req^G$ then $\gamma\hat{\ }o \in \mathrm{Dom}(T)$ where $o \in \{0, 1\}$.
3. If $\beta \in \mathrm{Dom}(T)$ and $T(\beta) \in Req^S_R$ then $\beta\hat{\ }o \in \mathrm{Dom}(T)$ where $o \in \{\infty_X, \langle \infty_Y, k\rangle, \langle k, l\rangle, d_0 | k, l \in \mathbb{N}\}$.
4. For all $\delta \in \mathrm{Dom}(T)$ such that the length $lh(\delta)$ is even $T(\delta) \in Req^G$.
5. If $\alpha \in \mathrm{Dom}(T)$ is a R-strategy then for each subrequirement S^W there is a S^W-strategy $\beta \in \mathrm{Dom}(T)$, a substrategy of α, such that $\alpha \subset \beta$.
6. If β is a S^W-strategy, a substrategy of α, then $\alpha \subseteq \beta$ and under $\beta\hat{\ }\infty_X$ and $\beta\hat{\ }\langle \infty_Y, k\rangle$ there aren't any other substrategies of α.
7. For each infinite path h in T and each R- or G-requirement there is a node $h \upharpoonright n$ along the path which is a R- or G-strategy respectively. For every S^W-requirement, subrequirement of R, there is also a node $h \upharpoonright n$ which is an S^W-strategy, unless there is already a higher priority S^W-strategy $h \upharpoonright m$ belonging to the same requirement R and $h(m+1) = \infty_X$ or $h(m+1) = \langle \infty_Y, k\rangle$.

2.3 Interaction between strategies

In order to have any organization whatsoever we make use of a global parameter, a counter b, whose value will be an upper bound to the numbers that have appeared in the construction up to the current moment.

1. First we will examine the interaction between an S^W-strategy β and a G^W-strategy γ. The interesting cases are when $\gamma \supseteq \beta\hat{\ }\infty_X$ and similarly when $\gamma \supseteq \beta\hat{\ }\langle \infty_Y, k\rangle$.

Let $\gamma \supseteq \beta\hat{}\infty_X$ and suppose β is of length n. Suppose β is visited on stage $s > n$ and adds an element k to the set U. There is an axiom $\langle k, E'\rangle \in \Theta_0$ which is currently valid, i.e. $E' \subseteq A^n_s$. The strategy β will keep a list $\mathbb{U}$ of the axioms from Θ_0 that it assumes to be valid when enumerating new elements in U. It is possible that later (even on the same stage) γ chooses a string μ_γ and extracts a member of E' from A. If there aren't any other axioms for k in the corresponding approximation of Θ_0, we have an error in U. On the next β-true stage, s_1 say, β will find this error, choose an agitator for k and move on to the right with outcome $\langle\infty_Y, k\rangle$. It is possible that later a new axiom for k is enumerated in the corresponding approximation of Θ_0 and thus the error in U is corrected. On the next β-true stage s_2, β returns to its initial aim to prove that X is c.e. But then another G^W-strategy $\gamma_1 \supseteq \gamma$ chooses a string μ_{γ_1} and again takes k out of U by extracting an element that invalidates the new axiom for k. If this situation appears infinitely many times, ultimately we will claim to have $X = U$ but k will be taken out of X infinitely many times and thus our claim would be wrong. Then this S^W requirement will not be satisfied. This is why we will have to ensure some sort of stability for the elements that we put in U, more precisely for the corresponding axioms in $\mathbb{U}$ that we assume to be valid. This is how the idea for *applying an axiom* arises. We apply an axiom $\langle k, E'\rangle$ by changing the value of the global parameter b so that it is larger than the elements of the axiom and then by initializing those strategies that might take k out of X. The first thing that comes to mind is to initialize *all* strategies $\delta \supseteq \beta\hat{}\infty_X$. This way we would avoid errors at all. If the set X is infinite though, we would never give a chance to strategies $\delta \supseteq \beta\hat{}\infty_X$ to satisfy their requirements. This problem is solved with the notion of local priority. Every G^W-strategy $\gamma \supseteq \beta\hat{}\infty_X$ will have a fixed local priority regarding β. This priority is given by a computable bijection $\sigma_\beta : \Gamma \to \mathrm{N}$ where Γ is the set of all G^W-strategies in the subtree of $\beta\hat{}\infty_X$. If $\gamma \subset \gamma_1$ then $\sigma_\beta(\gamma) < \sigma_\beta(\gamma_1)$. A strategy $\gamma \supseteq \beta\hat{}\infty_X$ has local priority $\sigma_\beta(\gamma)$ in relation to β. When we apply the axiom $\langle k, E'\rangle$ only strategies γ with $\sigma_\beta(\gamma)$ greater than k will be initialized. Then as the stages grow so do the elements that we put into U and with them grows the number of G^W-strategies that we preserve. Ultimately all strategies will get a chance to satisfy their requirements.

2. Now let us examine the interactions between two S^W-strategies β and β_1. The interesting case is $\beta \supseteq \beta_1\hat{}\infty_X$ and $\alpha \subset \alpha_1$ where α and α_1 are the corresponding mother strategies. Suppose that on stage s_1 the strategy β chooses its agitators E_k and F^k_l and takes them out of A. Note that it is important to keep both agitators in A or both agitators out of A to preserve the equality in the sets $\Phi^\alpha_0(X)$ and $\Phi^\alpha_1(Y)$ constructed at level α. Suppose now that on the next β_1-true stage s_2 the strategy β_1 decides to build its own agitators and in them it includes members from only one of the agitators that β selected at stage s_1, causing a difference in the sets $\Phi^\alpha_0(X)$ and $\Phi^\alpha_1(Y)$. To avoid this β_1 will choose its agitators carefully: along with the elements needed to form the agitator with the requested control property it will add also all elements of all agitators that were chosen and out of A on the previous

β_1-true stage s_1. Thus the two agitators of β will not be separated and will not cause an error such as $d \notin \Phi_0^\alpha(X)$ and $d \in \Phi_1^\alpha(Y)$.
Unfortunately this will not solve the problem completely. It is possible that on a later stage s_3 a new axiom is enumerated in Θ_0 for k or a new axiom is enumerated in Θ_1 for l, causing one of the agitators to lose its control property and creating a difference between the sets $\Phi_0^\alpha(X)$ and $\Phi_1^\alpha(Y)$ at the element d. If β is visited again then it would fix this mistake by discarding the false witness d. If not, the error would stay unfixed and the R-strategy α might not satisfy its requirement. Therefore we will attach a new parameter to α: a list $Watched_\alpha$ through which α will keep track of all its S^W-substrategies. The list will contain entries for all substrategies including information on what their agitators are. If α sees that one of the agitators loses its control property then it will go ahead with the actions on discarding the false witness and correcting the mistake in the operators Φ_0 and Φ_1 in advance. This action will not interfere with β's work. In fact if β is ever visited again it will cancel the witness and give up the agitator that has lost its control property. In that sense α is just pre-empting the actions of β.

2.4 The Construction

We will begin the description of the construction by listing again all parameters that are connected with each strategy. Their purpose was explained intuitively in the previous two sections. While describing the parameters we will suppress the superscripts that indicate the strategy to which they belong. The superscripts will appear only when more than one strategies are involved in a discussion and we need to distinguish between their parameters.

We have one global parameter b, common to all strategies, which is an upper bound to all elements that have appeared so far in the construction. Its initial value is 0.

In addition every strategy δ visited on stage s will have two more parameters E_s and F_s. The set E_s contains all elements restrained out of A on this stage s by strategies $\delta' \subset \delta$. The set F_s contains all elements that are restrained in A by strategies of higher priority $\delta'' < \delta$. Note that these elements may have been restrained on a previous stage.

Each G^W-strategy γ will have two parameters: finite binary strings λ and μ, with initial value the empty string $\varnothing$.

Each R-strategy α has a list $Watched$ with entries of the form $\langle \beta : \langle E, E_k, F_l^k \rangle, d \rangle$ where β is a substrategy of α, E_k and F_l^k are β's current agitators, the set E contains information needed to assess if the agitators still have the control property and d is the witness that must be *cancelled* in case one of the agitators loses its control property. The initial value of the list is $\emptyset$. Also α has the parameters Φ_0 and Φ_1, the enumeration operators that α and all its substrategies β construct together. Their initial value is $\emptyset$ as well.

Each S-strategy β inherits the two parameters Φ_0 and Φ_1 from its mother strategy. In addition it has c.e. sets U and V_k for all k, initially all empty. Then corresponding to them lists $\mathbb{U}$ and $\mathbb{V}_k$, with initial values the empty list. During the construction β might form agitators E_k for all k and F_l^k for all k and l or choose a witness d, but initially the agitators are empty and the witness is undefined.

On stage $s = 0$ all nodes of the tree are initialized, $b_0 = 0$, $\delta_0 = \varnothing$ and $A_0 = \mathbb{N}$.

On each stage $s > 0$ we will have $A_s^0 = \mathbb{N}$, $\delta_s^0 = \varnothing$ and $b_s^0 = b_{s-1}^{s-1}$.

Let's assume that we have already built δ_s^n, A_s^n and b_s^n. If $n = s$ then go on to the next stage $s+1$. Otherwise $n < s$ the strategy δ_s^n makes some actions as described below and has an outcome o. Then $\delta_s^{n+1} = \delta_s^n \hat{\ } o$.

I. δ_s^n is a G^W-strategy γ.
   (a) If $\lambda = \varnothing$ then define λ to be the binary string of length $b_s^n + 1$ such that

$$\lambda(a) \simeq 0 \text{ iff } a \in E_s$$

   and increase the value of the counter to $b_s^{n+1} = b_s^n + 1$.
   (b) If $\mu = \varnothing$ then ask if $\exists \mu(\ \lambda\hat{\ }\mu \ \in W\ )$. If the answer is **No** then $A_s^{n+1} = A_s^n$. All elements for which $\lambda(a) = 1$ are restrained by γ in A and the outcome is $o = 1$.
   If the answer is **Yes** then let μ be the least such binary string so that $\lambda\hat{\ }\mu \in W$ and increase the value of the counter to
   $b_s^{n+1} = \max(b_s^{n+1}, lh(\lambda\hat{\ }\mu)+1)$. All a such that $\lambda\hat{\ }\mu(a) = 1$ are restrained in A by γ. All a such that $a \geq lh(\lambda)$ and $\lambda\hat{\ }\mu(a) = 0$ are restrained out of A by γ.
   $A_s^{n+1} = A_s^n \setminus \{a | a \text{ is restrained out of } A \text{ by } \gamma\}$ and the outcome is $o = 0$.

II. δ_s^n is a R-strategy α.
   Then scan all entries in the list $Watched_\alpha$. For each $\langle \beta : \langle E, E_k, F_l^k \rangle, d \rangle \in Watched$ check if there is an axiom $\langle k, E' \rangle \in \Theta_0$ such that $E' \cap (E \cup E_k) = \emptyset$ or $\langle l, F' \rangle \in \Theta_1$ such that $F' \cap (E \cup E_k \cup F_l^k) = \emptyset$. If there is such an axiom then **cancel** d by enumerating in both sets Φ_0 and Φ_1 the axiom $\langle d, \emptyset \rangle$. $A_s^{n+1} = A_s^n$ and $o = 0$.

III. δ_s^n is a S^W-strategy β, a substrategy of α .
   First check if β is watched by α and delete the corresponding entry from $Watched_\alpha$ if there is one. Unless otherwise specified $b_s^{n+1} = b_s^n$. The actions that β makes depend on the outcome o^- that it had on the previous β-true stage s^-. If this is the first β-true stage in the construction, let $o^- = \infty_X$
   (a) The outcome o^- is ∞_X.
      1. Choose the least $k \in X \setminus U$. Here $X = \Theta_0^s(A_s^n)$. If there is such an element then there is an axiom $\langle k, E' \rangle \in \Theta_0^s$ with $E' \subseteq A_s^n$. Enumerate k in the set U and its relevant axiom $\langle k, E' \rangle$ in the list $\mathbb{U}$. **Apply** this axiom by initializing all strategies $\delta \supseteq \beta\hat{\ }\infty_X$ such that there is a G^W-strategy $\gamma \subseteq \delta$ of local $\beta\hat{\ }\infty_X$-priority with value greater than k and by setting $b_s^{n+1} = \max(b_s^n, E')$.
      2. Proceed through the elements of U until an element draws attention or until all elements are scanned. An element $k \in U$ draws attention if there isn't an applicable axiom for it.

Definition 4. *An axiom* $\langle k, E'\rangle \in \Theta_0$ *is* applicable *if:*
1. $E' \cap E_s^\beta = \emptyset$,
2. $E' \cap Out1_s = \emptyset$ *where* $Out1_s$ *is the set of all elements restrained out of* A *by some strategy* $\delta \supseteq \beta\hat{}\infty_X$ *such that:*

i. $\delta \subseteq \delta_{s^-}$,
ii. All G^W*-strategies* $\gamma \subseteq \delta$ *have local* $\beta\hat{}\infty_X$*-priority with value less than* k *(the ones that cannot be initialized when applying an axiom for* k*).*

The intuition behind this definition is that it is plausible that the axiom will end up valid. Note that the set Out_1 includes all elements that are restrained by G^W strategies with local priority less than k along what seems to be the true path. When we find a valid axiom that has not been applied, we will apply it thereby initializing all strategies below the first G^W-strategy with local priority greater than k along each path. We will not however initialize S^W-strategies above some G^W-strategy with local priority less than k. These S^W-strategies may have already chosen an agitator that may remain permanent. Therefore we must respect their choice and ask that an applicable axiom does not include any such elements.
For each element $k \in U$ act as follows:

- If k doesn't draw attention, find an applicable axiom $\langle k, E'\rangle$ for k that has minimal code. If the entry for k in $\mathbb{U}$ is different, replace it with $\langle k, E'\rangle$. If the axiom $\langle k, E'\rangle$ is not yet applied, **apply** it. If there aren't any elements k that draw attention then let $A_s^{n+1} = A_s^n$ and $o = \infty_X$.
- If k draws attention:
  A. Initialize all strategies $\delta \supseteq \beta\hat{}\infty_X$ such there is a G^W-strategy $\gamma \subseteq \delta$ of local $\beta\hat{}\infty_X$-priority with value greater than k.
  B. Examine all strategies in the subtree with root $\beta\hat{}\infty_X$. If β' was visited on stage s^-, had outcome $\langle l', k'\rangle$ and was not initialized after stage s^- then add to the list $Watched_{\alpha'}$ where α' is the mother strategy of β' an element of the following structure: $< \beta' : \langle E_{s^-}^{\beta'}, E_{k'}^{\beta'}, F_{l'}^{k',\beta'}\rangle, d^{\beta'} >$.
  Then define the agitator for k as $E_k = Out1_s \backslash E_s^\beta$. All elements $a \in E_k$ are restrained out of A by β. Let $A_s^{n+1} = A_s^n \backslash E_k$ and $o = \langle \infty_Y, k\rangle$.

(b) The outcome o^- is $\langle \infty_Y, k\rangle$.
  1. Check if there is an axiom $\langle k, E'\rangle \in \Theta_0$ such that $E' \cap (E_s^\beta \cup E_k) = \emptyset$. If so then act as in d.1.
  2. Choose the least element $l \in Y \backslash V_k$. If there is such an element then there is $\langle l, F'\rangle \in \Theta_1^s$ with $F' \subseteq A_s^n \backslash E_k$. Enumerate the element l in V_k and its corresponding axiom $\langle l, F'\rangle$ in $\mathbb{V}_k$. **Apply** this axiom by initializing all strategies $\delta \supseteq \beta\hat{}\langle \infty_Y, k\rangle$ such there is a G^W-strategy $\gamma \subseteq \delta$ of local $\beta\hat{}\langle \infty_Y, k\rangle$-priority with value greater than l and by setting $b_s^{n+1} = \max(b_s^n, F')$.

3. Proceed through the elements of V_k until all are scanned or until an element draws attention. An axiom $\langle l, F' \rangle \in \Theta_1$ is defined to be applicable similarly to case a.2:

Definition 5. *An axiom $\langle l, F' \rangle \in \Theta_1$ is* applicable *if:*
1. $F' \cap E_s^\beta = \emptyset$,
2. $F' \cap Out2_s = \emptyset$ where $Out2_s$ is the set of all elements restrained out of A by some strategy $\delta \supseteq \beta\hat{}\langle \infty_Y, k \rangle$ such that:

i. $\delta \subseteq \delta_{s^-}$,
ii. All G^W-strategies $\gamma \subseteq \delta$ have local $\beta\hat{}\langle \infty_Y, k \rangle$-priority with value less than l,

3. $F' \cap E_k = \emptyset$.

For each element $l \in V_k$ act as follows
- If l doesn't draw attention, find an applicable axiom with minimal code $\langle l, F' \rangle$. If the entry for l in $\mathbb{V}_k$ is different, replace it with $\langle l, F' \rangle$. If the axiom $\langle l, F' \rangle$ is not yet applied, **apply** it.
  If none of the elements draw attention then let $A_s^{n+1} = A_s^n \backslash E_k$ and $o = \langle \infty_Y, k \rangle$.
- If l draws attention:
  A. Initialize all strategies $\delta \supseteq \beta\hat{}\langle \infty_Y, k \rangle$ such that there is a G^W-strategy $\gamma \subseteq \delta$ of local $\beta\hat{}\langle \infty_Y, k \rangle$-priority with value greater than l.
  B. Examine all strategies in the subtree with root $\beta\hat{}\langle \infty_Y, k \rangle$. If β' was visited on stage s^-, had outcome $\langle l', k' \rangle$ and was not initialized after stage s^- then add to the list $Watched_{\alpha'}$ where α' is the mother strategy of β' an element of the following structure: $< \beta' : \langle E_{s^-}^{\beta'}, E_{k'}^{\beta'}, F_{l'}^{k', \beta'} \rangle, d^{\beta'} >$.
  The agitator for l is $F_l^k = Out2_s \backslash (E_s^\beta \cup E_k)$. All elements $a \in (E_k \cup F_l^k)$ are restrained in A by β.
  Find the least element d that has not been used in the definition of Φ_0 yet. This will be a witness β. Enumerate the axiom $\langle d, \{k\} \rangle$ in Φ_0 and the axiom $\langle d, \{l\} \rangle$ in Φ_1. Let $A_s^{n+1} = A_s^n$ and $o = d_0$.

(c) The outcome o^- is d_0. Check if the witness d has been enumerated in the c.e. set W. That is, check if $d \in W_s$.
If the answer is **Yes** then β restrains all elements $a \in (E_k \cup F_l^k)$ out of A. Let $A_s^{n+1} = A_s^n \backslash (E_k \cup F_l^k)$ and $o = \langle l, k \rangle$.
If the answer is **No** then let $A_s^{n+1} = A_s^n$ and $o = d_0$.

(d) The outcome o^- is $\langle l, k \rangle$. Then the agitators E_k and F_l^k and the witness d are defined.
1. Check for an axiom $\langle k, E' \rangle \in \Theta_0$ such that $E' \cap (E_s^\beta \cup E_k) = \emptyset$, that is E_k has lost its control property. If there is one then **cancel** d and let $V_k = \mathbb{V}_k = E_k = F_l^k = \emptyset$. Replace the entry for k in $\mathbb{U}$ with $\langle k, E' \rangle$. **Apply** the axiom $\langle k, E' \rangle$. The strategy β stops restraining elements $a \in E_k \cup F_l^k$. Let $A_s^{n+1} = A_s^n$ and $o = \beta\hat{}\infty_X$.

2. Check for an axiom $\langle l, F' \rangle \in \Theta_1$ such that $F' \cap (E_s^\beta \cup E_k \cup F_l^k) = \emptyset$. If there is one then **cancel** d and let $F_l^k = \emptyset$. Replace the entry for l in $\mathbb{V}_k$ with $\langle l, F' \rangle$. **Apply** the axiom $\langle l, F' \rangle$. The strategy β stops restraining elements $a \in F_l^k$. Let $A_s^{n+1} = A_s^n \backslash E_k$ and $o = \beta\hat{\ }\langle \infty_Y, k \rangle$.
3. If neither of the above two cases hold, hence both agitators still have their control property, then let $A_s^{n+1} = A_s^n \backslash (E_k \cup F_l^k)$ and $o = \langle l, k \rangle$.

2.5 Proof

The proof of the theorem is divided into four groups of lemmas. The first group is about the restrictions. It gives a clear idea about which elements are restrained at different stages. The second group of lemmas is about the agitator sets. Its purpose is to prove that the agitators have the intended control properties that we claim. Then follows the group of lemmas about the true path. Finally we prove that the requirements are indeed satisfied.

Restriction Lemmas The restriction lemmas are basic tools for the rest of the proof. We will establish some basic rules about the restriction that will help us later determine properties of the characteristic function of A. Note that, since the tree is infinitely branching, we could have infinite activity to the right of the true path. The following lemmas ensure that this activity does not have any undesired effect on A.

We start off with a simple property of the agitator sets that will be helpful for the rest of the restriction lemmas.

Proposition 1. *Let β be a strategy that is visited and chooses an agitator Ag on stage s. Then the elements of the agitator Ag were restrained out of A by some G^W-strategy $\gamma \supset \beta$ on some previous stage $s_0 < s$ after β was last initialized.*

Proof. The proof is by induction on s. Suppose the lemma is true for all strategies visited on stages $t < s$ and let β be visited on stage s. Assume β chooses its agitator and let a be an element from this agitator. Finally let s' be the stage on which β was last initialized before stage s. We will concentrate on the case when β chooses E_k; the case when it chooses F_l^k is similar. Then $a \in Out1_s$ and hence is restrained out of A on stage s^- by some strategy in the subtree with root $\beta\hat{\ }\infty_X$. Obviously $s^- > s'$, otherwise $Out1_s = \emptyset$ because all strategies that extend β would also be initialized and would not restrain any elements out of A.

If a was restrained out of A by a G^W-strategy on stage s^- the lemma is proved. Suppose a was restrained by a S^W-strategy $\beta' \supseteq \beta\hat{\ }\infty_X$. Then a is in the agitator $Ag^{\beta'}$ of β'. This agitator was chosen on some stage $t \leq s^-$ and β' was not initialized on stages between t and s^- as otherwise the agitator would be cancelled. So if we assume that β' was cancelled for the last time before stage s on stage t', we have $s' \leq t' < t \leq s^-$.

Then according to our induction hypothesis for t we have that a was restrained by a G^W-strategy on stage t_0 such that $t' < t_0 < t$. In particular $s' < t_0 < s$. This concludes the induction and the proof of the lemma. □

Lemma 1 (Preserving the Restrictions Lemma). *Let s_1 and s_2 be two consecutive δ-true stages. If δ is not initialized on any intermediate stage t such that $s_1 < t \leq s_2$ then $E^\delta_{s_1} = E^\delta_{s_2}$.*

Proof. We will prove the lemma by induction on the length of δ.

If δ is of length 0 then $\delta = \varnothing$ and $E^\delta_{s_1} = E^\delta_{s_2} = \emptyset$. So let us assume that the statement is true for strategies δ of length n. We will prove that it holds for $\delta\hat{\ }o$.

Suppose $\delta_1 = \delta\hat{\ }o$ is visited on stages s_1 and s_2 and not initialized on stages t such that $s_1 < t \leq s_2$. Then δ is also visited on stages s_1 and s_2 and is not initialized on any stage t such that $s_1 < t \leq s_2$. The induction hypothesis gives us $E^\delta_{s_1} = E^\delta_{s_2}$. So we only need to prove that the elements that δ restrains on stages s_1 and s_2 are the same. Indeed on each stage the set E_{δ_1} is obtained from E_δ by adding the elements that δ restrains out of A on that stage.

We will examine the different cases:

Case 1. If δ is a R-strategy, a G^W-strategy with $o = 1$ or a S^W-strategy with $o = \infty_X$ or $o = d_0$ then δ does not restrain any elements on stages s_1 and s_2.

Case 2. Suppose δ is a G^W-strategy with outcome $o = 0$. Then the value of δ's parameters λ and μ are the same on stages s_1 and s_2, as they can change only after initialization. Therefore the elements that δ restrains on both stages s_1 and s_2 are the same as well, namely the elements $a > lh(\lambda^\delta)$ such that $\lambda^\delta\hat{\ }\mu^\delta(a) = 0$.

Case 3. Suppose δ is a S^W-strategy with outcome $o = \langle\infty_Y, k\rangle$. Then the elements that δ restrains out of A on stages s_1 and s_2 are the ones in $(E_k)_{s_1}$ and $(E_k)_{s_2}$ respectively. If we assume that $(E_k)_{s_1} \neq (E_k)_{s_2}$ then on some stage t such that $s_1 < t \leq s_2$ we would have had an outcome $o = \infty_X$. Indeed δ can only choose a value for its agitator E_k if it had outcome ∞_X on the previous true stage. Once the value is chosen it can only be changed if the strategy is initialized or if the agitator loses its control property. In the latter case δ would have outcome ∞_X. But $\infty_X <_L \langle\infty_Y, k\rangle$ and therefore δ_1 would be initialized on stage t.

Case 4. Suppose δ is a S^W-strategy with outcome $o = \langle l, k\rangle$. Then the elements that δ restrains on stages s_1 and s_2 are the ones in $(E_k)_{s_1} \cup (F^k_l)_{s_1}$ and $(E_k)_{s_2} \cup (F^k_l)_{s_2}$ respectively. If we assume that $(E_k)_{s_1} \neq (E_k)_{s_2}$ or $(F^k_l)_{s_1} \neq (F^k_l)_{s_2}$ then on some stage t such that $s_1 < t \leq s_2$ we would have had an outcome $o' = \infty_X$ or $o' = \langle\infty_Y, k\rangle$ to the left of o and δ_1 would be initialized on stage t. □

Proposition 2. *If s is a δ-true stage and $a \in E^\delta_s$ then δ cannot restrain a (in or out of A) on this stage.*

Proof. 1. Let δ be a G^W-strategy. Let $s_0 \leq s$ be the earliest stage on which δ is visited such that δ is not initialized between stages s_0 and s. According to Lemma 1, $E^\delta_{s_0} = E^\delta_s$ and therefore $a \in E^\delta_{s_0}$. The value of the parameter λ^δ is chosen on stage s_0 and remains the same until stage s. Then $a < lh(\lambda_\delta)$ and $\lambda_\delta(a) = 0$, hence δ does not restrain a on stage s.

2. Let δ be a S^W-strategy. Then δ restrains only elements in its agitators. Let $s_0 \leq s$ be the stage on which δ is visited and chooses an agitator Ag. According

to Lemma 1 $E^{\delta}_{s_0} = E^{\delta}_{s}$ and therefore $a \in E^{\delta}_{s_0}$. According to the construction $Ag \cap E^{\delta}_{s_0} = \emptyset$. Therefore δ does not restrain a. $\square$

Lemma 2. *If s is a δ-true stage and $a \in F^{\delta}_{s}$ then δ can not restrain a out of A on stage s.*

Proof. Assume that a is restrained in A by $\delta_1 < \delta$ on stage $s_1 \leq s$. Note that $a \in F^{\delta}_{s}$ until δ_1 is initialized or is visited and stops restraining a in A. Hence δ_1 is not initialized until stage s. Let $s_2 \geq s_1$ be the first stage after the imposition of the restraint on which δ is visited. We will prove that s_2 is the first visit of δ after an initialization.

Case 1. $\delta_1 <_L \delta$. Then δ is initialized on stage s_1.
Case 2. $\delta_1 \subset \delta$.

a. δ_1 is a G^W-strategy. Then s_1 is the earliest stage after δ_1's last initialization, say on stage t, on which it picks a value for one of it parameters λ or μ.
If δ_1 chooses λ^{δ_1} on stage s_1 then s_1 is the first stage after the initialization on stage t on which δ_1 is visited. But δ was also initialized on stage t. If δ_1 chooses μ^{δ_1} on stage s_1 then it has outcome 0 and will have outcome 0 on each visit until it is initialized again (if ever). As δ is visited on stage s we can conclude that $\delta \supseteq \delta_1\hat{\ }0$. On the other hand the nodes that extend $\delta_1\hat{\ }0$ are visited for the first time after δ_1's last initialization on stage t not sooner than on stage s_1.

b. δ_1 is a S^W-strategy then on stage s_1 it has outcome d_0. This is the only case when a S^W-strategy restrains elements in A. Furthermore δ_1 had outcome $\langle\infty_Y, k\rangle$ on its previous visit on stage s_1^- and has outcome d_0 on each visit after s_1 while it is restraining the element in A. In particular it has this outcome on stage s. Hence $\delta \supseteq \delta_1\hat{\ }d_0$ and was initialized on stage s_1^-, when δ_1 had outcome $\langle\infty_Y, k\rangle$.

So, if $\gamma \supseteq \delta$ is a G^W-strategy then for any λ_γ that γ chooses on stages after stage s_1 we have $a < lh(\lambda_\gamma)$ and γ cannot restrain a out of A.

If δ is a S^W-strategy and we assume that δ restrains a out of A then a is included in some agitator Ag. As we proved in Proposition 1, any element of the agitator has been restrained out of A by some G_W-strategy $\gamma \supset \delta$ after $\delta's$ last initialization. But we just proved that no such γ restrains a out of A. Hence $a \notin Ag$. $\square$

Lemma 3. *Suppose that on stage s we visit δ_1. Suppose that δ_1 restrains out of A an element a that is currently restrained in A by a lower priority strategy $\delta_2 \supset \delta_1$. Then δ_2 is initialized on stage s.*

Proof. We will make the proof by induction on the distance $d(\delta_1, \delta_2) = lh(\delta_2) - lh(\delta_1)$. We know that $d > 0$. Let us assume that the statement is true for all pairs of strategies with distance $d < n$. Let $d(\delta_1, \delta_2) = n$.

On stage s_0 we have visited δ_2 which restrained a in A. Then from stage s_0 up until the substage on which we visit δ_1, the element is still restrained in A, hence δ_2 has not been initialized since stage s_0. Then neither is the strategy δ_1.

It follows from Proposition 2 that a was not restrained out of A by δ_1 on stage s_0. So on stage s the elements that δ_1 restrains out of A are different from the ones it restrained on stage s_0.

If δ_1 is a G^W-strategy, this could only happen if it had outcome 1 on stage s_0 and outcome 0 on stage s. The parameter λ^{δ_1} does not change value between stages s_0 and s, as δ_1 is not initialized. So only if the parameter μ changed value, could δ_1 restrain new elements out of A. But this means that $\delta_2 \supseteq \delta_1\hat{}1$ and is initialized on stage s.

If δ_1 is a S^W-strategy then a is included in some agitator Ag. This agitator was chosen on stage $t \leq s$ and is extracted from A on stage s, but was not extracted from A on stage s_0.

The easy case is $\delta_2 \supseteq \delta_1\hat{}d_0$. Then on stage s, δ_1 has outcome $\langle l, k\rangle$ and initializes δ_2.

Whenever δ_1 has outcome $\langle l, k\rangle$ both agitators are extracted from A. In particular if this is the outcome on s_0, as the elements extracted by δ_1 on stages s_0 and s are different, δ_1 must have had outcome ∞_X or $\langle\infty_Y, k\rangle$ on an intermediate stage when it changed the values of at least one of the agitators. On that stage δ_2 would be initialized.

This leaves us with $\delta_2 \supseteq \delta_1\hat{}\infty_X$ or $\delta_2 \supseteq \delta_1\hat{}\langle\infty_Y, k\rangle$. In the first case $Ag = E_k$, as elements that enter F^k_l are restrained by G^W-strategies below $\delta_1\hat{}\langle\infty_Y, k\rangle$ by Proposition 1. These are initialized on stage s_0 and can not restrain a out of A by Lemma 2. In the second case $Ag = F^k_l$ as E_k is already extracted from A on stage s_0 and does not change until stage s, or δ_1 would have outcome ∞_X on an intermediate stage and δ_2 would be initialized.

In both cases the agitator is chosen on stage $t > s_0$ and after that δ_1 has outcome to the right. Then by the definition of an agitator the element a was restrained out of A by some $\sigma \supset \delta_1$ on stage $t^- \geq s_0$. We claim that $\sigma \subset \delta_2$ and $s_0 < t^-$ so by the induction hypothesis δ_2 would be initialized on stage t^-, contradicting our assumptions.

Indeed $\sigma <_L \delta_2$ would initialize δ_2 on stage t^- and $\delta_2 < \sigma$ would not allow σ to restrain a out of A. So $\sigma \subset \delta_2$ and furthermore $s_0 \neq t^-$ or by Proposition 2 δ_2 cannot restrain a at all on stage s_0. □

Corollary 1. $\forall s \forall \delta (E^\delta_s \cap F^\delta_s = \emptyset)$.

Proof. Assume for a contradiction that $\exists s \exists \delta (E^\delta_s \cap F^\delta_s \neq \emptyset)$. Let s be the least stage and δ be the least strategy for which our assumption holds. Let $a \in E^\delta_s \cap F^\delta_s$. Then when we visit δ, a is restrained out of A by $\delta_1 \subset \delta$ and a is restrained in A by $\delta_2 < \delta$. We will examine the possible positions of δ_1 and δ_2:

1. $\delta_1 > \delta_2$. But then $a \in F^{\delta_1}_s$ and δ_1 can not restrain a out of A.
2. $\delta_1 < \delta_2$. Then $\delta_1 \subset \delta_2$. We know that δ_1 restrains a on stage s. According to Lemma 3 δ_2 is initialized on stage s. But then it stops restraining elements and a is not restrained by δ_2 when we visit δ. This contradicts our choice of δ_2. □

2.6 Lemmas about the Agitators

Let's take a closer look at the agitators. Suppose β chooses an agitator at stage s. Then $o^- = \infty_X$, in which case $Ag = Out1_s \backslash E_s^\beta$, or $o^- = \langle \infty_Y, k \rangle$, in which case $Ag = Out2_s \backslash (E_k \cup E_s^\beta)$. It follows from Lemma 1 that $E_s^\beta = E_{s^-}^\beta$. Any element a in $Out1$ or $Out2$ was restrained on stage s^- by a strategy $\delta \supset \beta$ and hence $E_{s^-}^\delta \supseteq E_{s^-}^\beta$. So $Out1_s \cap E_s^\beta = Out2_s \cap E_s^\beta = \emptyset$. Also in the second case $E_{s^-}^\delta \supseteq E_k$, so $Out2_s \cap E_k = \emptyset$. Hence the agitators have a simpler definition, namely $Ag = Out1_s$ in the first case and $Ag = Out2_s$ in the second case.

Suppose $\beta' \supset \beta$ is a S^W-strategy and on stage s^- it was visited and had outcome $\langle l', k' \rangle$. Then let $E_{\beta'} = E_{\beta'}^\beta \cup E_{k'} \cup F_{l'}^{k'}$ where $E_{\beta'}^\beta = E_{s^-}^{\beta'} \backslash E_{s^-}^\beta$: the elements that are restrained out of A by strategies below β, but above β'. If β' is not initialized on stage s then $E_{\beta'} \subset Ag$.

Similarly if $\beta' \supset \beta$ is a S^W-strategy and on stage s^- it was visited and had outcome $\langle \infty_y, k' \rangle$ then let $E_{\beta'} = E_{\beta'}^\beta \cup E_{k'}$ where $E_{\beta'}^\beta = E_{s^-}^{\beta'} \backslash E_{s^-}^\beta$. If β' is not initialized on stage s then $E_{\beta'} \subset Ag$.

Now that we have established these basic facts about the agitators we can proceed with the proof of some of their more complicated properties. Note that every S^W-strategy may have influence on the operators Φ_0 and Φ_1 that it helps construct, even if it is visited finitely many times. The following lemmas give us information on what that influence might be.

Lemma 4. *1. Let β be a strategy that is visited on stage t_0 and chooses an agitator E_k for k. If the node $\beta\hat{\ }\infty_X$ is never again initialized or visited on any stage $t > t_0$ and $E_k \subseteq A$ then $k \in X$.*

2. Let β be a strategy that is visited on stage t_0 and chooses an agitator F_l^k for l. If the node $\beta\hat{\ }\langle \infty_Y, k \rangle$ is not initialized or visited on any stage $t > t_0$ and $F_l^k \subseteq A$ then $l \in Y$.

Proof. We will prove the first clause of the lemma; the second clause is proved similarly. To prove that $k \in X = \Theta_0(A)$ we need to find an axiom $\langle k, E' \rangle \in \Theta_0$ with $E' \subset A$.

Consider the axiom $\langle k, E' \rangle$ for k listed in $\mathbb{U}$ on stage t_0. We will prove that it has that property. It was applied not later than on stage t_0. Furthermore it was valid when it entered $\mathbb{U}$ hence $E' \cap E_{t_0}^\beta = \emptyset$ according to Lemma 1.

The strategy β chooses an agitator for k on stage t_0. Hence we initialize all strategies δ such that $\beta <_L \delta$. Furthermore $o_{t_0^-} = \infty_X$, hence on stage t_0^- we have initialized all strategies δ' such that $\beta\hat{\ }\infty_X <_L \delta'$. The strategies $\delta' \supset \beta$ such that $\beta\hat{\ }\infty_X <_L \delta'$ are initialized on stage t_0^- and are not visited again before stage t_0.

Therefore all nodes δ such that $\beta\hat{\ }\infty_X <_L \delta$ cannot restrain elements from E' out of A. And the only strategy that can extract elements from E' out of A on stage t_0 is β.

For a contradiction assume that an element $a \in E'$ is extracted from A on infinitely many stages t. Let t_1 be the first stage after t_0 on which $a \notin A_{t_1}$. Let δ restrain a out of A on stage t_1. We know that $\beta\hat{\ }\infty_X \not<_L \delta$. Also our assumptions on β, namely that $\beta\hat{}\infty_X$ is never again visited or initialized, give us that $\delta \not\supseteq \beta\hat{}\infty_X$ and $\delta \not<_L \beta\hat{}\infty_X$. This leaves us with the following two possibilities:

a. $\delta = \beta$. If β itself extracts the element a out of A, then a must be an element of one of β's agitators. The F_l^k agitators are all empty at stage t_0, and when they are defined at later stages they will contain elements restrained out of A by strategies extending $\beta\hat{}\langle\infty_Y, k\rangle$. We have already established such elements cannot be from the set E', so a must be in some version of E_k defined at or after stage t_0. However, E_k will not change its value after t_0, because otherwise we will have a $\beta\hat{}\infty_X$-true stage, contradicting our assumption. As we have also assumed $E_k \subset A$, we have reached the desired contradiction.

b. $\delta \subset \beta$. We treat G^W and S^W-strategies separately.

If δ is a G^W-strategy then in order to restrain elements out of A on stage t_1 it must have outcome $o = 0$. It cannot be that $\beta \supseteq \delta\hat{}1$ or it would be initialized on stage t_1. Hence $\delta\hat{}0 \subseteq \beta$ and δ is not initialized on stages t such that $t_0 < t \leq t_1$. Therefore $a \in E_{t_0}^\beta$ and $a \notin E'$.

If δ is a S^W-strategy then a is included in some agitator Ag which is taken out of A on stage t_1. Whenever a S^W-strategy chooses an agitator it moves on to the right. If the agitator is formed on stage $t \leq t_0$ then, since on stage t_0 the strategy β is visited and sees a in A, we can conclude that $\beta \supseteq \delta\hat{}d_0$, but then on stage t_1 it must be initialized.

Suppose Ag is formed on stage $t > t_0$. Then a was extracted from A on the previous δ-true stage t^- by one of the strategies extending δ. Our choice of t_1 as the first stage after t_0 on which a is extracted from A guarantees that $t^- = t_0$. But we know that the only strategy that can extract a on stage t_0 is β, hence $a \in E_k \subset A$. □

Lemma 5. *Let $\beta\hat{}\langle l, k\rangle$ be visited on stage t_0. If β is not initialized or visited on stages $t > t_0$ and $(E_k \cup F_l^k) \not\subset A$ then $(E_k \cup F_l^k \cup E_{t_0}^\beta) \cap A = \emptyset$.*

Proof. Let $(E_k \cup F_l^k) \not\subset A$. First we will prove that $(E_k \cup F_l^k) \cap A = \emptyset$. Let $a \in E_k \cup F_l^k$. Then a is restrained out of A by some G^W-strategy $\gamma \supset \beta$ on some stage $t' < t_0$ after $\beta's$ last initialization as we established in Proposition 1. As β is not initialized or visited anymore, no other G^W-strategy can restrain the element a out of A. Indeed G^W-strategies of higher priority than β would initialize β if they restrained a new element. The ones to the right of β are initialized on stage t' and choose their parameter λ to be of length greater than a. So if $a \notin A_t$ then a is restrained out of A by some S^W-strategy $\delta \subset \beta$. We can even say that $\delta\hat{}\infty_X \subseteq \beta$, if a is included in some agitator $E_{k'}$, and $\delta\hat{}\langle\infty_Y, k\rangle \subseteq \beta$, if a is included in some agitator $F_{l'}^{k'}$, again using the result from Proposition 1. Moreover the agitator is chosen on stage $t_1 > t_0$, as after the strategy δ chooses its agitator it has outcomes to the right of β until the agitator is cancelled.

Suppose a is taken out of A on stage $t > t_0$ by $\beta_1 \subset \beta$. Then a is included in the agitator Ag_1 of β_1, chosen on stage $t_1 > t_0$. So $a \notin A_{t_1^-}$ and $t_1^- \geq t_0$. If $t_1^- = t_0$ then $E_k \cup F_l^k \subseteq Ag_1$. If $t_1^- > t_0$ then there is another strategy β_2 such that $\beta_1 \subset \beta_2 \subset \beta$ and a is included in one of its agitators Ag_2. With a similar argument we get a monotone decreasing sequence of stages $t_1 > t_2 > \dots$ bounded by t_0, hence finite.

Therefore always when $a \notin A_t$, we have a finite sequence of S^W-strategies $\beta_1 \subset \beta_2 \subset \dots \subset \beta$ and a corresponding monotone sequence of their agitators $Ag_1 \supset Ag_2 \supset \dots \supset (E_k \cup F_l^k)$ such that Ag_1 is restrained out of A on stage t. If $a \notin A_t$ and $t > t_0$ then $(E_k \cup F_l^k) \cap A_t = \emptyset$ and ultimately $(E_k \cup F_l^k) \cap A = \emptyset$.

Let us assume now that $b \in E_{t_0}^{\beta} \cap A \neq \emptyset$. Then there is a stage t_b such that $b \in A_t$ for all $t > t_b$. Let t' be a stage for which $(E_k \cup F_l^k) \cap A_t = \emptyset$ and $t' > t_b$. Then there is a series of S^W strategies $\beta_1 \subset \beta_2 \subset \dots \subset \beta_n \subset \beta$ and a corresponding series of their agitators $Ag_1 \supset Ag_2 \supset \dots \supset (E_k \cup F_l^k)$. According to Lemma 1, we can express $E_{t_0}^{\beta}$ in the following way:

$E_{t_0}^{\beta} = E_{t_1}^{\beta_{t'}} \cup (E_{\beta_2}^{\beta_1})_{t_2} \cup \dots \cup (E_{\beta}^{\beta_n})_{t_0}$.

If $b \in (E_{\beta_2}^{\beta_1})_{t_2} \cup \dots \cup (E_{\beta}^{\beta_n})_{t_0}$ then $b \in Ag_1$ and therefore $b \notin A_{t'}$ contradicting the choice of $t' > t_b$. Therefore $E_{t_0}^{\beta} \cap A = \emptyset$. □

2.7 The True Path

The true path will ultimately be the path along which each strategy satisfies its requirement. It will be as usual the leftmost path visited infinitely often. It is not obvious that such a path exists, as our tree of strategies is infinitely branching. Fortunately we can prove the following:

Lemma 6. *There exists an infinite path f in T with the following properties:*

1. $\forall n \exists^{\infty} t (f \upharpoonright n \subseteq \delta_t)$ - *the infinite property,*
2. $\forall n \exists t_n \forall t > t_n (f \upharpoonright n \not<_L \delta_t)$ - *the leftmost property.*

Proof. We will define f by induction on n and simultaneously prove that it has the desired properties. First $f \upharpoonright 0 = \varnothing$ obviously has both properties. It is visited on every stage and $t_0 = 0$. Now let's assume we have defined $f \upharpoonright n$ with the desired properties. We will define $f \upharpoonright n+1 = (f \upharpoonright n)\hat{\ }o$ where o is an outcome of the strategy $f \upharpoonright n$. We will refer to this outcome as *the true outcome.*

I. If $f \upharpoonright n$ is a R-strategy then $o = 0$. We always visit $f \upharpoonright (n+1)$ when we visit $f \upharpoonright n$, hence infinitely often and $t_{n+1} = t_n$.
II. If $f \upharpoonright n$ is a G^W-strategy then the possible outcomes are 0 and 1. As we visit $f \upharpoonright n$ infinitely many times, at least one of the two outcomes will also be visited infinitely many times. If

$$\exists^{\infty} t[(f \upharpoonright n)\hat{\ }0 \subseteq \delta_t]$$

then $o = 0$. As this is the leftmost possible outcome $t_{n+1} = t_n$.
Otherwise $(f \upharpoonright n)\hat{\ }0$ is visited only finitely many times and there exists a stage t_1 such that $\forall t > t_1 [f \upharpoonright n \subseteq \delta_t \Rightarrow (f \upharpoonright n)\hat{\ }1 \subseteq \delta_t]$. Then $o = 1$ and $t_{n+1} = \max(t_n, t_1)$.
III. If $f \upharpoonright n$ is a S^W-strategy then:

(a) If

$$\exists^\infty t[(f \restriction n)\hat{\ }\infty_X \subseteq \delta_t]$$

then $o = \infty_X$ and $t_{n+1} = t_n$.
Otherwise there exists a least $f \restriction n$-true stage t_1 such that

$$\forall t \geq t_1[f \restriction n \subseteq \delta_t \Rightarrow (f \restriction n)\hat{\ }\infty_X \not\subseteq \delta_t].$$

On stage t_1 the strategy $f \restriction n$ chooses an agitator E_k and has outcome $\langle \infty_Y, k\rangle$. Then for all stages greater than t_1 the possible outcomes are $\langle \infty_Y, k\rangle$, $\{\langle l, k\rangle | l \in \mathbb{N}\}$ and d_0.

(b) If

$$\exists^\infty t[(f \restriction n)\hat{\ }\langle \infty_Y, k\rangle \subseteq \delta_t]$$

then $o = \langle \infty_Y, k\rangle$ and $t_{n+1} = \max(t_n, t_1)$.
Otherwise there exists a least $f \restriction n$-true stage $t_2 \geq t_1$ such that

$$\forall t \geq t_2[f \restriction n \subseteq \delta_t \Rightarrow (f \restriction n)\hat{\ }\infty_X \not\subseteq \delta_t \wedge (f \restriction n)\hat{\ }\langle \infty_Y, k\rangle \not\subseteq \delta_t].$$

Then on stage t_2 the strategy $f \restriction n$ chooses a second agitator F_l^k and has outcome d_0. For all stages $t > t_2$ the possible outcomes are d_0 and $\langle l, k\rangle$.
If on some stage $t_3 > t_2$ we have an outcome $\langle l, k\rangle$ then on all stages $t \geq t_3$ we would have this outcome, because you can't return from outcome $\langle l, k\rangle$ back to d_0 without passing through $\langle \infty_Y, k\rangle$ or ∞_X.

(c) If the outcome $\langle l, k\rangle$ never occurs, that is

$$\forall t \geq t_2[f \restriction n \subseteq \delta_t \Rightarrow (f \restriction n)\hat{\ }d_0 \subseteq \delta_t],$$

then $o = d_0$ and $t_{n+1} = \max(t_n, t_2)$.

(d) Otherwise there is a stage t_3 such that

$$\forall t \geq t_3[f \restriction n \subseteq \delta_t \Rightarrow (f \restriction n)\hat{\ }\langle l, k\rangle \subseteq \delta_t].$$

Then $o = \langle l, k\rangle$ and $t_{n+1} = \max(t_n, t_3)$. □

Unfortunately the leftmost property does not guarantee that the strategies along the true path will be initialized only finitely many times and will be able to satisfy their requirements eventually. This is due to the second case of initialization. That is why we need to prove this separately.

Lemma 7 (Stability Lemma). *For every S^W-strategy β the following statement is true:*

1. If $\beta\hat{\ }\infty_X \subseteq f$ then for every $k \in U$ there exists an axiom $\langle k, E'\rangle \in \Theta_0$ and a stage t_k such that if $t > t_k$ and β is visited on t with $o^- = \infty_X$ then $\langle k, E'\rangle$ is applicable for k and therefore k does not draw attention. Furthermore $E' \subseteq A$.

2. If $\beta\hat{\ }\langle \infty_Y, k\rangle \subseteq f$ then for every $l \in V_k$ there exists an axiom $\langle l, F'\rangle \in \Theta_1$ and a stage t_l such that if $t > t_l$ and β is visited on t with $o^- = \langle \infty_Y, k\rangle$ then $\langle l, F'\rangle$ is applicable for l and therefore l does not draw attention. Furthermore $F' \subseteq A$.

Proof. Assume that this is not the case and choose $\beta \subseteq f$ as the least strategy for which the proposition is false. Suppose $\beta\hat{}\infty_X \subseteq f$. The case $\beta\hat{}\langle\infty_Y, k\rangle \subseteq f$ is similar. Let $k \in U^\beta$ be the least element such that k draws attention infinitely many times.

Let $\Gamma = \{\gamma \supseteq \beta\hat{}\infty_X | \gamma$ is a G^W-strategy with local priority less than k $\}$.

We choose a stage t so big that:

a. If $\beta' \subset \beta$ is a S^W-strategy such that $\beta'\hat{}\infty_X \subseteq \beta$ then the elements of $U_{\beta'}$ which are less than or equal to the local β'-priority of any $\gamma \in \Gamma$ are already in $U_{\beta'}$ and do not draw attention any more. For these elements there is an applicable axiom and let all axioms with a smaller code that get applied at some stage be already applied. According to our choice of β as the least strategy for which the proposition is not true, this choice of t is satisfiable.
b. Similarly if $\beta' \subset \beta$ is a S^W-strategy such that $\beta'\hat{}\langle\infty_y, k'\rangle \subseteq \beta$ then the elements of $V^{\beta'}_{k'}$ which are less than or equal than the local β'-priority of any $\gamma \in \Gamma$ are already in $V^{\beta'}_{k'}$, do not draw attention anymore and do not apply any new axioms.
c. For all elements $m \in U$ such that $m \leq k$ we have $m \in U_t$.
d. All elements $m < k$ do not draw attention on stages $s > t$ and do not apply any axioms.
e. Let $M = \max\{lh(\gamma) | \gamma \in \Gamma\} + 2$. Let t_M be the stage for which $\forall s > t_M(\delta_s \not<_L f \upharpoonright M)$ from the leftmost property of f. Then $t > t_M$.

According to our choice of t, precisely conditions a, b and e, it is true that for all $s > t$, β does not get initialized on stage s. Then Lemma 1 gives us that E^β_s is the same on all β-true stages $s > t$. We can therefore omit the index s in further discussions and refer to this set as E^β.

Let $t_1 > t$ be a stage on which $f \upharpoonright M$ is visited. On the next β-true stage t_1^+ the previous outcome is ∞_X. We scan the elements of U and change their corresponding elements in $\mathbb{U}$ if needed. The elements $m < k$ do not draw attention anymore, but it is still possible that k draws attention.

1. If k does not draw attention then for the axiom $\langle k, E'\rangle$ in $\mathbb{U}$ we have that:
   (a) $E' \cap E^\beta = \emptyset$,
   (b) $E' \cap Out1_{t_1^+} = \emptyset$.
2. If k does draw attention on stage t_1^+ then we define an agitator $E_k = Out1_{t_1^+}$ and move to the right of the true path. Let t_2 be the next stage on which $\beta\hat{}\infty_X$ is visited. On this stage we must have found an axiom $\langle k, E''\rangle$ for which again:
   (a) $E'' \cap E^\beta = \emptyset$,
   (b) $E'' \cap Out1_{t_1^+} = \emptyset$.

In both cases we have an axiom $\langle k, E^0\rangle$ for which the two conditions hold. Let $t_3 > t_1$ be a $f \upharpoonright M$-true stage by which this strategy is applied. We will prove that no strategy extracts elements from E^0 on stages $s > t_3$. Hence this axiom will be the one we are searching for.

Note that after this axiom has been applied, none of the strategies that have been initialized during or after this application can ever restrain any elements of E^0 out of A, including all strategies below $f \restriction M$. At stage t_1 all axioms to the right of $f \restriction M$ have been initialized. In the first case the axiom is applied not later than on stage t_1^+. The strategies to the right of $\beta\hat{\ }\infty_X$ are initialized on that stage and the strategies below $\beta\hat{\ }\infty_X$ that are to the right of $f \restriction M$ are not visited after their initialization until t_1^+.

In the second case the axiom is applied on stage t_2 and again strategies to the right of $\beta\hat{\ }\infty_X$ are initialized on that stage and the strategies below $\beta\hat{\ }\infty_X$ that are to the right of $f \restriction M$ are not visited after their initialization until t_2.

Strategies to the left of $f \restriction M$ are not visited after stage $t_1 < t_3$ and can not restrain elements from E^0 out of A at any later stage.

The only danger is that a strategy δ along $f \restriction M$ restrains an element from E^0 out of A on stage $s > t_3$. We will prove that this also does not happen.

First of all if δ is a G^W-strategy, by stage t_1 its outcome is final and so are all elements that it restrains out of A. These elements are in E^β if $\delta \subset \beta$ or in $Out1_{t_1^+}$ if $\delta \supset \beta$. In particular a is not restrained by δ out of A on any stage $s > t_3$.

If δ is an S^W-strategy then the elements it restrains out of A are the ones in its agitators. We need to consider the possible ways that such agitators might be constructed. So suppose that δ has an agitator Ag that it extracts on stage $s > t_3$.

Notice first that our approximation of the true path δ_s never goes left of $f \restriction M$ after stage t_1. Thus δ does not have outcomes to the left of the outcome it had on stage t_1.

Suppose that δ had already chosen this agitator Ag by stage t_1, that is Ag has already a value on stage t_1 and does not change its value until stage s on which it is out of A. If $Ag \subset A_{t_1}$ then δ has outcome d_0 on stage t_1. This is the rightmost outcome and as δ does not have outcomes to the left of it on further stages it will not extract Ag on stage s. Thus Ag is restrained out of A on stage t_1. Hence $Ag \subset E^\beta \cup Out1^\beta_{t_1^+}$ and does not contain elements from E^0.

We are left with the case when δ chooses Ag after stage t_1. This limits the possibilities for the true outcome of δ. We can have $\delta\hat{\ }\infty_X \subset f$ in which case each agitator that δ ever chooses is eventually cancelled. We can also have $\delta\hat{\ }\langle\infty_Y, k\rangle \subset f$ in which case the agitator E_k is chosen before stage t_1 and does not contain elements from A, as we have just established, and each agitator F_l^k is eventually cancelled.

We will prove that agitators formed after stage t_3 cannot contain elements from E^0. This will show that the elements from E^0 can be extracted from A only finitely many times and hence $E^0 \subset A$.

It is convenient to consider each S^W-strategy $\delta \subset f \restriction M$ in order of its length, starting from the longest. The reason is that strategies of lower priority determine the elements that enter agitators of higher priority strategies.

Let δ be the longest S^W-strategy along $f \restriction M$. Suppose δ chooses an agitator Ag on stage $s > t_3$. All of Ag's elements were restrained by strategies extending

δ on the previous δ-true stage $s^- \leq t_3$. These are either strategies that were initialized when the axiom $\langle k, E^0 \rangle$ was applied and hence cannot restrain elements from E^0, or G^W-strategies $\gamma \subset f \upharpoonright M$ which as we already proved do not restrain elements from E^0.

By induction we can prove the same for the shorter S^W-strategies. $\square$

Corollary 2. *Every strategy along the true path is eventually not initialized.*

Proof. We will prove by induction on n that there is a $f \upharpoonright n$-true stage t_n^* such that $f \upharpoonright n$ is not initialized on any stage $t > t_n^*$. We will refer to this stage t_n^* in the rest of the proof.

The case $n = 0$ is trivial because $f \upharpoonright 0 = \varnothing$ is never initialized and is visited on every stage, so $t_0^* = 0$.

Assume that $f \upharpoonright n$ is visited on stage t_n^* and not initialized on stages $t > t_n^*$. If $f \upharpoonright (n+1)$ is a R- or S^W-strategy then $f \upharpoonright n$ is a G^W-strategy and it does not initialize strategies in its subtree at all. So let t_{n+1}^* be the first stage on which $f \upharpoonright (n+1)$ is visited after $\max\{t_n^*, t_{n+1}\}$ where t_{n+1} is the stage from the leftmost property of the true path (second property of Lemma6). Then $f \upharpoonright (n+1)$ is not initialized on stages $t > t_{n+1}^*$.

If $f \upharpoonright (n+1)$ is a G^W-strategy then we choose t_{n+1}^* so that the following conditions hold

1. $t_{n+1}^* > t_n^*$.
2. $t_{n+1}^* > t_{n+1}$ where t_{n+1} is the stage from the leftmost property of the true path.
3. For every S^W-strategy β with $\beta\hat{\ }\infty_X \subseteq f \upharpoonright (n+1)$ and every $k \in U_\beta$ less than the local β-priority of $f \upharpoonright (n+1)$, we have an applicable axiom $\langle k, E_0 \rangle$ which is applicable on every stage after t_k. There are finitely many axioms with a code that is less than that of E_0. Let t_{n+1}^* be so big that all axioms with a code that is smaller than the code of E_0 and that get applied at some point are already applied.
4. For every S^W-strategy β with $\beta\hat{\ }\langle \infty_Y, k \rangle \subseteq f \upharpoonright (n+1)$ and every $l \in V_k^\beta$ less than the local β-priority of $f \upharpoonright (n+1)$, we have an applicable axiom $\langle l, F_0 \rangle$ which is applicable on every stage after t_l. There are finitely many axioms with a code that is less than that of F_0. Let t_{n+1}^* be so big that all axioms with a code that is smaller than the code of F_0 and that get applied at some stage are already applied.
5. $f \upharpoonright (n+1)$ is visited on stage t_{n+1}^*.

It follows from Lemmas 6 and 7 that this choice of t_{n+1}^* is satisfiable. Clause 2 ensures that $f \upharpoonright (n+1)$ will not be initialized by strategies to the left. Clause 1 ensures that it won't initialized due to the initialization of G^W-strategies that $f \upharpoonright (n+1)$ extends and finally clauses 3 and 4 ensure that $f \upharpoonright (n+1)$ won't be initialized due to S^W-strategies that it extends. $\square$

2.8 Satisfaction of The Requirements

Lemma 8. *Every R requirement is satisfied.*

Proof. Fix a R-requirement. Let α be the corresponding R-strategy on the true path. We will prove that $\Theta_0(A) = X$ and $\Theta_1(A) = Y$ do not form a minimal pair. The proof is divided into the following three cases depending on the true outcomes of the S^W-substrategies of α along the true path:

1. All S^W-strategies $\hat{\beta} \subset f$, substrategies of α, have true outcomes d_0 or $\langle k, l \rangle$. Then we will prove that $\Phi_0(X) = \Phi_1(Y) = D$ and D is not c.e.
2. There is a S^W-strategy $\hat{\beta} \subset f$, substrategy of α, with true outcome ∞_X. Then X will be c.e.
3. There is a S^W-strategy $\hat{\beta} \subset f$, substrategy of α, with true outcome $\langle \infty_Y, k \rangle$. Then Y will be c.e.

We will treat each case separately.

1. For all S^W strategies $\hat{\beta} \subset f$, substrategies of α,

$$\exists k \exists l (\hat{\beta}\hat{\ }\langle l, k \rangle \subset f) \vee \hat{\beta}\hat{\ }d_0 \subset f.$$

We start by proving that $\Phi_0^\alpha(X) = \Phi_1^\alpha(Y)$. Now the properties of the agitators proved in Section 2.6 will play an important role as the operators Φ_0 and Φ_1 are constructed by *all* of α's substrategies, not only the ones along the true path. So we have to prove that $\Phi_0(X)(d^\beta) = \Phi_1(Y)(d^\beta)$, for every witness d^β that any substrategy β has ever used.
We automatically have this equality for any witness d^β that is cancelled. Cancelling the witness involves enumerating the axiom $\langle d^\beta, \emptyset \rangle$ in both operators. So $\Phi_0(X)(d^\beta) = \Phi_1(Y)(d^\beta) = 1$.
This means that strategies β to the right of the true path will not cause problems. Strategies to the left of and on the true path may have witnesses that are never cancelled. So let β be a substrategy of α and d^β be a witness chosen on stage t_0 that is never cancelled. Then β has outcome d_0 on stage t_0. After stage t_0 the strategy β is not initialized and does not have outcomes ∞_X or $\langle \infty_Y, k \rangle$, as in those cases we would cancel β's witness d. Let the corresponding agitators for d be E_k and F_l^k, so we have axioms $\langle d, \{k\} \rangle \in \Phi_0$ and $\langle d, \{l\} \rangle \in \Phi_1$. Also note that the length of the node β is necessarily less than t_0, as according to the construction a strategy acts only on stages s greater than its length.
We have the following three possibilities:
(a) $\beta <_L f$. Then let $t \geq t_0$ be the last stage on which β is visited.
If $\beta\hat{\ }\langle l, k \rangle \subseteq \delta_t$ then the conditions of Lemma 5 are true. Therefore if $(E_k \cup F_l^k) \not\subseteq A$ then $(E_k \cup F_l^k \cup E_t^\beta) \cap A = \emptyset$. If $(E_k \cup F_l^k) \subseteq A$ then according to Lemma 4 we have $k \in X$ and $l \in Y$ and therefore $\Phi_0(X)(d) = \Phi_1(Y)(d) = 1$.
If $(E_k \cup F_l^k \cup E_t^\beta) \cap A = \emptyset$ then from the proof of Lemma 5 we can conclude that there is an entry $\langle \beta : \langle E_t, E_k, F_l^k \rangle, d \rangle \in Watched_\alpha$. In this case we claim $\Phi_0(X)(d) = \Phi_1(Y)(d) = 0$. Suppose for a contradiction that this is not true, say $\Phi_0(X)(d) = 1$. Then the only axiom in Φ_1 for d is true, so $k \in X = \Theta_0(A)$. Therefore there is an axiom $\langle k, E' \rangle \in \Theta_0$ such

that $E' \subseteq A$ and hence $E' \cap (E_k \cup E_t^\beta) = \emptyset$. It appears in Θ_0^s on some stage s. The strategy $\alpha \subset f$ is visited on some stage $s' > s$. According to the construction α will spot this axiom while examining the entry for β in *Watched* and cancel d. Similarly we may prove that $\Phi_1(Y)(d) = 0$. If $\beta\hat{\ }d_0 \subseteq \delta_t$, as β is not initialized on stages $s' > t$, we have that $E_k \cup F_l^k$ is restrained in A by β. From Lemmas 2 and 3 it follows that $E_k \cup F_l^k \subset A$. Lemma 4 gives us $k \in X$ and $l \in Y$. Hence $\Phi_0(X)(d) = \Phi_1(Y)(d) = 1$.

(b) Suppose $\beta\hat{\ }d_0 \subseteq f$. By Lemma 6 and the fact that d is not cancelled whenever we visit β outcome d_0 from stage t_0 on. Therefore by Lemmas 2 and 3 $E_k \cup F_l^k \subset A$. Lemma 4 gives us $k \in X$ and $l \in Y$. Hence $\Phi_0(X)(d) = \Phi_1(Y)(d) = 1$.

(c) If $\beta\hat{\ }\langle l, k\rangle \subseteq f$ then by Lemma 6 there is a stage $t_1 > t_0$ such that on β-true stages $t > t_1$ the strategy β always has this outcome and $E_k \cup F_l^k$ is extracted from A_t. Also by Lemma 1 $E_t^\beta = E_{t_1}^\beta$ for all β-true stages $t > t_1$ and we will refer to this set as E^β. As β is visited on infinitely many stages $(E_k \cup F_l^k \cup E^\beta) \cap A = \emptyset$. We claim that in this case $\Phi_0(X)(d) = \Phi_1(Y)(d) = 0$.
Assume for a contradiction that this is not true, say $\Phi_0(X)(d) = 1$. Then there is an axiom $\langle k, E'\rangle \in \Theta_0$ with $E' \subseteq A$ and therefore $E' \cap (E_k \cup F_l^k \cup E^\beta) = \emptyset$. The axiom appears in $\Theta_0^{s_1}$ on some stage s_1 . Let s be a β-true stage such that $s > \max(s_1, t_1)$. According to the construction on stage s the strategy β will have outcome ∞_X contradicting the choice of t_1. Similarly we may prove that $\Phi_1(Y)(d)$ cannot equal 1.

This gives us a set $D = \Phi_0(X) = \Phi_1(Y)$. To prove that D is not c.e. let W be any c.e. set and consider the S^W-substrategy $\hat{\beta}$ along the true path. Let $n = lh(\hat{\beta})$. After stage t_{n+1} from Lemma 6 $\hat{\beta}$ always has its true outcome whenever it is visited and a permanent witness $\hat{d}$. This witness will prove $W \neq D$.

If $\hat{\beta}\hat{\ }\langle l, k\rangle \subset f$ then $W(\hat{d}) = 1$. The witness $\hat{d}$ is not cancelled by α. Indeed if α cancels the witness at stage t due to some axiom $\langle k, E'\rangle \in \Theta_0$ or $\langle l, F'\rangle \in \Theta_1$ then when we visit $\hat{\beta}$ on stage $t_1 \geq \max(t, t_{n+1})$ the strategy $\hat{\beta}$ would see this axiom and have outcome ∞_X or $\langle \infty_Y, k\rangle$ contradicting our choice of stage t_1. We just proved that in this case $D(\hat{d}) = 0$. Therefore $D \neq W$.

If $\hat{\beta}\hat{\ }d_0 \subset f$. Then the witness will not be cancelled by α as there will not be an entry for it in the list $Watched_\alpha$. We proved that $D(\hat{d}) = 1$. It follows that $W(\hat{d}) = 0$ as otherwise there would be a stage $s > t_{lh(n)+1}$ on which $\hat{d} \in W_s$. Then on the next $\hat{\beta}$-true stage we would have an outcome $\langle l, k\rangle <_L d_0$. Therefore $D \neq W$.

2. There is a strategy $\hat{\beta}$, substrategy of α, with $\hat{\beta}\hat{\ }\infty_X \subseteq f$. Let $n = lh(\hat{\beta})$. We will prove that $U_{\hat{\beta}} = X$ and so X is c.e. Assume for a contradiction that there is an element $k \in X \backslash U$ and choose the least one. Then there is an axiom $\langle k, E'\rangle \in \Theta_0$ such that $E' \subset A$. Note that on all $\hat{\beta}$-true stages $s > t_{n+1}^*$ by Lemma 1 $E_s^{\hat{\beta}} = E_{t_{n+1}^*}^{\hat{\beta}}$ and $E_s^{\hat{\beta}} \subset \overline{A}$. So $E' \cap E_s^{\hat{\beta}} = \emptyset$. Let $t > t_{n+1}^*$ be a stage on which all elements smaller than k that ever enter U are already in

U and all elements that are in E' are not taken out of A anymore. Then k will enter U on the next $\hat{\beta}$-true stage on which $o^- = \infty_X$, if not before. According to Lemma 7 for every $k \in U$ there is an axiom $\langle k, E' \rangle \in \Theta_0$ for which $E' \subseteq A$, therefore $k \in X$ and $U \subseteq X$. Ultimately we get $X = U$.

3. There is a S^W-strategy $\hat{\beta}$ which is a substrategy of α with $\hat{\beta}\hat{\ }\langle \infty_Y, k \rangle \subset f$ for some k. We show in this case that $V_k^{\hat{\beta}} = Y$ and therefore Y is c.e. The proof is similar to part 2. □

Lemma 9. *Every G^W requirement is satisfied.*

Proof. Fix a c.e. set W and consider the G^W-strategy $\gamma \subset f$. Let $n = lh(\gamma)$. Let λ and μ denote the values of γ's parameters on stage t^*_{n+1} from Corollary 2. It follows from the construction that these values remain the same on further stages. Indeed λ changes value only after initialization and μ changes value only when γ switches to outcome 0. We will prove that $\lambda\hat{\ }\mu \subset A$ and that $\lambda\hat{\ }\mu \in W$ or for every extension $\tau \supseteq \lambda\hat{\ }\mu$ we have $\tau \notin W$ and so the requirement G^W is satisfied.

By Lemma 1 the value of the set E^γ_t does not change on γ-true stages $t > t^*_{n+1}$ and we will refer to it as E^γ. Finally γ has always its true outcome on true stages $t > t^*_{n+1}$.

If $\lambda\hat{\ }\mu(a) = 1$ then a is restrained in A by γ and by Lemmas 2 and 3 $a \in A$. If $\lambda\hat{\ }\mu(a) = 0$ and $a < lh(\lambda)$ then $a \in E^\gamma \subset \overline{A}$ so $A(a) = 0$. If $\lambda\hat{\ }\mu(a) = 0$ and $a \geq lh(\lambda)$ then a is extracted on every γ-true stage $t \geq t^*_{n+1}$ and $A(a) = 0$. Therefore $\lambda\hat{\ }\mu \subset A$.

If $\gamma\hat{\ }0 \subset f$ then this outcome was visited after we saw that $\lambda\hat{\ }\mu \in W_{t^*_{n+1}} \subset W$. If $\gamma\hat{\ }1 \subset f$ then $\mu = \varnothing$ and for all extensions $\tau \supseteq \lambda$ we have $\tau \notin W$. Indeed if there were an extension of λ, $\tau \in W$, then it would appear in the approximation of W on some finite stage and on the next γ-true stage we would have outcome 0 contradicting the choice of t^*_{n+1}.

This concludes the proof of the lemma and the theorem. □

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