A generic set that does not bound a minimal pair

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Abstract. The structure of the semi lattice of enumeration degrees has been investigated from many aspects. One aspect is the bounding and nonbounding properties of generic degrees. Copestake proved that every 2-generic enumeration degree bounds a minimal pair and conjectured that there exists a 1-generic set that does not bound a minimal pair. In this paper we verify this longstanding conjecture by constructing such a set using an infinite injury priority argument. The construction is explained in detail. It makes use of a priority tree of strategies.

1 Introduction

In contrast to the Turing case where every 1-generic degree bounds a minimal pair as proved in [5] we construct a 1-generic set, whose e-degree does not bound a minimal pair in the semi-lattice of the enumeration degrees.

In her paper [1] Copestake examines the n-generic sets for every $n < \omega$. She proves that every 2-generic set bounds a minimal pair and states that there is a 1-generic set that does not bound a minimal pair. Her proof of the statement does not appear in the academic press. In their paper [2] Cooper, Sorbi, Lee and Yang show that every Δ^0_2 set bounds a minimal pair, and construct a Σ^0_2 set that does not bound a minimal pair. In the same paper the authors state that their construction can be used to build a 1-generic set that does not bound a minimal pair. Initially the goal of this paper was to build a 1-generic set with the needed properties by following the construction from [2]. In the working process it turned out that significant modifications of the construction had to be made in order to get the desired 1-generic set. The 1-generic set that is constructed is also Σ^0_2, and generalizes the result from [2].

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2 Constructing a 1-generic set that does not bound a minimal pair

Definition 1. *A set A is 1-generic if for every c.e. set X of strings*

$$\exists \tau \subset A(\tau \in X \vee \forall \rho \supseteq \tau(\rho \notin X))$$

An enumeration degree is 1-generic, if it contains a 1-generic set.

Definition 2. *Let a and b be two enumeration degrees. We say that a and b form a minimal pair in the semi-lattice of the enumeration degrees if:*

1. $a > 0$ *and* $b > 0$.
2. *For every enumeration degree* c $(c \leq a \wedge c \leq b \rightarrow c = 0)$.

Theorem 1. *There exists a 1-generic enumeration degree a, that does not bound a minimal pair in the semi-lattice of the enumeration degrees.*

We will use the priority method with infinite injury to build a set A, whose degree will have the intended properties. The construction involves a priority tree of strategies. For further definitions of both computability theoretic and tree notations and terminologies see [3] and [4].

2.1 Requirements

We will construct a set A, satisfying the following requirements:

1. A is generic, therefore for all c.e. sets W we have a requirement:

$$G^W : \exists \tau \subseteq \chi_A(\tau \in W \vee \forall \mu \supseteq \tau(\mu \notin W)),$$

   where τ and μ are finite parts.
   Let Req^G be the set of all G^W requirements.
2. A does not bound a minimal pair, therefore for each pair of c.e. sets Θ_0 and Θ_1 we will have a requirement:

$$R^{\Theta_0\Theta_1} : \Theta_0(A) = X - c.e. \vee \Theta_1(A) = Y - c.e. \vee$$

$$\vee \exists \Phi_0 - c.e. \Phi_1 - c.e.((\Phi_0(X) = \Phi_1(Y) = D) \wedge \forall W - c.e.(W \neq D))$$

   Let Req^R be the set of all $R^{\Theta_0\Theta_1}$ requirements.
   For each requirement $R^{\Theta_0\Theta_1}$ let $X = \Theta_0(A)$, $Y = \Theta_1(A)$ and let Φ_0 and Φ_1 be the c.e. sets defined above. In order for $R^{\Theta_0\Theta_1}$ to be satisfied we will make sure that the following subrequirements S^W for each c.e. set W are satisfied:

$$S^W : (X - c.e. \vee Y - c.e. \vee (\Phi_0(X) = \Phi_1(Y) = D \wedge \exists d(W(d) \neq D(d))))$$

   Let $Req^S_{R^{\Theta_0\Theta_1}}$ be the set of all S^W subrequirements of $R^{\Theta_0\Theta_1}$.

2.2 Priority Tree of Strategies

For every requirement we will have a different strategy. The strategy aims to fulfill the requirement according to the current situation, giving different outcomes. Let O be the set of all possible outcomes. We define a tree of strategies - a subset of O^*, closed under extensions. Each node α is labelled with a requirement Req, we say that α is a Req-strategy.

1. Let γ be a G^W-strategy. The actions that γ makes are the following:
   (a) γ chooses a finite part λ_γ, according to rules that insure compatibility with strategies of higher priority.
   (b) If there is a finite part μ, such that $\lambda_\gamma\hat{\ }\mu \in W$, then γ remembers the shortest one – μ_γ, and has outcome 0. If not, then $\mu_\gamma = \varnothing$, the outcome is 1. The order between the two outcomes is $0 < 1$. The strategy is successful if we insure that $\lambda_\gamma\hat{\ }\mu_\gamma \subseteq A$. γ will restrain some elements out of and in A to ensure this.
2. Let α be a $R^{\Theta_0\Theta_1}$ - strategy. It is like a mother strategy to all its substrategies. It insures that they work correctly. We assume that on this level the two sets Φ_0 Φ_1 are built. They are common to all substrategies of α. This type of strategy has one outcome: 0.
3. Let β be a S^W- strategy. It is a substrategy of one fixed $R^{\Theta_0\Theta_1}$- strategy - α, for which $\alpha \subset \beta$ holds. The actions that β makes are the following:
   (a) First it tries to make the set X c.e. In order to accomplish this β builds a set U, which should turn out equal to X. On each step it adds elements to U and then looks if any errors have occurred in the set. While there are no errors the outcome is ∞_X.
   (b) If an error occurs, then some element, that was assumed to be in the set X has come out of the set. The strategy can not fix the error in U because we want U to be c.e. In this case it gives up on our desire to make X c.e., it finds the smallest error $k \in U\backslash X$ and forms a set E_k, which is called an agitator for k. The agitator has the following property: $k \in X \Leftrightarrow E_k \subseteq A$. The strategy now turns its attention to Y, trying to make it c.e., constructing a similar set V_k, that would turn out equal to Y. It makes similar actions, checking at the same time if the agitator for k preserves the desired property. While there is no mistake in V_k the outcome is $\langle\infty_Y, k\rangle$.
   (c) If an error is found in V_k, the strategy chooses the smallest error $l \in V_k\backslash Y$ and forms an agitator F^k_l for l with the following property: $l \in Y \Leftrightarrow F^k_l \subseteq A$. Now β has control over the sets X and Y. It adds axioms $\langle d, \{k\}\rangle \in \Phi_0$ and $\langle d, \{l\}\rangle \in \Phi_1$, for some witness d, constructing a difference between D and W. If $d \in W\backslash D$, the outcome is $\langle l, k\rangle$. Otherwise: $d \in D\backslash W$, the outcome is d_0.

   And so the possible outcomes of a S^W- strategy are:

$$\infty_X < T_0 < T_1 < \ldots < T_k < \ldots < d_0,$$

   where T_k is the following group of outcomes:

$$\langle\infty_Y, k\rangle < \langle 0, k\rangle < \langle 1, k\rangle < \ldots < \langle l, k\rangle \ < \ldots$$

The priority tree of strategies is a computable function T with $\mathrm{Dom}(T) \subseteq \{0, 1, \infty_X, \langle\infty_Y, k\rangle, \langle l, k\rangle, d_0 | k, l \in N\}^*$ and $\mathrm{Range}(T) = Req^G \cup Req^R \cup (\bigcup_{R\in Req_R} Req^S_R)$, for which the following properties hold:

1. For every infinite path f in T $\mathrm{Range}(T \restriction f) = \mathrm{Range}(T)$.

2. If $\alpha \in \mathrm{Dom}(T)$ and $T(\alpha) \in Req^R$, then $\alpha\hat{}0 \in \mathrm{Dom}(T)$.
3. If $\gamma \in \mathrm{Dom}(T)$ and $T(\gamma) \in Req^G$, then $\gamma\hat{}o \in \mathrm{Dom}(T)$, where $o \in \{0, 1\}$.
4. If $\beta \in \mathrm{Dom}(T)$ and $T(\beta) \in Req^S_R$, then $\beta\hat{}o \in \mathrm{Dom}(T)$, where $o \in \{\infty_X, \langle\infty_Y, k\rangle, \langle k, l\rangle, d_0 | k, l \in N\}$.
5. If $\alpha \in \mathrm{Dom}(T)$ is a R-strategy, then for each subrequirement S^W there is a S^W-strategy $\beta \in \mathrm{Dom}(T)$, a substrategy of α, such that $\alpha \subset \beta$.
6. If β is a S^W-strategy, substrategy of α, then $\alpha \subseteq \beta$ and under $\beta\hat{}\infty_X$ and $\beta\hat{}\langle\infty_Y, k\rangle$ there aren't any other substrategies of α.

The construction is on stages - on each stage we construct a set A_s – approximating A and a string $\delta_s \in \mathrm{dom}(T)$ of length s. For every visited node $\delta \subseteq \delta_s$ of length $n \leq s$ we will build a corresponding set A^n_s, and then $A_s = A^s_s$. Ultimately the set A will be the set of all natural numbers a, for which there exists a step t_a, such that $\forall t > t_a(a \in A_t)$. At the end of step s we initialize all strategies $\delta > \delta_s$.

2.3 Interaction between strategies

In order to have any organization whatsoever we make use of a global parameter – a counter b, whose value will be an upper bound of the numbers, that have appeared in the construction up to the current moment.

1. First we will examine the interaction between a S^W-strategy β and a G^W-strategy γ. The interesting cases are when $\gamma \supseteq \beta\hat{}\infty_X$ and its similar one – when $\gamma \supseteq \beta\hat{}\langle\infty_Y, k\rangle$.
Let $\gamma \supseteq \beta\hat{}\infty_X$. When we visit β we add an element k to the set U. For it there is an axiom $\langle k, E'\rangle$, recorded in a corresponding set U, and $E' \subseteq A$ holds. It is possible that even on the same stage γ chooses a string μ_γ which takes out of A an element from E'. If there aren't any other axioms for k in the corresponding approximation of Θ_0, we we have an error in U. On the next stage when we visit β we will find this error, choose an agitator for k and move on to the right with outcome $\langle\infty_Y, k\rangle$. It is possible that later a new axiom for k is enumerated in the corresponding approximation of Θ_0 and thus the error in U is corrected. We return to our desire to make X-c.e. But then another G^W-strategy $\gamma_1 \supseteq \gamma$ chooses a string μ_{γ_1} and again takes k out of U. If this process continues infinitely many times, ultimately we will claim to have $X = U$, but k will be taken out of X infinitely many times and thus our claim would be wrong. Then this S^W requirement will not be satisfied. This is why we will have to ensure some sort of stability for the elements, that we put in U. This is how the idea for applying an axiom arises. When we apply an axiom $\langle k, E'\rangle$ – we change the value of the global parameter b, so that it is larger than the elements of the axiom. Then we initialize those strategies, that might take k out of X.
The first thing that we can think of is to initialize all strategies $\delta \supseteq \beta\hat{}\infty_X$. This way we would avoid errors at all. If the set X is infinite though, we would never give a chance to strategies $\delta \supseteq \beta\hat{}\infty_X$ to get satisfied. This problem is solved with a new idea - local priority. Every G^W- strategy $\gamma \supseteq \beta\hat{}\infty_X$

will have a fixed local priority regarding β, given by a computable bijection $\sigma : \Gamma \to \mathrm{N}$,where $\Gamma = \left\{\gamma - G^W \text{ strategy} | \gamma \supseteq \beta\hat{}\infty_X\right\}$ such that if $\gamma \subset \gamma_1$, then $\sigma(\gamma) < \sigma(\gamma_1)$. $\gamma \supseteq \beta\hat{}\infty_X$ has local priority $\sigma(\gamma)$ in relation to β.
When we apply the axiom $\langle k, E' \rangle$, only strategies γ of local priority lower than k will be initialized. Then as the value of the stage increases, so do the elements that we put into U, and with them grows the number of G^W - strategies, that we do not initialize. Ultimately all strategies will get a chance to satisfy their requirements.

2. Now let us examine the interactions between two S^W - strategies β β_1. An interesting cases is again $\beta_1 \supseteq \beta\hat{}\infty_X$. Therefore let $\beta_1 \supseteq \beta\hat{}\infty_X$,let β_1 be a substrategy of α_1 and $\alpha_1 \subset \beta$. It is possible that β_1 chooses agitators E_{k_1} and $F^{k_1}_{l_1}$ and takes them out of A. The next stage on which β is visited, β might like to build its own agitators that may include elements from E_{k_1} or $F^{k_1}_{l_1}$, causing an error in the sets $\Phi^{\alpha_1}_0$ and $\Phi^{\alpha_1}_1$. If β_1 is visited again then it would fix this mistake, by discarding the false witness. If not, the error would stay unfixed - and the R- strategy α_1 will not satisfy its requirement. In order to avoid this situation we do two things. First we choose our agitators carefully: along with the elements, needed two form the agitator with the requested property, we will add also all elements of all agitators that were chosen and out of A on the previous β - true stage. Thus the two agitators of β_1 will not be separated and will not cause an error like $d_1 \notin \Phi^{\alpha_1}_0(X)$ and $d_1 \in \Phi^{\alpha_1}_1(Y)$ in the corresponding sets. It is possible that on a later stage a new axiom for k or l in the corresponding approximations of $\Theta^{\alpha_1}_0$ or $\Theta^{\alpha_1}_1$ appears, causing one of the agitators to loose its control. If this happens - we might again have the same error in $\Phi^{\alpha_1}_0(X)$ and $\Phi^{\alpha_1}_1(Y)$. Therefore we will connect a structure with α_1 - a list $Watched_{\alpha_1}$ in which we will keep track of all S^W - substrategies of α_1 that do not have control over their agitator sets. Through this list α_1 can avoid any errors.

If a strategy δ is visited on a stage s, we connect to δ the set E^δ_s, that contains all elements restrained out of A on this stage s by strategies $\delta' \subset \delta$.

2.4 The Construction

At the beginning all nodes of the tree are initialized, $b_0 = 0$, $\delta_0 = \varnothing$, $A_0 = \mathrm{N}$.
On each stage $s > 0$ we will have $A^0_s = \mathrm{N}$, $\delta^0_s = \varnothing$ and $b^0_s = b^{s-1}_{s-1}$.
Lets assume that we have already built δ^n_s, A^n_s and b^n_s.
The strategy δ^n_s makes some actions and has an outcome o. Then $\delta^{n+1}_s = \delta^n_s\hat{}o$.

I. δ^n_s is a G^W strategy γ.
$b^{n+1}_s = b^n_s$.
 (a) If γ has been initialized on some stage after its last visit $\lambda_\gamma = \varnothing$. Then define λ_γ so: λ_γ is a string of length $b^n_s + 1$ and
 $\lambda_\gamma(a) \simeq 0$, iff $a \in E^\gamma_s$
 $b^{n+1}_s = b^{n+1}_s + 1$

(b) Ask if: $\exists\mu(\ \lambda_\gamma\hat{}\mu\ \in W\ )$. If the answer is "No" then: $\chi_\gamma=\lambda_\gamma$, $A_s^{n+1}=A_s^n$, all elements for which $\chi_\gamma(a)=1$ are restrained from γ in A, the outcome is $o=1$.
If the answer is "Yes", then $\mu_\gamma=$ the least μ, such that $\lambda_\gamma\hat{}\mu\in W$. $\chi_\gamma=\lambda_\gamma\hat{}\mu_\gamma$. $b_s^{n+1}=\max(b_s^{n+1},lh(\chi_\gamma+1))$. All $a\in\mathrm{Dom}(\chi_\gamma)$ and such that $\chi_\gamma(a)=1$ are restrained in A from γ. All $a\in\mathrm{Dom}(\chi_\gamma)$ such that $a\geq lh(\lambda_\gamma)$ and $\chi_\gamma(a)=0$ are restrained out of A from γ. $A_s^n=A_s^{n+1}\backslash\{a|$ is restrained out of A from $\gamma\}$, the outcome is $o=0$.

II. δ_s^n is a R strategy α.
Then scan all substrategies β ,for which there is an element in the list $Watched_\alpha$.
Let $\langle\beta,E,E_k,F_l^k,d\rangle\in Watched_\alpha$. Check if there is an axiom $\langle k,E'\rangle\in\Theta_0$, such that $E'\cap(E\cup E_k)=\emptyset$ or $\langle l,F'\rangle\in\Theta_1$, such that $F'\cap(E\cup E_k\cup F_l^k)=\emptyset$. if there is such an axiom then **cancel** $d:\Phi_0=\Phi_0\cup\{\langle d,\emptyset\rangle\}$, $\Phi_1=\Phi_0\cup\{\langle d,\emptyset\rangle\}$. $A_s^{n+1}=A_s^n$, $o=0$.

III. δ_s^n is a S^W strategy β, substrategy of α .
First check if β is watched by α and delete the corresponding element from $Watched_\alpha$ if there is one.
$b_s^{n+1}=b_s^n$
The outcome β depends on what the previous outcome $o-$ was on the previous β- true stage $s-$.
(1) The outcome $-$ is ∞_X
  a. Let k_0 be the least $k\in X\backslash U$. Here $X=\Theta_0^s(A_s^n)$. If there is such an element, then there is an axiom $\langle k_0,E'\rangle\in\Theta_0^s$ with $E'\subseteq A_s^n$. Then $U=U\cup\{k_0\}$ and $U=U\cup\{\langle k_0,E'\rangle\}$.
  b. Proceed through the elements of U, until an elements that draws attention, or until all elements are scanned.

Definition 3. *An axiom $\langle k,E'\rangle\in\Theta_0$ is applicable, if:*
1. $E'\cap E_s^\beta=\emptyset$
2. Let Γ be the set of these elements a, that are restrained out of A from G^W strategies $\gamma\supseteq\beta\hat{}\infty_X$ of higher local priority than k. Let $Out1_s^\beta=\Gamma\backslash A_{s-}$. Then $E'\cap Out1_s=\emptyset$.

An element $k\in U$ draws attention, if there isn't an applicable axiom for it.
For each element $k\in U$ act as follows:
  A. If k doesn't draw attention, find an applicable axiom for k - $\langle k,E'\rangle$, that has a minimal code. If the element for k in U is different, replace it with $\langle k,E'\rangle$. If the axiom $\langle k,E'\rangle$ is not yet applied, apply it.
  If there aren't any elements k that draw attention, then: $A_s^{n+1}=A_s^n$, $o=\infty_X$.
  B. If k draws attention:
  1.Examine all strategies
  $\beta'\in O_1=\{\beta'|\beta'\supseteq\beta\hat{}\infty_X\wedge\beta'\hat{}\langle\infty_Y,k'\rangle\subseteq\delta_{s-}\}$

β' is visited on stage $s-$ and an agitator $E_{k'}$ is defined for it. Let $E_{\beta'} = E^{\beta}_{\beta'} \cup E'_k$, where $E^{\beta}_{\beta'} = {}^{\beta'}_{s-}\backslash E^{\beta}_{s-}$ - the elements,that are restrained out of A from strategies below β, but above β'.
2. Examine all strategies
$\beta' \in O_2 = \{\beta' | \beta' \supseteq \beta\hat{\ }\infty_X \wedge \beta'\hat{\ }\langle l', k'\rangle \subseteq \delta_{s-}\}$
β' is visited on stage $s-$ and both agitators $E_{k'}$ and $F^{k'}_{l'}$ and a witness d' are defined. Then let $E_{\beta'} = E^{\beta}_{\beta'} \cup E_{k'} \cup F^{k'}_{l'}$, where $E^{\beta}_{\beta'} = {}^{\beta'}_{s-}\backslash E^{\beta}_{s-}$.
Add to the list $Watched_{\alpha'}$, where α' is the superstrategy of β' an element of the following structure:
$< \beta' : \langle {}^{\beta'}_{s-}, E_{k'}, F^{k'}_{l'}\rangle, d' >$
The agitator for k is defined as follows:
$E_k = (Out1^{\beta}_s \cup \bigcup_{\beta' \in O_1 \cup O_2} E_{\beta'})\backslash E^{\beta}_s$
All elements $a \in E_k$ are restrained out of A from β. $A^{n+1}_s = A^n_s \backslash E_k$ and $o = \langle \infty_Y, k\rangle$

(2) The outcome $-$ is $\langle \infty_Y, k\rangle$.

a. Check if there is an axiom $\langle k, E'\rangle \in \Theta_0$, such that $' \cap (E^{\beta}_s \cup E_k) = \emptyset$. If so then act as in 4.a.
b. Let l_0 be the least $l \in Y \backslash V_k$. If there is such an element, then there is $\langle l_0, F'\rangle \in \Theta^s_1$ with $F' \subseteq A^n_s \backslash E_k$. $V_k = V_k \cup \{l_0\}$, $V_k = V_k \cup \{\langle l_0, F'\rangle\}$.
c. Proceed throuhg the elements of V_k, until all are scanned, or until an element that draws attention.
An axiom $\langle l, F'\rangle \in \Theta_1$ is defined to be applicable similarly to case 2.b with the additional requirement that $F' \cap E_k = \emptyset$.
For each element $l \in V_k$:
A. If it doesn't draw attention, find an applicable axiom with minimal code $\langle l, F'\rangle$. If the element for l in V_k is different, replace it with $\langle l, F'\rangle$. If the axiom $\langle l, F'\rangle$ is not yet applied, apply it.
If none of the elements draw attention, then: $A^{n+1}_s = A^n_s \backslash E_k$
$o = \langle \infty_Y, k\rangle$
B. If l draws attention:
1.Examine all strategies
$\beta' \in O_1 = \{\beta' | \beta' \supseteq \beta\hat{\ }\langle \infty_Y, k\rangle \wedge \beta'\hat{\ }\langle \infty_Y, k'\rangle \subseteq \delta_{s-}\}$
β' is visited on stage $s-$ and an agitator $E_{k'}$ is defined for it. Let $E_{\beta'} = E^{\beta}_{\beta'} \cup E'_k$, where $E^{\beta}_{\beta'} = {}^{\beta'}_{s-}\backslash E^{\beta}_{s-}$ - the elements, that are restrained out of A from strategies below β,but above β'.
2. Examine all strategies
$\beta' \in O_2 = \{\beta' | \beta' \supseteq \beta\hat{\ }\langle \infty_Y, k\rangle \wedge \beta'\hat{\ }\langle l', k'\rangle \subseteq \delta_{s-}\}$
β' is visited on stage $s-$ and both agitators $E_{k'}$ and $F^{k'}_{l'}$ and a witness d' are defined. Then let $E_{\beta'} = E^{\beta}_{\beta'} \cup E_{k'} \cup F^{k'}_{l'}$, where $E^{\beta}_{\beta'} = {}^{\beta'}_{s-}\backslash E^{\beta}_{s-}$.
Add to the list $Watched_{\alpha'}$, where α' is the superstrategy of β' an element of the following structure:
$< \beta' : \langle {}^{\beta'}_{s-}, E_{k'}, F^{k'}_{l'}\rangle, d' >$

The agitator for l is:
$F_l^k = (Out2_s^\beta \cup \bigcup_{\beta' \in O_1 \cup O_2} E_{\beta'}) \backslash (E_s^\beta \cup E_k)$
All elements $a \in (E_k \cup F_l^k)$ are restrained in A from β.
Find the least $d \notin L(\Phi_0)$. This will be a witness for the strategy.
$\Phi_0 = \Phi_0 \cup \{\langle d, \{k\}\rangle\}$, $\Phi_1 = \Phi_1 \cup \{\langle d, \{l\}\rangle\}$.
$A_s^{n+1} = A_s^n$, $o = d_0$.

(3) The outcome $o-$ is d_0. Check if the witness d has been enumerated in the c.e. set W.
If the answer is "YES", β restrains all elements $a \in (E_k \cup F_l^k)$ out of .
$A_s^{n+1} = A_s^n \backslash (E_k \cup F_l^k)$, $o = \langle l, k\rangle$.
If the answer is "NO" then:
$A_s^{n+1} = A_s^n$, $o = d_0$.

(4) The outcome $o-$ is $\langle l, k\rangle$. Then there are agitators E_k and F_l^k and a witness d.
  a. Check for an axiom $\langle k, E'\rangle \in \Theta_0$, such that $E' \cap (E_s^\beta \cup E_k) = \emptyset$. If there is: **cancel** d, $V_k = \emptyset$. Replace the element for k in U with $\langle k, E'\rangle$. **Apply** the axiom $\langle k, E'\rangle$. All elements $a \in E_k \cup F_l^k$ are not restrained from β anymore. $A_s^{n+1} = A_s^n$, $o = \beta\hat{\ }\infty_X$. Proceed to the next step.
  b. Check for an axiom $\langle l, F'\rangle \in \Theta_1$, such that $F' \cap (E_s^\beta \cup E_k \cup F_l^k) = \emptyset$. If there is: **cancel** d. Replace the element for l in V_k with $\langle l, F'\rangle$. **Apply** the axiom $\langle l, F'\rangle$. β stops restraining elements $a \in F_l^k$. $A_s^{n+1} = A_s^n \backslash E_k$, $o = \beta\hat{\ }\langle \infty_Y, k\rangle$. Proceed to the next step.
  c. If not, then the agitators are still valid: $A_s^{n+1} = A_s^n \backslash (E_k \cup F_l^k)$, $o = \langle l, k\rangle$

2.5 Proof

The proof of the theorem is divided into a number of groups of lemmas. The first group concerns the construction. The lemmas from this group are more like facts, that help the reader to get a more clear picture of the construction. The second group of lemmas is about the restrictions – it gives a clear idea about which elements are restrained and how this changes on the different stages. The third group of lemmas is about the agitator sets. Its purpose is to prove that the agitators have indeed the properties that we claim. Then follows the group of lemmas about the true path. Finally come the lemmas that prove, that the requirements are indeed satisfied. Here the final part of the proof will be summarized.

The true path f is defined inductively as the most left infinite path in the tree of strategies, for which

$$\forall n \overset{\infty}{\exists} t (f \upharpoonright n \subseteq \delta_t)$$

As usual a second property of the true path is:

Lemma 1 (Most Left Lemma). $\forall n \exists t_n \forall t > t_n (f \upharpoonright n \not<_L \delta_t)$

Unfortunately these two properties are not sufficient for the proof. The problem comes from the application of axioms. Even when a stage comes, at which we are sure that for a certain n, $f \upharpoonright n$ can not be initialized from a strategy to the left, strategies that are above it can still initialize it at a later stage. Therefore another lemma is proved:

Lemma 2 (Stability Lemma). *For every S^W strategy β the following statement is true:*

1.If $\beta\hat{}\infty_X \subseteq f$, then for every $k \in U$ there exists an axiom $\langle k, E'\rangle \in \Theta_0$ and a stage t_k, such that if $t > t_k$ and β is accessible on t with $o- = \infty_X$, then $\langle k, E'\rangle$ is applicable for k and therefore k does not draw attention. For this axiom $\langle k, E'\rangle$: $E' \subseteq A$.

2.If $\beta\hat{}\langle\infty_Y, k\rangle \subseteq f$, then for every $l \in V_k$ there exists an axiom $\langle l, F'\rangle \in \Theta_1$ and a stage t_l, such that if $t > t_l$ and β is accessible on t with $o- = \langle\infty_Y, k\rangle$, then $\langle l, F'\rangle$ is applicable for l and therefor l does not draw attention. For this axiom $\langle l, F'\rangle$: $F' \subseteq A$.

Corollary 1.

$\forall n \exists t_n^* (f \upharpoonright n$ is accessible on stage $t_n^* \wedge \forall t > t_n^* (f \upharpoonright n$ is not initialized on stage $t))$

Finally we are ready to prove the most important result.

Lemma 3. *Every R requirement is satisfied.*

Proof. Let us look at one R requirement. Let α is the corresponding R - strategy on the true path. The proof of the lemma is divided into the following three cases:

1. For all S^W strategies β,that are substrategies of α
$$\beta \subset f \Rightarrow (\exists k \exists l(\beta\hat{}\langle l, k\rangle \subset f) \vee \beta\hat{}d_0 \subset f)$$
In this case we prove $\Phi_0^\alpha(X) = \Phi_0^\alpha(Y) = D$ and for each c.e. set W we have a witness d such that with $W(d) \neq D(d)$.
2. There is a S^W strategy β, that is a substrategy of α for which:
$$\beta \subset f \wedge \beta\hat{}\infty_X \subset f$$
In this case we prove that $X = U$, end hence X is c.e.
3. There is a S^W strategy β, which is a substrategy of α for which:
$$\beta \subset f \wedge \exists k(\beta\hat{}\langle\infty_Y, k\rangle \subset f)$$
In this case we prove that $Y = V_k$ and hance Y is c.e.

Lemma 4. *Every G^W requirement is satisfied.*

Proof. Let us look at a fixed G^W requirement and let γ be the corresponding G^W strategy on the true path. On each stage $t > t^*_{lh(\gamma)}$, γ in not initialized and has a constant string χ_γ. It can be proved that $\chi_\gamma \subset A$. Then

1. If $\gamma\hat{}1 \subseteq f$, then $\chi_\gamma = \lambda_\gamma \subseteq \chi_A$ and there is no extension χ_γ, that is in W.
2. If $\gamma\hat{}0 \subseteq f$, then $\chi_\gamma \in W$.

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