ENUMERATION REDUCIBILITY AND COMPUTABLE STRUCTURE THEORY

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1. Introduction

In classical computability theory the main underlying structure is that of the natural numbers or equivalently a structure consisting of some constructive objects, such as words in a finite alphabet. In the 1960's computability theorists saw it as a challenge to extend the notion of *computable* to arbitrary structure. The resulting subfield of computability theory is commonly referred to as *computability on abstract structures*. One approach towards this is the theory of computability in admissible sets of the hereditarily finite superstructure $\mathbb{HF}(\mathfrak{A})$ over a structure $\mathfrak{A}$. The development of computability on ordinals was initiated by Kreisel and Sacks [43, 42], who investigated computability notions on the first incomputable ordinal, and then further developed by Kripke and Platek [44, 58] on arbitrary admissible ordinals and by Barwise [6], who considered admissible sets with urelements. The notion of Σ-definability on $\mathbb{HF}(\mathfrak{A})$, introduced and studied by Ershov [16, 17] and his students Goncharov, Morozov, Puzarenko, Stukachev, Korovina, etc., is a model of nondeterministic computability on $\mathfrak{A}$. A survey of results on $\mathbb{HF}$-computability and on abstract computability based on the notion of Σ-definability can be found in [18, 95]. Montague [53] took a model theoretic approach to generalized computability theory, considering computability as definability in higher order logics.

The approach towards abstract computability that ultimately lead to the results discussed in this article starts with searching for ways in which one can identify abstract computability on a structure internally. Let $\mathfrak{A}$ be an arbitrary abstract structure. There are many different internal ways to define a class of functions that can be considered as the analog of classical computable functions. Different models of computation on $\mathfrak{A}$ give rise to different classes of computable functions: $PC(\mathfrak{A})$ denotes the functions that are *prime computable* in $\mathfrak{A}$, introduced by Moschovakis [54]. $REDS(\mathfrak{A})$ is the set of functions computable by means of *recursively enumerable definitional schemes*, introduced by Friedman and Shepherdson [21, 65]. Finally, we have the *search computable* functions, denoted by $SC(\mathfrak{A})$, and also introduced by Moschovakis [54]. Gordon [34] proved the equivalence of search computability with Montague's approach and with computability in admissible sets. Prime computability has a deterministic (sequential) character. REDS is nondeterministic (parallel) and allows searches on the set of natural numbers. Search computability is also nondeterministic, however here one is allowed to perform a

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search among arbitrary elements of the domain of the structure. For every structure $\mathfrak{A}$ we have $PC(\mathfrak{A}) \subseteq REDS(\mathfrak{A}) \subseteq SC(\mathfrak{A})$. In general these inclusions do not reverse.

Another natural way to study computability on a countable first-order structure is to consider an external approach. Every enumeration of the domain of a structure gives rise to an isomorphic structure on the natural numbers, called its *representation*. Fraisse [19] and Lacombe [45] suggest the notion of $\forall$-recursiveness: a function falls in this class if every enumeration of the domain of the given structure transforms this function into a function on the natural numbers that is recursive in the diagram of the corresponding representation. The equivalence between $\forall$-recursiveness and search computability on countable (total) structures with equality is proved by Moschovakis [56].

In the 1970s Skordev initiated the development of algebraic recursion theory, presented in his monograph [66]. The main goal of this program is to further clarify the connections between the two basic approaches to abstract computability: the internal approach, based on specific models of computation, and the external approach, which defines the computable functions through invariance relative to all enumerations of a structure, in the more general setting of partial structures, structures whose functions and relations can be partial. To find natural external analogues for partial structures we must extend classical relative computability to partial functions. Here as well, there are two different approaches: one corresponding to deterministic computational procedures and one corresponding to arbitrary effective ones. The first one can be mathematically described as relative *μ-recursiveness*: a partial function φ is μ-recursive relative to partial functions $\varphi_1, \dots, \varphi_n$ if φ can be obtained from $\varphi_1, \dots, \varphi_n$, the constant 0, the successor function S, and the projection functions, using superposition, primitive recursion and the minimization operation μ. The other notion is called *relative partial recursiveness* and it can be described via enumeration reducibility: the graph of φ is enumeration reducible to the graphs of $\varphi_1, \dots, \varphi_n$. If we restrict these notions to total functions then they coincide. However there are easy examples of partial objects for which they do not. Let φ be the characteristic function of the complement of the halting set $\overline{K}$ and ψ be the partial function that equals zero when the argument is in $\overline{K}$ and is not defined otherwise. Then φ is partial recursive relative to ψ but not μ-recursive relative ψ.

In 1977 Skordev conjectures that the partial functions which are invariantly computable in all computable presentations of a countable partial structure $\mathfrak{A}$ on the natural numbers are exactly the ones that are search computable on $\mathfrak{A}$. Soskov [69, 71, 70] modifies and extends this hypothesis to give a full classification. He proves that the invariantly partial computable functions in all total representations of $\mathfrak{A}$ are exactly $SC(\mathfrak{A})$, the invariantly partial computable functions in all partial representations of $\mathfrak{A}$ are exactly $REDS(\mathfrak{A})$ and the invariantly μ-recursive functions in all partial representations of $\mathfrak{A}$ are exactly $PC(\mathfrak{A})$.

The next theme investigated in this context is a reducibility between a certain class of abstract structures, considered natural for the purposes of abstract computability. These are partial two-sorted relational structures, with an abstract sort and the sort of the natural numbers. Partial functions can be represented through their graphs, provided that we have included equality and non-equality among the basic predicates. The reducibility is defined between structures with the same abstract sort: a structure $\mathfrak{A}$ is s-reducible to a structure $\mathfrak{B}$ if all the basic predicates of $\mathfrak{A}$ are search computable in $\mathfrak{B}$. The properties of this reducibility are very similar to the properties of enumeration reducibility. The obtained results [5, 7, 36] about the structure of the s-degrees have natural analogs in the enumeration degrees. On the other hand many of the techniques used in this area, could be adapted to study the enumeration degrees. This leads Soskov to transfer his focus towards degree theory, where he explored the ideological connections between one of the models of abstract computability, search computability, and enumeration reducibility. Soskov and his students [72, 73, 74, 77] develop the theory of regular enumerations and apply it to the enumeration degrees, obtaining a series of new results, mainly in relation to the enumeration jump.

The relationship between enumeration degrees and abstract models of computability inspires a new direction in the field of computable structure theory. Computable structure theory uses the notions and methods of computability theory in order to find the effective contents of some mathematical problems and constructions. One of the fundamental problems is to characterize the abstract structures from the point of view of their computability theoretic complexity and definability strength. A well studied measure of the computability theoretic complexity of a given structure is the notion of Turing degree spectrum. The Turing degree spectrum, introduced by Jockusch and Richter [60, 61], is the set of all Turing degrees of the diagrams of the representations (the isomorphic copies) of the structure. In recent years the Sofia school in computability lead by Soskov has been exploring computable structure theory in the more general setting obtained by considering partial structures with the underlying computation model given by enumeration reducibility and measure of complexity given by their enumeration degree spectra. In this article we will outline this line of research.

2. Enumeration reducibility

Enumeration reducibility gives a general way to compare the positive information in two sets of natural numbers. It is introduced by Friedberg and Rogers [20] in 1959. Enumeration reducibility relates to relative partial recursiveness in the same way that Turing reducibility relates to relative μ-recursiveness, the reducibility that captures both positive and negative information between two sets.

A set A is enumeration reducible to a set B if there is an effective uniform way, given by an *enumeration operator*, to obtain an enumeration of A given any enumeration of B. The enumeration operators are interesting in themselves, as they give the semantics of the type free λ-calculus in graph models, suggested by Plotkin [59] in 1972. The interest in enumeration reducibility is also supported by the fact that the structure of the enumeration degrees contains the structure of the Turing degrees without being elementary equivalent to it. Contemporary definability results [8, 30, 29, 92] in the theory of the enumeration degrees show that the structure is useful for the study of the structure of Turing degrees.

Definition 1. *Let A and B be sets of natural numbers. The set A is* enumeration reducible to *the set B, written $A \leq_e B$, if there is a c.e. set W, such that:*

$$A = W(B) = \{x \mid (\exists D)[\langle x, D\rangle \in W \ \&\ D \subseteq B]\},$$

where D is a finite set coded in the standard way.

The definition above associates an effective operator on sets to every c.e. set, the aforementioned enumeration operator. The set $A \oplus B = \{2n \mid n \in A\} \cup \{2n+1 \mid n \in B\}$ is a least upper bound of A and B with respect to $\leq_e$. Two sets A and B are *enumeration equivalent* ($A \equiv_e B$) if $A \leq_e B$ and $B \leq_e A$. The equivalence class of a set A under this relation is its *enumeration degree* $d_e(A)$. The set $\mathcal{D}_e$ consisting of all enumeration degrees, together with the naturally induced partial order and least upper bound operation is the *upper semi-lattice of the enumeration degrees*. It has a least element $\mathbf{0}_e$ consisting of all computably enumerable sets.

Let $A^+ = A \oplus \overline{A}$. The set A^+ codes in a positive way the positive and negative information about a set A. This suggests a relationship between Turing reducibility, enumeration reducibility and the relation "c.e. in", formally expressed as follows.

Proposition 1. *Let A and B be sets of natural numbers.*

(1) *$A \leq_T B$ if and only if $A^+ \leq_e B^+$.*
(2) *A is c.e. in B if and only if $A \leq_e B^+$.*

A set A is called *total* if and only if $A \equiv_e A^+$. Examples of total sets are the graphs of total functions. Proposition 1 gives rise to a natural embedding of the Turing degrees into the enumeration degrees $\iota : \mathcal{D}_T \to \mathcal{D}_e$, defined by $\iota(d_T(A)) = d_e(A^+)$ [49, 57]. An enumeration degree is *total* if it contains a total set. The enumeration degrees in the range of ι coincide with the total enumeration degrees.

The pioneering work on the enumeration degrees dates back to Case [9] and Medvedev [49]. In particular, Case shows that $\mathcal{D}_e$ is not a lattice as a consequence of the exact pair theorem and Medvedev proves the existence of quasi-minimal degrees: a degree is *quasi-minimal* if it bounds no nonzero total enumeration degree. The following theorem by Selman shows that the total enumeration degrees play an important role in the structure: an enumeration degree can be characterized by the set of total degrees above it.

Theorem 1. [64] *For any $A, B \subseteq \mathbb{N}$ the following are equivalent:*

(1) *$A \leq_e B$;*
(2) *$\{X \mid B$ is c.e. in $X\} \subseteq \{X \mid A$ is c.e. in $X\}$;*
(3) *$\{\mathbf{x} \in \mathcal{D}_e \mid \mathbf{x}$ is total & $d_e(B) \leq \mathbf{x}\} \subseteq \{\mathbf{x} \in \mathcal{D}_e \mid \mathbf{x}$ is total & $d_e(A) \leq \mathbf{x}\}$.*

Finally, we give the definition of a jump operator for the enumeration degrees, originally due to Cooper and studied by McEvoy [12, 48]. Let $E_A = \{\langle e, x\rangle \mid x \in W_e(A)\}$. The set $A' = E_A^+$ is called the enumeration jump of A and $d_e(A)' = d_e(A')$. The enumeration jump is monotone and agrees with the Turing jump J_T in the following sense: $J_T(A)^+ \equiv_e (A^+)'$.

We will use Soskov's jump inversion theorem for the enumeration jump:

Theorem 2. [73] *For every enumeration degree $\mathbf{a}$ there exists a total enumeration degree $\mathbf{b}$, such that $\mathbf{a} \leq \mathbf{b}$ and $\mathbf{a}' = \mathbf{b}'$.*

We can iterate the enumeration jump along all computable ordinals. We will identify every ordinal with its notation. In particular we will write $\alpha < \beta$ instead of $\alpha <_o \beta$. If α is a limit ordinal then by $\{\alpha(p)\}_{p\in\mathbb{N}}$ we will denote the unique strongly increasing sequence of ordinals with limit α, determined by the notation of α, and write $\alpha = \lim \alpha(p)$. For every computable ordinal α the α-th iteration of the enumeration jump $\mathbf{a}^{(\alpha)}$ is defined in a way similar to that one used in the definition the α-th iteration of the Turing jump, see [74]. Let $A^{(\alpha+1)} = (A^{(\alpha)})'$, and if $\alpha = \lim \alpha(p)$ is a limit ordinal then $A^{(\alpha)} = \{\langle p, x\rangle \mid x \in A^{(\alpha(p))}\}$. Again it turns out that both definitions are consistent on the total enumeration degrees. Using the technique of regular enumerations Soskov and Baleva extend Theorem 2 for the computable ordinals. Here is a simple version of their result.

Theorem 3. [74] *Let B be a set of natural numbers and let Q be a total set, such that $Q \geq_e B^{(\alpha)}$. Let A be such that $A^+ \leq_e Q$ and $A \not\leq_e B^{(\beta)}$ for some $\beta < \alpha$. There exists a total set F such that:*

(1) $B \leq_e F$,
(2) $A \not\leq_e F^{(\beta)}$, *and*
(3) $F^{(\alpha)} \equiv_e Q$.

3. Enumeration degree spectra

The enumeration degree spectrum $DS(\mathfrak{A})$ of a countable structure $\mathfrak{A}$ is introduced by Soskov [75] as the set of all enumeration degrees generated by the presentations (homomorphic copies) of $\mathfrak{A}$ on the set of the natural numbers. Let $\mathfrak{A} = (\mathbb{N}; R_1, \dots, R_k)$ be a countable relational structure. Here we consider the relations as sets instead of zero-one-valued functions. In the context of enumeration reducibility this corresponds to partial functions, i.e. the relations are true on certain elements and not defined on others. As $\mathfrak{A}$ is countable we may assume that the domain of $\mathfrak{A}$ is $\mathbb{N}$. An enumeration of $\mathfrak{A}$ is a total surjective mapping of $\mathbb{N}$ onto $\mathbb{N}$. Given an enumeration f of $\mathfrak{A}$ and a subset of A of $\mathbb{N}^a$, let

$$f^{-1}(A) = \{\langle x_1, \dots, x_a\rangle \mid (f(x_1), \dots, f(x_a)) \in A\}.$$

Denote by $f^{-1}(\mathfrak{A}) = f^{-1}(R_1) \oplus \dots \oplus f^{-1}(R_k) \oplus f^{-1}(=) \oplus f^{-1}(\neq)$. If f is the identity then we refer to $f^{-1}(\mathfrak{A})$ as $D(\mathfrak{A})$—the positive atomic diagram of $\mathfrak{A}$.

Definition 2. [75] *The* enumeration degree spectrum of $\mathfrak{A}$ *is the set*

$$DS(\mathfrak{A}) = \{d_e(f^{-1}(\mathfrak{A})) \mid f \text{ is an enumeration of } \mathfrak{A}\}.$$

If $\mathbf{a}$ *is the least element of* $DS(\mathfrak{A})$*, then* $\mathbf{a}$ *is called* the enumeration degree (e-degree) of $\mathfrak{A}$.

One noticeable difference with the standard definition of Turing degree spectra is that in the definition of the enumeration spectra we use the surjective enumerations, instead of bijective enumerations. Consider the structure $\mathfrak{A} = (\mathbb{N}; =, \neq)$ if we define the degree spectrum of $\mathfrak{A}$ by taking into account only the bijective enumerations, then it will be equal to $\{\mathbf{0}_e\}$, while if we take all surjective enumerations, then $DS(\mathfrak{A})$ will consist of all total enumeration degrees. Fortunately, this difference does not affect the notion of e-degree of a structure since for every enumeration f of $\mathfrak{A}$ there exists a bijective enumeration g of $\mathfrak{A}$ such that $g^{-1}(\mathfrak{A}) \leq_e f^{-1}(\mathfrak{A})$. On the other hand it allows us to show that the enumeration degree spectrum is always *closed upwards with respect to total degrees*, i.e. if $\mathbf{a} \in DS(\mathfrak{A})$, $\mathbf{b}$ is a total e-degree and $\mathbf{a} \leq \mathbf{b}$ then $\mathbf{b} \in DS(\mathfrak{A})$. This can be seen as follows: if g is an enumeration of $\mathfrak{A}$ and F is a total set such that $g^{-1}(\mathfrak{A}) \leq_e F$ then we can define a new enumeration f of $\mathfrak{A}$, which mimics g on the even numbers: $f(n/2) = g(n)$ and codes F on the odd numbers, by mapping all of them to one of two distinct members of $\mathfrak{A}$ depending on membership in F. In general, however, the enumeration degree spectra are not closed upwards as we shall see next.

Just like Turing reducibility can be expressed in terms of enumeration reducibility, the Turing degree spectrum [61, 40] of a structure $\mathfrak{A}$ corresponds to the enumeration degree spectrum of a structure, denoted by $\mathfrak{A}^+$, which codes in a positive way both the positive and negative facts about the predicates in $\mathfrak{A}$. If $\mathfrak{A} = (\mathbb{N}, R_1, \dots, R_k)$ then let $\mathfrak{A}^+ = (\mathbb{N}, R_1, \dots, R_k, \overline{R}_1, \dots, \overline{R}_k)$. The image of the Turing degree spectrum of $\mathfrak{A}$ is exactly $DS(\mathfrak{A}^+)$.

Note, that $DS(\mathfrak{A}^+)$ consists only of total enumeration degrees. A structure $\mathfrak{A}$ is called *total* if for every enumeration f of $\mathfrak{A}$ the set $f^{-1}(\mathfrak{A})$ is total. In general, if $\mathfrak{A}$ is a total structure then $DS(\mathfrak{A}) = \iota(DS_T(\mathfrak{A}))$, so if $\mathfrak{A}$ is a total structure then $\mathfrak{A}$ and $\mathfrak{A}^+$ have the same enumeration degree spectrum. Note that, however, not all structures whose degree spectrum consist only of total enumeration degrees are total. Consider for example, the structure $\mathfrak{A} = (\mathbb{N}; G_S, K)$, where G_S is the graph of the successor function and K is the halting set. Then $DS(\mathfrak{A})$ consists of all total degrees. On the other hand if $f = \lambda x.x$, then $f^{-1}(\mathfrak{A})$ is a c.e. set. Hence $\overline{K} \not\leq_e f^{-1}(\mathfrak{A})$. Clearly $\overline{K} \leq_e (f^{-1}(\mathfrak{A}))^+$, so $f^{-1}(\mathfrak{A})$ is not a total set.

A natural question arises here: if $DS(\mathfrak{A})$ consists of total degrees does there exist a total structure $\mathfrak{B}$ such that $DS(\mathfrak{A}) = DS(\mathfrak{B})$? In his last paper [81] Soskov proves the following general result, giving a much stronger relationship between Turing degree spectra and enumeration degree spectra:

Theorem 4. [81] *For every structure $\mathfrak{A}$ there exists a total structure $\mathfrak{M}$ such that* $DS(\mathfrak{M}) = \{\mathbf{a} \mid \mathbf{a} \text{ is total } \wedge (\exists \mathbf{x} \in DS(\mathfrak{A}))(\mathbf{x} \leq \mathbf{a})\}$.

We will return to explain the methods developed for the proof of this result in the last section of this paper. Here we turn to some important examples of degree spectra.

Slaman [67] and independently Wehner [101] give an example of a structure whose Turing degree spectrum consists of all nonzero Turing degrees. Translated into our terms this gives a structure $\mathfrak{A}$ such that $DS(\mathfrak{A}) = \{\mathbf{a} \mid \mathbf{a}$ is total and $\mathbf{0}_e < \mathbf{a}\}$. Kalimullin [39], building on Wehner's result, transfers these ideas to enumeration degree spectra.

Theorem 5. [39] *There is a structure $\mathfrak{A}$ such that* $DS(\mathfrak{A}) = \{\mathbf{a} \mid \mathbf{a} \in \mathcal{D}_e \mathbin{\&} \mathbf{a} > \mathbf{0}_e\}$.

Kalimullin has a different definition of enumeration degree spectra : for a countable structure $\mathfrak{A}$ he considers the set of the enumeration degrees of full diagrams of isomorphic copies of $\mathfrak{A}$ with domain *a subset of* $\mathbb{N}$. He denotes this set by $e\text{-}SP(\mathfrak{A})$ and shows that for every structure $\mathfrak{A}$ there is a structure $P(\mathfrak{A})$ such that $DS(P(\mathfrak{A}))$ is the upwards closure of $e\text{-}SP(\mathfrak{A})$ in the enumeration degrees.

Following Knight [40] we define the α-th jump spectrum and α-th jump degree of a structure for computable ordinals α:

Definition 3. *Let $\alpha < \omega_1^{CK}$. Then the α-th jump spectrum of $\mathfrak{A}$ is the set*

$$DS_\alpha(\mathfrak{A}) = \{d_e(f^{-1}(\mathfrak{A})^{(\alpha)}) \mid f \text{ is an enumeration of } \mathfrak{A}\}.$$

If $\mathbf{a}$ is the least element of $DS_\alpha(\mathfrak{A})$, then $\mathbf{a}$ is called the α-th jump degree *of $\mathfrak{A}$.*

We will leave examples of structures with or without α-th jump degree for Section 6, where we also investigate the possibilities of defining the jump of a structure. Next we consider the co-spectrum of a structure, a characteristic that plays especially well with enumeration degrees.

4. Co-spectra

Let $\mathcal{A}$ be a nonempty set of enumeration degrees the *co-set of* $\mathcal{A}$ is the set $co(\mathcal{A})$ of all lower bounds of $\mathcal{A}$. Namely

$$co(\mathcal{A}) = \{\mathbf{b} \mid \mathbf{b} \in \mathcal{D}_e \ \&\ (\forall \mathbf{a} \in \mathcal{A})(\mathbf{b} \leq \mathbf{a})\}.$$

For every $\mathcal{A} \subseteq \mathcal{D}_e$ the set $co(\mathcal{A})$ is a countable ideal. We will see that every countable ideal can be represented as co-set of the spectrum of some structure $\mathfrak{A}$.

Definition 4. *Let $\mathfrak{A}$ be a countable relational structure.*

(1) *The* co-spectrum $CS(\mathfrak{A})$ *of a structure $\mathfrak{A}$ is the co-set of $DS(\mathfrak{A})$, i.e. the set of all lower bounds of the enumeration degree spectrum of the structure $\mathfrak{A}$. If $CS(\mathfrak{A})$ has a greatest element, then it is the* co-degree *of $\mathfrak{A}$.*
(2) *For every $\alpha < \omega_1^{CK}$ the co-set of $DS_\alpha(\mathfrak{A})$ is $CS_\alpha(\mathfrak{A})$,* the α-th jump co-spectrum *of $\mathfrak{A}$. If $CS_\alpha(\mathfrak{A})$ has a greatest element, then it is the* α-th jump co-degree *of $\mathfrak{A}$.*

4.1. **Examples.** If a structure $\mathfrak{A}$ has a degree $\mathbf{a}$ then $\mathbf{a}$ is also its co-degree. The reverse is not always true. We have already seen one such example: Kalimullin's structure $\mathfrak{A}$ with degree spectrum $DS(\mathfrak{A})$ consisting of all nonzero enumeration degrees clearly has no enumeration degree, but has co-degree $\mathbf{0}_e$. As a second example, consider Richter's [61] result on linear orderings: the Turing degree spectrum $DS_T(\mathfrak{A})$ always contains a minimal pair. Thus the co-degree of $DS(\mathfrak{A}^+)$ is always $\mathbf{0}_T$, and non-computable linear orderings have co-degree but no degree. (In fact, Richter gives conditions in terms of enumeration reducibility for when a first order theory has a model with no degree). Knight [40] extends Richter's result to show that the only possible first jump Turing degree of a linear ordering is $\mathbf{0}'_T$. An analysis of her proof shows that the first jump co-spectrum of a linear ordering consists of all Σ^0_2 enumeration degrees, and so the first jump co-degree is always $\mathbf{0}'_e$, even though not every linear ordering has a first jump degree.

There are also structures with no co-degree. For example, consider $\mathfrak{A} = (\mathbb{N}; G_\Psi, P)$, where Ψ is a function such that $\Psi(\langle n, x\rangle) = \langle n, x+1\rangle$ and the relation $P(x)$ is defined and true if $(\exists t)(x = \langle 0, t\rangle)$ or $(\exists n)(\exists t)(x = \langle n+1, t\rangle \ \&\ t \in \emptyset^{(n+1)})$. For every $X \subseteq \mathbb{N}$ we have that $d_e(X) \in CS(\mathfrak{A})$ iff $(\exists n)(X \leq_e \emptyset^{(n)})$. It follows that $CS(\mathfrak{A})$ consists of all arithmetical degrees and hence has no greatest element, i.e. $\mathfrak{A}$ has no co-degree.

The co-degree of a structure is closely related to what Knight [41] and Montalbán [52] call the "enumeration degree of a structure". A set $X \subseteq \mathbb{N}$ is the "enumeration degree" of a structure $\mathfrak{A}$ if every enumeration of X computes a copy of $\mathfrak{A}$, and every copy of $\mathfrak{A}$ computes an enumeration of X. Thus by Selman's theorem the enumeration degree of X is the co-degree of the structure $\mathfrak{A}^+$. This co-degree, however has an additional property: $DS(\mathfrak{A}^+)$ is exactly the set of total enumeration degrees above $d_e(X)$. Thus, examples of structures with "enumeration degree" translate to examples of structures with co-degree and there are many of those: Given $X \subseteq \mathbb{N}$, consider the group $G_X = \bigoplus_{i \in X} \mathbb{Z}_{p_i}$, where p_i is the i-th prime number. Then G_X has "enumeration degree" X, as we can easily build G_X given any enumeration of X, and for the reverse direction, we have that $n \in X$ if and only if there is an elements $g \in G_X$ of order p_n. Montalbán [52] proves that if a class K of structures is axiomatized by some computable infinitary Π^c_2 sentence and every structure $\mathfrak{A}$ in K is existentially atomic, i.e. an atomic structure with all types generated by existential formulas, then every structure in K has "enumeration degree" given by its $\exists$-theory.

A further example of this sort is given, when one considers torsion free abelian groups of rank 1, i.e. subgroups of $(\mathbb{Q}, +, =)$. Downey and Jockusch [13] analyze the computability theoretic properties of such groups. Using results that go back to Baer, they discover a way to associate a set $S(G)$, called the characteristic of G, to every torsion free abelian group G of rank 1, so that the Turing degree spectrum of G is precisely $\{d_T(Y) \mid S(G) \text{ is c.e. in } Y\}$. On the other hand, they show that for every set of natural numbers S there is a torsion free abelian group G of rank 1, such that $S(G) \equiv_1 S$. They knew from Richter [61] that this meant that not all such groups have a degree. Coles, Downey and Slaman [11] use a forcing construction to show that, however, every such group has first jump degree.

Soskov [75] considers the problem from the point of view of enumeration reducibility. Any subgroup of the rationals can be seen as a total structure, as the only relation involved is the graph of addition, which is a total function. Let G be such a group and let $\mathbf{s}_b = d_e(S(G))$. It follows that

$$DS(G) = \{\mathbf{b} \mid \mathbf{b} \text{ is total and } \mathbf{s}_b \leq_e \mathbf{b}\}.$$

It is an easy consequence of Selman's theorem that $\mathbf{s}_b$ is the co-degree of G. Furthermore, G has degree if and only if $\mathbf{s}_b$ is total. The result of Coles, Downey and Slaman now follows from Theorem 2. There is a total enumeration degree $\mathbf{f} \geq \mathbf{s}_b$ with $\mathbf{f}' = \mathbf{s}_b'$ and so the first jump spectrum of G consists of all total enumeration degrees greater than or equal to $\mathbf{s}_b'$, in particular $\mathbf{s}_b'$ is the first jump degree of G.

Another consequence of this example is that every principal ideal of enumeration degrees is a co-spectrum of a structure, namely the co-spectrum of some torsion free abelian group of rank one. To generalize this result to arbitrary countable ideals we need a characterization of the co-spectrum of a structure.

4.2. **Normal forms.** Soskov [75] gives two characterizations of $CS_\alpha(\mathfrak{A})$ in terms of the structure $\mathfrak{A}$, one in terms of forcing and one in terms of definability. The first characterization is inspired by the fact that the members of $CS(\mathfrak{A})$ are exactly the degrees of the domains of the search computable functions ranging over the natural numbers and by the well known results by Ash, Knight, Manasse and Slaman [4] and by Chisholm [10]. We note that independently Ash and Knight [3] also characterize the elements of the co-spectrum for certain structures: they showed that for a computable structure $\mathfrak{A}$ a set $A \subseteq \mathbb{N}$ is c.e. relative to $f^{-1}(\mathfrak{A})$ for every bijective enumeration f of $\mathfrak{A}$ if and only if for some tuple $\overline{a}$ in $\mathfrak{A}$, the set A is enumeration reducible to the existential type of $\overline{a}$.

The natural forcing partial order associated with enumerations of a given structure $\mathfrak{A}$ with domain $\mathbb{N}$ consists of finite functions from $\mathbb{N}$ to $\mathbb{N}$ ordered by extension, called *finite parts*. An enumeration f of $\mathfrak{A}$ is *α-generic* for a computable ordinal α if for every computable ordinal $\beta < \alpha$ and for every set S of finite parts such that $S \leq_e D(\mathfrak{A})^{(\beta)}$ the enumeration f meets or avoids S. By transfinite induction Soskov then defines the relations $\tau \Vdash_\alpha F_e(x)$ and $\tau \Vdash_\alpha \neg F_e(x)$ for every computable ordinal α, so that if f is α-generic then $x \in (f^{-1}(\mathfrak{A}))^{(\alpha)}$ if and only if there is a finite function $\tau \preceq f$, such that $\tau \Vdash_\alpha F_e(x)$ and if f is $(\alpha+1)$-generic then $x \notin (f^{-1}(\mathfrak{A}))^{(\alpha)}$ if and only if there is a finite function $\tau \preceq f$, such that $\tau \Vdash_\alpha \neg F_e(x)$.

Definition 5. *A set $A \subseteq \mathbb{N}$ is* forcing α-definable in the structure $\mathfrak{A}$ *if there exist finite part δ and a natural number e s.t.*

$$A = \{x \mid (\exists \tau \supseteq \delta)(\tau \Vdash_\alpha F_e(x))\}.$$

Soskov shows that $CS_\alpha(\mathfrak{A})$ consists of the enumeration degrees of the forcing α-definable sets.

Theorem 6. [75] *A set $A \subseteq \mathbb{N}$ is forcing α-definable in $\mathfrak{A}$ if and only if $A \leq_e f^{-1}(\mathfrak{A})^{(\alpha)}$ for every enumeration f of $\mathfrak{A}$.*

The second characterization uses positive computable infinitary Σ_α formulas, denoted by Σ^+_α, whose structure follows that of the forcing relation. These formulas can be considered as a modification of the ones introduced by Ash [2]. Let $\mathcal{L}$ be the first order language of the structure $\mathfrak{A}$. A Σ^+_α formula with free variables among $X_1, \dots, X_l$ is a c.e. infinitary disjunction of *elementary* Σ^+_α formulas with free variables among $X_1, \dots, X_l$ which are defined by transfinite induction on α as follows. The elementary Σ^+_0 formulas are those of the form $\exists Y_1 \dots \exists Y_m \theta(X_1, \dots, X_l, Y_1, \dots, Y_m)$ where θ is a finite conjunction of atomic predicates of $\mathcal{L}$. For $\alpha = \beta+1$ an elementary Σ^+_α formula is of the form $\exists Y_1 \dots \exists Y_m \Psi(X_1, \dots, X_l, Y_1, \dots, Y_m)$, where Ψ is a finite conjunction of Σ^+_β formulas and negations of Σ^+_β formulas with free variables among $X_1, \dots, X_l, Y_1, \dots, Y_m$.

For $\alpha = \lim \alpha(p)$ a limit ordinal the elementary Σ^+_α formulas are of the form $\exists Y_1 \dots \exists Y_m \Psi(X_1, \dots, X_l, Y_1, \dots, Y_m)$, where Ψ is a finite conjunction of $\Sigma^+_{\alpha(p)}$ formulas with free variables among $X_1, \dots, X_l, Y_1, \dots, Y_m$.[1]

Definition 6. *A set $A \subseteq \mathbb{N}$ is* formally α-definable *in a structure $\mathfrak{A}$ if there exists a computable function $f(x)$ with values indices of Σ^+_α formulas $\Phi_{f(x)}$ with free variables among $W_1, \dots, W_r$ and parameters $t_1, \dots, t_r \in |\mathfrak{A}|$ such that for every natural number x the following equivalence holds:*

$$x \in A \iff \mathfrak{A} \models \Phi_{f(x)}(W_1/t_1, \dots, W_r/t_r).$$

Theorem 7. [75] *A set $A \subseteq \mathbb{N}$ is forcing α-definable in a structure $\mathfrak{A}$ iff it is formally α-definable in $\mathfrak{A}$.*

Using these normal forms, as promised, we can represent every countable ideal of enumeration degrees I as the co-spectra of a structure. Fix such an ideal I, and let $\mathbf{b}_0 \leq \mathbf{b}_1 \leq \dots \leq \mathbf{b}_k \dots$ be a countable sequence, generating I. Fix $B_k \in \mathbf{b}_k$, for each k. Consider the structure $\mathfrak{A} = (\mathbb{N}; G_f, \sigma, =, \neq)$, where

$f(\langle i, n\rangle) = \langle i+1, n\rangle$ and $\sigma = \{\langle i, n\rangle \mid n = 2k+1 \vee n = 2k \ \&\ i \in B_k\}$.

To show that $I \subseteq CS(\mathfrak{A})$ it is sufficient to see that $B_k \leq_e g^{-1}(\mathfrak{A})$ for every enumeration g of $\mathfrak{A}$ and each k. For every x using the pre-image of G_f we can find the pre-image of the natural number $\langle x, 2k\rangle$ and enumerate x in B_k if the pre-image of $\langle x, 2k\rangle$ is in the pre-image of σ. The reverse direction requires quite a bit more work, and relies on an analysis of the formally 0-definable in $\mathfrak{A}$ sets.

Theorem 8. [75] *Every countable ideal I of enumeration degrees is a co-spectrum of a structure.*

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[1] Note, that this indexing does not quite match the usual definition of computable infinitary formulas, namely level zero in this definition corresponds to level one in the usual definition.

4.3. **Structural properties of spectra and co-spectra.** Now that we know that every countable ideal of enumeration degrees is the co-spectrum of a structure, we might wonder if we can characterize spectra in a similar way: is every set of degrees that is upwards closed with respect to total elements the enumeration spectrum of a structure? The answer is, of course, 'No'. One way to see this is via the notion of a *base* and its relationship to the existence of a degree. A subset $\mathcal{B} \subseteq \mathcal{A}$ of a set of enumeration degrees $\mathcal{A}$ is *a base of* $\mathcal{A}$ if $(\forall \mathbf{a} \in \mathcal{A})(\exists \mathbf{b} \in \mathcal{B})(\mathbf{b} \leq \mathbf{a})$. Using generic enumerations and an argument much like that used in Selman's theorem we can show the following.

Theorem 9. [75] *A structure $\mathfrak{A}$ has an e-degree if and only if $DS(\mathfrak{A})$ has a countable base.*

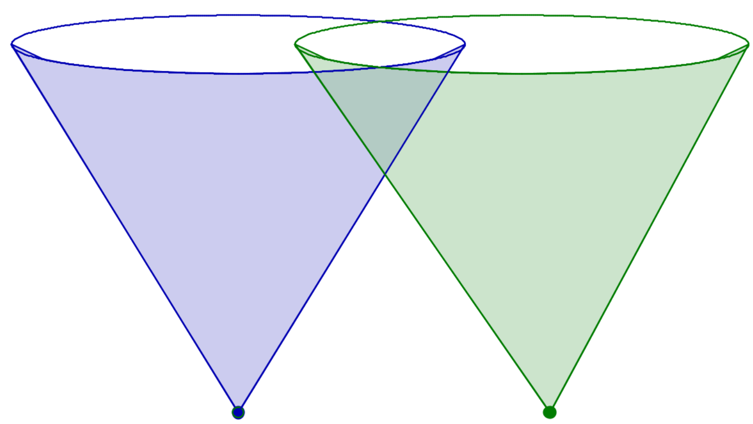

FIGURE 1. An upwards closed set with respect to total degrees which is not a degree spectra of a structure

In particular the union of two cones above incomparable degrees cannot be the enumeration degree spectrum of a structure (just like it cannot be the Turing degree spectrum of a structure). Nevertheless, degree spectra play well with co-spectra and behave structurally with respect to their elements just like the cone of total degrees above a fixed enumeration degree. This is not too surprising, as a further easy application of Selman's theorem shows that the co-spectrum of $\mathfrak{A}$ depends only on the total elements of the spectrum of $\mathfrak{A}$, i.e $CS(\mathfrak{A}) = co(DS(\mathfrak{A})_t)$, where $DS(\mathfrak{A})_t = \{\mathbf{a} \mid \mathbf{a} \text{ is total } \& \ \mathbf{a} \in DS(\mathfrak{A})\}$.

Our first more elaborate example of this phenomenon is an analogue, and in fact a generalization, of a result of Rozinas [62], stating that for every $\mathbf{a} \in \mathcal{D}_e$ there exist total $\mathbf{f_1}, \mathbf{f_2}$ below $\mathbf{a}''$ which are a minimal pair above $\mathbf{a}$.

Theorem 10. [75] *Let $\alpha < \omega_1^{CK}$ and let $\mathbf{b} \in DS_\alpha(\mathfrak{A})$. There exist total elements $\mathbf{f_0}$ and $\mathbf{f_1}$ of $DS(\mathfrak{A})$ such that :*

(1) ${\mathbf{f_0}}^{(\alpha)} \leq \mathbf{b}$ *and* ${\mathbf{f_1}}^{(\alpha)} \leq \mathbf{b}$*;*
(2) ${\mathbf{f_0}}^{(\beta)}$ *and* ${\mathbf{f_1}}^{(\beta)}$ *do not belong to* $CS_\beta(\mathfrak{A})$ *for* $\beta < \alpha$*;*

(3) $co(\{\mathbf{f_0}^{(\beta)}, \mathbf{f_1}^{(\beta)}\}) = CS_\beta(\mathfrak{A})$ *for every* $\beta + 1 < \alpha$.

This property does not hold for arbitrary sets that are upwards closed with respect to total degrees. Consider the finite lattice L consisting of the elements $\mathbf{a}$,

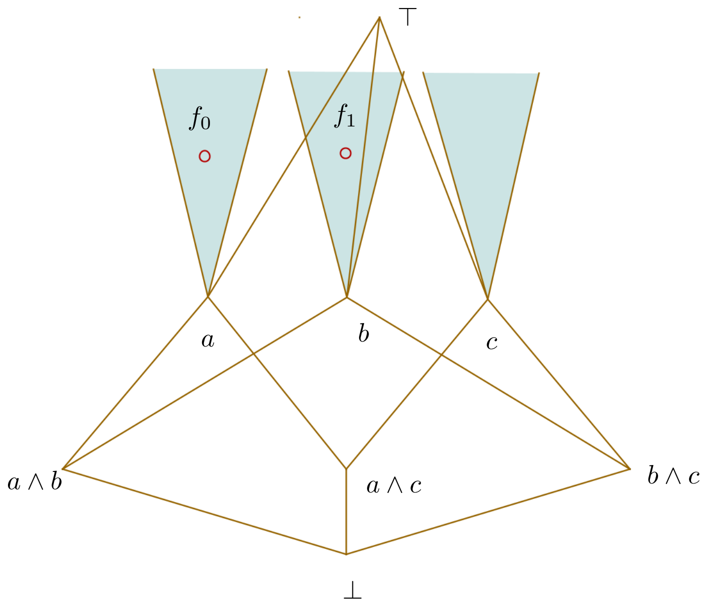

Figure 2. An upwards closed set with no minimal pair

$\mathbf{b}$, $\mathbf{c}$, $\mathbf{a} \wedge \mathbf{b}$, $\mathbf{a} \wedge \mathbf{c}$, $\mathbf{b} \wedge \mathbf{c}$, $\top$, $\bot$ such that $\top$ and $\bot$ are the greatest and the least element of L, respectively, $\mathbf{a} > \mathbf{a} \wedge \mathbf{b}$, $\mathbf{a} > \mathbf{a} \wedge \mathbf{c}$, $\mathbf{b} > \mathbf{a} \wedge \mathbf{b}$, $\mathbf{b} > \mathbf{b} \wedge \mathbf{c}$, $\mathbf{c} > \mathbf{a} \wedge \mathbf{c}$ and $\mathbf{c} > \mathbf{b} \wedge \mathbf{c}$. The lattice L can be embedded in the enumeration degrees (see for example [46]). Then $\mathcal{A} = \{\mathbf{d} \in \mathcal{D}_e \mid \mathbf{d} \geq \mathbf{a} \vee \mathbf{d} \geq \mathbf{b} \vee \mathbf{d} \geq \mathbf{c}\}$ is a set that does not satisfy the minimal pair property, because $co(\mathcal{A}) = \{\bot\}$, but no pair of elements in $\mathcal{A}$ has greatest lower bound $\bot$.

The next property is analogue of the existence of a quasi-minimal enumeration degree proved by Medvedev [49]. Let $\mathcal{A}$ be a set of enumeration degrees. The degree $\mathbf{q}$ is *quasi-minimal with respect to* $\mathcal{A}$ if:

- $\mathbf{q} \notin co(\mathcal{A})$.
- If $\mathbf{a}$ is total and $\mathbf{a} \geq \mathbf{q}$, then $\mathbf{a} \in \mathcal{A}$.
- If $\mathbf{a}$ is total and $\mathbf{a} \leq \mathbf{q}$, then $\mathbf{a} \in co(\mathcal{A})$.

Theorem 11. [75] *For every structure* $\mathfrak{A}$ *there exists a quasi-minimal with respect to* $DS(\mathfrak{A})$ *degree.*

To prove this theorem Soskov introduces the notion of a *partial generic enumeration* φ of $\mathfrak{A}$, generic enumeration in the forcing partial order consisting of finite functions from $\mathbb{N}$ to $\mathbb{N} \cup \{\bot\}$, where $\bot$ represents partiality. He then shows that if φ is a partial generic enumeration of $\mathfrak{A}$ then $d_e(\varphi^{-1}(\mathfrak{A}))$ is quasi-minimal with respect to $DS(\mathfrak{A})$.

Since every countable ideal of enumeration degrees is a co-spectrum of a structure as a corollary we receive a result of Slaman and Sorbi :

Corollary 1. [68] *Let I be a countable ideal of enumeration degrees. There exists an enumeration degree* $\mathbf{q}$ *such that*

(1) *If* $\mathbf{a} \in I$ *then* $\mathbf{a} <_e \mathbf{q}$.
(2) *If* $\mathbf{a}$ *is total and* $\mathbf{a} \leq_e \mathbf{q}$ *then* $\mathbf{a} \in I$.

The technique of partial generic enumerations is further developed by Ganchev, Soskov and A. Soskova in [22, 24, 84]. Soskov and A. Soskova also investigate further properties of the notion of a quasi-minimal degree in [91]. They show that for every countable structure $\mathfrak{A}$ there are uncountably many quasi-minimal degrees with respect to $DS(\mathfrak{A})$. The proof relies on a diagonalization: for every countable sequence $\{X_i\}$ of sets that are not forcing 0-definable, (such as the members of a quasi-minimal degree), there is a partial generic enumeration of the structure omitting all X_i. Their main find is however a characterization of the first jump spectra in terms of the jumps of quasi-minimal degrees:

Theorem 12. [91] *The first jump spectrum of every structure $\mathfrak{A}$ consists exactly of the enumeration jumps of the quasi-minimal with respect to $DS(\mathfrak{A})$ degrees.*

When one applies the theorem above to any computable structure, one obtains directly McEvoy's jump inversion theorem:

Corollary 2. [48] *For every total e-degree* $\mathbf{a} \geq_e \mathbf{0}'_e$ *there is a quasi-minimal degree* $\mathbf{q}$ *with* $\mathbf{q}' = \mathbf{a}$.

The final property of quasi-minimal degrees that we will mention here, is inspired by the well-known fact from enumeration degree theory, which states that every total enumeration degree is the least upper bound of two quasi-minimal e-degrees. One way to see this is to go through Jockusch's semi-recursive sets. Recall that a set is *semi-recursive* if it is a left cut in some computable linear ordering. Jockusch [37] showed that every nonzero Turing degree contains a semi-recursive set A, such that both A and $\overline{A}$ are not c.e. In the context of enumeration reducibility this translates to: every total enumeration degree $\mathbf{a}$ is the least upper bound of two nonzero e-degrees $d_e(A)$ and $d_e(\overline{A})$, where A is a semi-recursive set. Arslanov, Cooper and Kalimullin [1] showed that if A is a semi-recursive set such that A and $\overline{A}$ are not c.e., then the e-degrees of A and its complement $\overline{A}$ are quasi-minimal. If we restrict our attention only to total degrees above $\mathbf{0}'_e$ then once again, this property turns out to be a special case of a general fact about quasi-minimal degrees of structures:

Theorem 13. [91] *For every element* $\mathbf{a}$ *of the jump spectrum of a structure* $\mathfrak{A}$ *there exist quasi-minimal with respect to* $DS(\mathfrak{A})$ *degrees* $\mathbf{p}$ *and* $\mathbf{q}$ *such that* $\mathbf{a} = \mathbf{p} \vee \mathbf{q}$.

5. Abstract generalized enumeration reducibilities

5.1. **Definability on a structure.** Another way to characterize the complexity of a structure $\mathfrak{A}$ is to analyze the definable sets in $\mathfrak{A}$. This gives a finer measure as it may happen that two structures have the same degree spectra but greatly differ in their definability power and model theoretic properties. Let α be a computable ordinal and $A = |\mathfrak{A}|$. A set $B \subseteq A^a$ is $\Sigma^c_{\alpha+1}$ definable on a structure $\mathfrak{A}$ if there is a computable infinitary $\Sigma^c_{\alpha+1}$ formula $\varphi(\bar{X}, \bar{Z})$ and parameters $\bar{t} \in A$ such that $B = \{\bar{s} \mid \mathfrak{A} \models \varphi(\bar{s}, \bar{t})\}$. A set $B \subseteq A^a$ is relatively intrinsically $\Sigma^0_{\alpha+1}$ in a structure $\mathfrak{A}$ if for each $(\mathfrak{B}, X) \simeq (\mathfrak{A}, B)$ the set X is $\Sigma^0_{\alpha+1}$ in the atomic diagram $D(\mathfrak{B})$, which in our terms means that $f^{-1}(B) \leq_e f^{-1}(\mathfrak{A}^+)^{(\alpha)}$ for every enumeration f

of $\mathfrak{A}$. Ash, Knight, Manasse and Slaman [4] and independently Chisholm [10] prove that these two notions coincide. Soskov and Baleva [76] give an analogue of the relatively intrinsically Σ^0_α sets on a structure $\mathfrak{A}$ from the point of view of enumeration reducibility: For every computable ordinal α a set $B \subseteq A^a$, is *relatively α-intrinsic* on the structure $\mathfrak{A}$ if for every enumeration f of $\mathfrak{A}$ the set $f^{-1}(B)$ is enumeration reducible to $(f^{-1}(\mathfrak{A}))^{(\alpha)}$. Soskov and Baleva show that the α-intrinsic sets are exactly the ones definable by computable inifinitary $\Sigma^+_{\alpha+1}$ formulas with parameters.

Having moved to this setting, they go one step further and consider the following generalization in the spirit of Ash [2]. For two subsets B and C of A and two computable ordinals α and β Ash defines that B is relatively α, β-intrinsic on $\mathfrak{A}$ with respect to C if for all enumerations f such that $f^{-1}(C)$ is enumeration reducible to $f^{-1}(\mathfrak{A})^{(\beta)}$, $f^{-1}(B)$ is enumeration reducible to $f^{-1}(\mathfrak{A})^{(\alpha)}$. In other words, consider not all enumerations of $\mathfrak{A}$ but only those enumerations which "assume" that B is relatively β-intrinsic. Soskov and Baleva generalized this notion with respect to a sequence of sets $\{B_\gamma\}_{\gamma\leq\zeta}$ of subsets of A.

Definition 7. *A subset B of A^a is* relatively α-intrinsic on $\mathfrak{A}$ with respect to the sequence $\mathcal{B} = \{B_\gamma\}_{\gamma\leq\zeta}$ *if for every enumeration f of $\mathfrak{A}$ such that*

$(\forall\gamma \leq \zeta)(f^{-1}(B_\gamma) \leq_e (f^{-1}(\mathfrak{A}))^{(\gamma)})$ *uniformly in γ, the set $f^{-1}(B)$ is enumeration reducible to $(f^{-1}(\mathfrak{A}))^{(\alpha)}$.*

The authors give a normal form of these sets first in terms of a forcing construction. To give a syntactic characterization, they redefine the infinitary computable $\Sigma^+_{\alpha+1}$ formulas, taking into account the sequence $\mathcal{B}$. For every γ they add a new unary predicate P_γ for the set B_γ. This predicate is included positively at level γ of the hierarchy. For example for $\alpha = \beta + 1$ an elementary Σ^+_α formula is in the form $\exists Y_1 \dots \exists Y_m \Psi(X_1, \dots, X_l, Y_1, \dots, Y_m)$, where Ψ is a finite conjunction of $P_\alpha(X_i)$ or $P_\alpha(Y_j)$ or Σ^+_β formulas and negations of Σ^+_β formulas with free variables among $X_1, \dots, X_l, Y_1, \dots, Y_m$.

Theorem 14. [76] *A subset B of A^a is relatively α-intrinsic on $\mathfrak{A}$ with respect to the sequence $\mathcal{B} = \{B_\gamma\}_{\gamma\leq\zeta}$ if and only if B is definable in $\mathfrak{A}$ by a computable infinitary Σ^+_α-formula with parameters, constructed with respect to the sequence $\mathcal{B}$.*

The authors also give an abstract version of the Theorem 3. To formulate it we need the following definition :

Definition 8. *For any computable ordinal $\alpha \leq \zeta$ the jump sequence $\mathcal{P}(\mathcal{B}) = \{\mathcal{P}_\alpha\}_{\alpha<\zeta}$ of the sequence $\mathcal{B}$ is defined inductively as follows:*

- *$\mathcal{P}_0 = B_0$, for $\alpha = 0$;*
- *$\mathcal{P}_\alpha = (\mathcal{P}_\beta)' \oplus B_\alpha$, for $\alpha = \beta + 1$;*
- *For $\alpha = \lim \alpha(p)$, denote by $\mathcal{P}_{<\alpha} = \{\langle p, x\rangle : x \in \mathcal{P}_{\alpha(p)}\}$ and let $\mathcal{P}_\alpha = \mathcal{P}_{<\alpha} \oplus B_\alpha$.*

The abstract jump inversion says that for every $B \subseteq A$ which is not Σ^+_α- definable on $\mathfrak{A}$ and each total set $Q \geq_e A^+ \oplus \mathcal{P}_\xi$, where $\xi = \max(\alpha + 1, \zeta)$ there exists an enumeration f of $\mathfrak{A}$ satisfying the following conditions: $f \leq_e Q$, the enumeration degree of $f^{-1}(\mathfrak{A})$ is total, for all $\gamma \leq \zeta$, $f^{-1}(B_\gamma) \leq_e (f^{-1}(\mathfrak{A}))^{(\gamma)}$ uniformly in γ, $(f^{-1}(\mathfrak{A}))^{(\xi)} \equiv_e Q$ and $f^{-1}(B) \not\leq_e (f^{-1}(\mathfrak{A}))^{(\alpha)}$.

5.2. **Joint spectra and Relative spectra.** The results described so far lead Soskov and A. Soskova to the goal of generalizing the notion of degree spectrum of a structure to the degree spectrum of sequences of structures. Initially, they consider the case when the sequence is finite and introduce two generalizations: the joint spectrum [82, 83, 84] and the relative spectrum [85, 86].

Fix countable structures $\mathfrak{A}_0, \dots, \mathfrak{A}_n$.

Definition 9. *The* joint spectrum *of* $\mathfrak{A}_0, \dots, \mathfrak{A}_n$ *is the set* $DS(\mathfrak{A}_0, \mathfrak{A}_1, \dots, \mathfrak{A}_n) = \{\mathbf{a} \mid \mathbf{a} \in DS(\mathfrak{A}_0), \mathbf{a}' \in DS(\mathfrak{A}_1), \dots, \mathbf{a}^{(\mathbf{n})} \in DS(\mathfrak{A}_n)\}$.

So, the joint spectrum is the set of all enumeration degrees of the $DS(\mathfrak{A}_0)$, such that for all $i \leq n$ their ith enumeration jump is in $DS(\mathfrak{A}_i)$. The k-th jump joint spectrum $DS_k(\mathfrak{A}_0, \dots, \mathfrak{A}_n)$ and the k-th co-spectrum are defined similarly to $DS_k(\mathfrak{A})$ and $CS_k(\mathfrak{A})$. In this case as well $DS_k(\mathfrak{A}_0, \dots, \mathfrak{A}_n)$ is closed upwards with respect to total degrees. The k-th co-spectrum of the sequence $\mathfrak{A}_0, \dots, \mathfrak{A}_n$ depends only on the first k members.

Theorem 15. *For every $k \leq n$ we have that $CS_k(\mathfrak{A}_0, \dots, \mathfrak{A}_k) = CS_k(\mathfrak{A}_0, \dots, \mathfrak{A}_n)$. Moreover for every set B of natural numbers $d_e(B) \in CS_k(\mathfrak{A}_0, \dots, \mathfrak{A}_k)$ if and only if for every $k+1$ enumerations $f_0, \dots, f_k$, of $\mathfrak{A}_0, \dots, \mathfrak{A}_k$ respectively, the set $B \leq_e \mathcal{P}(f_0^{-1}(\mathfrak{A}_0), \dots, f_k^{-1}(\mathfrak{A}_k))$.*

Here $\mathcal{P}(f_0^{-1}(\mathfrak{A}_0), \dots, f_k^{-1}(\mathfrak{A}_k))$ is the kth jump sequence of the given sequence.

Soskov and A. Soskova [82] give a syntactical normal form for the members of the degrees in the set $CS_k(\mathfrak{A}_0, \dots, \mathfrak{A}_n)$. This time they use many-sorted Σ_k^+ infinitary computable formulas with different sorts for every structure $\mathfrak{A}_i$. A. Soskova [83, 84] shows that the structural properties of co-spectra are preserved. The analog of the minimal pair theorem holds here as well: for any sequence of structures $\mathfrak{A}_0, \dots, \mathfrak{A}_n$, there exist enumeration degrees $\mathbf{f}$ and $\mathbf{g}$ in $DS(\mathfrak{A}_0, \dots, \mathfrak{A}_n)$, such that for any enumeration degree $\mathbf{a}$ and $k \leq n$:

$$\mathbf{a} \leq \mathbf{f}^{(k)} \ \& \ \mathbf{a} \leq \mathbf{g}^{(k)} \Rightarrow \mathbf{a} \in CS_k(\mathfrak{A}_0, \dots, \mathfrak{A}_n).$$

Furthermore, A. Soskova proves the existence of quasi-minimal degree $\mathbf{q}$ with respect to $DS(\mathfrak{A}_0, \mathfrak{A}_1, \dots, \mathfrak{A}_n)$. The proof techniques are based on regular enumerations introduced in [73] and partial generic enumerations used in [75].

The second generalization defines the relative spectrum of a structure with respect to finitely many structures. Consider a structure $\mathfrak{A}$ and finitely many structures $\mathfrak{A}_1, \dots, \mathfrak{A}_n$. We will restrict the class of enumerations of $\mathfrak{A}$ to these enumerations of $\mathfrak{A}$ which “assume” that each $\mathfrak{A}_i$ is relatively intrinsically Σ^0_{i+1} in $\mathfrak{A}$: An enumeration f of $\mathfrak{A}$ is *n-acceptable* with respect to the structures $\mathfrak{A}_1, \dots, \mathfrak{A}_n$ if $f^{-1}(\mathfrak{A}_i)$ is enumeration reducible to $f^{-1}(\mathfrak{A})^{(i)}$ for each $i \leq n$.

Definition 10. *The* relative spectrum of the structure $\mathfrak{A}$ with respect to $\mathfrak{A}_1, \dots, \mathfrak{A}_n$ *is the set*

$RS(\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_n) = \{d_e(f^{-1}(\mathfrak{A})) \mid f$ *is an n-acceptable enumeration of* $\mathfrak{A}\}$.

The elements of the co-spectrum of the k-th relative spectrum are the enumeration degrees which contain a set which is enumeration reducible to the k-th jump sequence $\mathcal{P}_k^f$ of the sequence $f^{-1}(\mathfrak{A}), f^{-1}(\mathfrak{A}_1), \dots, f^{-1}(\mathfrak{A}_k)$, for every k-acceptable enumeration of $\mathfrak{A}$ with respect to the structures $\mathfrak{A}_1, \dots, \mathfrak{A}_k$. The normal form of these sets is given [85, 86] using a forcing construction. In this case as well there is an analog of the minimal pair theorem and the existence of quasi-minimal degree.

The co-spectra of the joint spectra and the relative spectra coincide, but there are examples of sequence of structures for which the k-th co-spectra for $k > 0$ differ.

As we have seen the structural properties of the degree spectra and co-spectra obtained remain true when one relativizes to consider finite sequences of structures. The main question here is whether these generalizations give rise to new sets of degrees, or is it the case that for every finite sequence of countable structures there exists one structure whose degree spectrum is exactly the relative spectrum or the joint spectrum of the given sequence. An answer to this question will be given in the last section of this paper.

5.3. **Omega-enumeration reducibility.** In 2006 Soskov initiates the study of uniform reducibility between sequences of sets and the induced structure of the ω-degrees. Soskov, Ganchev and M. Soskova obtain many results, providing substantial proof that the structure of the ω-degrees is a natural extension of the structure of the enumeration degrees, with a jump operation that has interesting properties and with natural new members, which turn out to be extremely useful for the characterization of certain classes of enumeration degrees. These investigations appear in [77, 78, 27, 26, 23, 25, 24, 28, 92, 79].

The jump class of the sequence $\mathcal{X} = \{X_n\}_{n<\omega}$ of sets of natural numbers is the set $J_{\mathcal{X}} = \{d_T(B) \mid (\forall n)(X_n \text{ is c.e. in } B^{(n)} \text{ uniformly in } n)\}$. The definition of ω-enumeration reducibility between sequences of sets is an analogue of Selman's characterization Theorem 1 of enumeration reducibility.

Definition 11. *The sequence* $\mathcal{X}$ *is* ω-enumeration reducible *to the sequence* $\mathcal{Y}$ *(*$\mathcal{X} \leq_\omega \mathcal{Y}$*) if* $J_{\mathcal{Y}} \subseteq J_{\mathcal{X}}$.

Let $\mathcal{X} = \{X_n\}_{n<\omega}$ and $\mathcal{Y} = \{Y_n\}_{n<\omega}$ be sequences of sets of natural numbers. $\mathcal{X} \leq_e \mathcal{Y}$ if for all n, $X_n \leq_e Y_n$ uniformly in n. This reducibility is useful in many considerations, however it does not quite characterize ω-enumeration reducibility. The true characterization was given by Soskov and Kovachev:

Theorem 16. [77] $\mathcal{X} \leq_\omega \mathcal{Y} \iff \mathcal{X} \leq_e \mathcal{P}(\mathcal{Y})$.

Clearly "$\leq_\omega$" is a reflexive and transitive relation on the set $\mathcal{S}$ of all sequences of sets of natural numbers and induces the equivalence relation $\equiv_\omega$. For every sequence $\mathcal{X}$ the set $d_\omega(\mathcal{X}) = \{\mathcal{Y} \mid \mathcal{Y} \equiv_\omega \mathcal{X}\}$ is the ω-enumeration degree of the sequence $\mathcal{X}$ and $\mathcal{D}_\omega = \{d_\omega(\mathcal{X}) \mid \mathcal{X} \in \mathcal{S}\}$ is the *structure of the ω-enumeration degrees.* The relation $\leq_\omega$ induces a partial ordering of $\mathcal{D}_\omega$ with least element $\mathbf{0}_\omega = d_\omega(\emptyset_\omega)$, where $\emptyset_\omega$ is the sequence with all members equal to $\emptyset$. $\mathcal{D}_\omega$ is further an upper semi-lattice, with least upper bound induced by $\mathcal{X} \oplus \mathcal{Y} = \{X_n \oplus Y_n\}_{n<\omega}$. There is a natural embedding of the enumeration degrees into the ω-enumeration degrees. Given a set A of natural numbers denote by $A \uparrow \omega$ the sequence $\{A_k\}_{k<\omega}$, where $A_0 = A$ and for all $k \geq 1$, $A_k = \emptyset$. The embedding is $\kappa : \mathcal{D}_e \to \mathcal{D}_\omega$ by $\kappa(d_e(A)) = d_\omega(A \uparrow \omega)$.

For every $\mathcal{X} \in \mathcal{S}$ *the ω-enumeration jump of* $\mathcal{X}$ is $\mathcal{X}' = \{\mathcal{P}_{n+1}(\mathcal{X})\}_{n<\omega}$. We have that $J'_{\mathcal{X}} = \{\mathbf{a}' \mid \mathbf{a} \in J_{\mathcal{X}}\}$. The jump operator is monotone and it induces a jump operation on the ω-enumeration degrees. It agrees with the jump operation on $\mathcal{D}_e$ and the embedding κ. It turns out that the ω-enumeration degrees behave in an unusual way with respect to the considered jump operation. In [27] Soskov and Ganchev prove the following strong jump inversion theorem: for every $n \in \mathbb{N}$ and for $\mathbf{a}^{(n)} \leq \mathbf{b}$ there exists a **least** ω-enumeration degree $\mathbf{x} \geq \mathbf{a}$ such that $\mathbf{x}^{(n)} = \mathbf{b}$. So we can define an operation $I^n_{\mathbf{a}}$ on the upper cone with a least element $\mathbf{a}^{(n)}$

such that $I^n_{\mathbf{a}}(\mathbf{b})$ is the least solution $\mathbf{x}$ of this system: $\mathbf{x} \geq \mathbf{a}$ such that $\mathbf{x}^{(n)} = \mathbf{b}$. Let $\mathbf{o}_n = I^n_{\mathbf{0}_\omega}(\mathbf{0}_\omega^{(n+1)})$, i.e. $\mathbf{o}_n$ denotes the least ω-enumeration degree, such that $\mathbf{o}_n^{(n)} = \mathbf{0}_\omega^{(n+1)}$. We have $\mathbf{0}'_\omega = \mathbf{o}_0 \geq \mathbf{o}_1 \geq \cdots \geq \mathbf{o}_n \geq \ldots$. The sequence is strictly decreasing but it does not converge to the least degree $\mathbf{0}_\omega$. The authors proved the existence of almost zero nontrivial degrees which are nonzero and below all $\mathbf{o}_n$. A nontrivial almost zero ω-enumeration degree contains a sequence $\mathcal{R}$ such that $(\forall n)(\mathcal{P}_n(\mathcal{R}) \equiv_e \emptyset^{(n)})$, but non-uniformly.

A. Soskova [89] generalizes the enumeration degree spectrum with respect to an infinite sequences of sets using ω-enumeration reducibility. Let $\mathcal{B} = \{B_n\}_{n<\omega}$ be a sequence of sets of natural numbers and $\mathfrak{A}$ be a countable structure on the natural numbers.

Definition 12. The ω-degree spectrum *of the structure* $\mathfrak{A}$ *with respect to the sequence* $\mathcal{B}$ *is the set*

$$DS(\mathfrak{A}, \mathcal{B}) = \{d_e(f^{-1}(\mathfrak{A})) \mid f \text{ - enumeration of } \mathfrak{A} \text{ s.t. } \{f^{-1}(B_n)\} \leq_\omega \{f^{-1}(\mathfrak{A})^{(n)}\}\}.$$

The ω-co-spectrum of $DS(\mathfrak{A}, \mathcal{B})$ is the set $Ocsp(\mathfrak{A}, \mathcal{B})$ of ω-enumeration degrees, which are lower bounds of the ω-spectrum.

Note that if $\mathcal{B}$ is the sequence of empty sets then $DS(\mathfrak{A}, \mathcal{B}) = DS(\mathfrak{A})$. The set $Ocsp(\mathfrak{A}, \mathcal{B})$ is in this case a new meaningful notion and we will denote it by $Ocsp(\mathfrak{A})$.

Most properties of co-spectra, such as the existence of minimal pairs and quasi-minimal degrees, hold for the ω-co-spectra, but not all. For every structure $\mathfrak{A}$ and $n > 0$ if $\mathbf{c} \in DS_n(\mathfrak{A})$ then $CS_n(\mathfrak{A})$ is the co-set of $\mathcal{A} = \{\mathbf{a} \mid \mathbf{a} \in DS(\mathfrak{A}) \;\&\; \mathbf{a}^{(n)} = \mathbf{c}\}$. Vatev [96] shows that there is a structure $\mathfrak{A}$, a sequence $\mathcal{B}$ and $\mathbf{c} \in DS_n(\mathfrak{A}, \mathcal{B})$ such that if $\mathcal{A} = \{\mathbf{a} \in DS(\mathfrak{A}, \mathcal{B}) \mid \mathbf{a}^{(n)} = \mathbf{c}\}$ then $CS(\mathfrak{A}, \mathcal{B}) \neq co(\mathcal{A})$.

A. Soskova gives a characterization of the k-th ω-co-spectrum of a structure (the co-set of the k-th jump ω-spectrum) in terms of definability via computable sequence $\{\Phi^{\gamma(n,x)}\}_{n,x<\omega}$ of formulas such that for every n, $\Phi^{\gamma(n,x)}$ is a Σ^+_{n+k} infinitary computable formula with parameters. This set is also characterized as the least ideal of ω-enumeration degrees containing the k-th jumps of elements of the ω-co-spectrum. The set $I = CS(\mathfrak{A}, \mathcal{B})$ is a countable ideal. By the minimal pair theorem there exist total enumeration degrees $\mathbf{f}, \mathbf{g}$ in $DS(\mathfrak{A}, \mathcal{B})$, such that $CS(\mathfrak{A}, \mathcal{B}) = I(\mathbf{f}_\omega) \cap I(\mathbf{g}_\omega)$ where $I(\mathbf{f}_\omega)$ and $I(\mathbf{g}_\omega)$ are the principal ideals of ω-enumeration degrees with greatest elements $\mathbf{f}_\omega = \kappa(\mathbf{f})$ and $\mathbf{g}_\omega = \kappa(\mathbf{g})$, the images of $\mathbf{f}$ and $\mathbf{g}$ under the embedding κ of $\mathcal{D}_e$ in $\mathcal{D}_\omega$. Denote by $I^{(k)}$ the least ideal, containing all k-th ω-jumps of the elements of I. Ganchev [23] proves that if $I = I(\mathbf{f}_\omega) \cap I(\mathbf{g}_\omega)$ then $I^{(k)} = I(\mathbf{f}_\omega^{(k)}) \cap I(\mathbf{g}_\omega^{(k)})$ for every k. But $I({\mathbf{f}_\omega}^{(k)}) \cap I(\mathbf{g}_\omega^{(k)}) = CS_k(\mathfrak{A}, \mathcal{B})$ for each k. Thus $I^{(k)} = CS_k(\mathfrak{A}, \mathcal{B})$, i.e. the k-th omega co-spectrum is a minimal ideal containing the k-th jumps of elements of the ω-co-spectrum.

Using this result Ganchev, A. Soskova and Vatev show another difference between co-spectra and ω-co-spectra: There is a countable ideal I of ω-enumeration degrees for which there is no structure $\mathfrak{A}$ and sequence $\mathcal{B}$ such that $I = CS(\mathfrak{A}, \mathcal{B})$. Let

$$\mathcal{A} = \{\mathbf{0}_\omega, \mathbf{0}'_\omega, \mathbf{0}''_\omega, \ldots, \mathbf{0}_\omega^{(n)}, \ldots\}.$$

and consider the countable ideal I generated by $\mathcal{A}$. Assume now that there is a structure $\mathfrak{A}$ and a sequence $\mathcal{B}$ such that $I = CS(\mathfrak{A}, \mathcal{B})$ and let $\mathbf{f}$ and $\mathbf{g}$ be a minimal pair of total enumeration degrees for $DS(\mathfrak{A}, \mathcal{B})$. It follows that $I^{(n)} =$

$I(\mathbf{f}_\omega^{(n)}) \cap I(\mathbf{g}_\omega^{(n)})$ for each n. But $\mathbf{f}_\omega \geq \mathbf{0}_\omega^{(n)}$ and $\mathbf{g}_\omega \geq \mathbf{0}_\omega^{(n)}$ for each n. If $F \in \mathbf{f}$ and $G \in \mathbf{g}$ are total, then $F \geq_T \emptyset^{(n)}$ and $G \geq_T \emptyset^{(n)}$ for each n. By Enderton and Putnam (1970) [15], Sacks (1971) [63] : $F'' \geq_T \emptyset^{(\omega)}$ and $G'' \geq_T \emptyset^{(\omega)}$ and hence $\mathbf{f}'' \geq_T \mathbf{0}_T^{(\omega)}$ and $\mathbf{g}'' \geq_T \mathbf{0}_T^{(\omega)}$. Then $\kappa(\iota(\mathbf{0}_T^{(\omega)})) \in I(\mathbf{f}_\omega'') \cap I(\mathbf{g}_\omega'')$, but $\kappa(\iota(\mathbf{0}_T^{(\omega)})) \notin I''$ since all elements of I'' are bounded by $\mathbf{0}_\omega^{(k+2)}$ for some k. Hence $I'' \neq I(\mathbf{f}_\omega'') \cap I(\mathbf{g}_\omega'')$, a contradiction.

Inspired by this Vatev [96] investigates the principal ideal case. He shows that for every principal ideal of ω-enumeration degrees I there is sequence $\mathcal{B}$ and a structure $\mathfrak{A}$ such that $I = CS(\mathfrak{A}, \mathcal{B})$.

6. Jump of a structure

The idea of the *jump* of a structure is first considered by Soskov and his student Baleva [5] in the context of s-reducibility between structures, a reducibility based on relative search computability. Given a structure $\mathfrak{A}$ the goal is to define a structure $\mathfrak{A}'$ so that $\mathfrak{A}'$ knows the sets definable by computable infinitary Σ_1^c formulas in $\mathfrak{A}$. The idea to define such a structure resurfaced in computable structure theory in the period 2002–2010 independently in the work of Soskov and Soskova [90], Montalbán [50] and Stukachev [93, 94]. Soskov and A. Soskova [90] define the jump $\mathfrak{A}'$ of the structure $\mathfrak{A}$ by considering the Moschovakis' extension of $\mathfrak{A}$ together with a predicate, an analogue of the halting set, which codes all sets definable by computable infinitary Σ_1^c formulas with parameters. This changes the domain of the structure, but keeps the language finite. Montalbán's approach was to keep the domain of the structure the same and to add a complete set of relations definable by computable infinitary Π_1^c formulas. This can possibly make the language infinite, however Montalbán [50, 51, 35] gives some examples of structures, such as linear orderings and Boolean algebras, where the complete set of relations is finite and natural. Stukachev's approach is in terms of Σ-definability in hereditarily finite extension of the structure. We will focus on the approach taken by Soskov and Soskova.

Let $\mathfrak{A} = (A; R_1, \dots, R_n)$ be a countable structure and let equality be among the predicates $R_1, \dots, R_s$. Following Moschovakis [55] we define an extension of $\mathfrak{A}$ as follows. Let $\bar{0}$ be a new element, such that $\bar{0} \notin A$ and let $A_0 = A \cup \{\bar{0}\}$. Let $\langle ., . \rangle$ be a pairing function such that none of the elements of A_0 is a pair. The set A^* is the closure of A_0 under $\langle ., . \rangle$ and functions $L(\langle s, t \rangle) = s$ and $R(\langle s, t \rangle) = t$ are decoding functions. We next represent the basic relations in $\mathfrak{A}^*$ by unary relations in $\mathfrak{A}^*$ as follows: $R_i^*(\langle s_1, \dots, s_{k_i} \rangle) = R_i(s_1, \dots, s_{k_i})$.

Definition 13. Moschovakis' extension [55] *of* $\mathfrak{A}$ *is the structure*

$$\mathfrak{A}^* = (A^*, R_1^*, \dots, R_n^*, A_0, G_{\langle ., . \rangle}, G_L, G_R).$$

It is straightforward to check that for any countable structure $\mathfrak{A}$ the structure $\mathfrak{A}^*$ has the same complexity as $\mathfrak{A}$, namely $DS(\mathfrak{A}) = DS(\mathfrak{A}^*)$. The advantage to considering $\mathfrak{A}^*$ is that in it we can code a copy of the natural numbers $\mathbb{N}^*$ in A^* by induction: $\bar{0}^* = \bar{0}$ and $\overline{(n+1)}^* = \langle \bar{0}, \bar{n}^* \rangle$. Using $\mathbb{N}^*$ we can now represent the graph of every finite part $\tau : \mathbb{N} \to A$ as an element τ^* of $\mathfrak{A}^*$. Let

$$K_{\mathfrak{A}} = \{\langle \delta^*, \bar{e}^*, \bar{x}^* \rangle : (\exists \tau \supseteq \delta)(\tau \Vdash_0 F_e(x))\}.$$

Soskov and A. Soskova define the jump only for total structure $\mathfrak{A}^+$. In light of Theorem 4 there is a natural way to extend this definition to non-total structures.

Definition 14. *The jump of the structure* $\mathfrak{A}^+$ *is the structure*

$$\mathfrak{A}' = ((\mathfrak{A}^+)^*, K_{\mathfrak{A}}, A^* \setminus K_{\mathfrak{A}}).$$

Note, that the structure $\mathfrak{A}'$ is also total. The next property can be viewed as a correctness statement: it reaffirms that this definition of the jump of a structure is truly an analog of the jump operator on sets of natural numbers. The main technique used in its proof is once again that of generic enumerations.

Theorem 17. [88, 90][2] *For every countable structure* $\mathfrak{A}$*,* $DS_1(\mathfrak{A}^+) = DS(\mathfrak{A}')$*.*

Another proof of this theorem was published independently by Montalbán [50]. Montalbán called it in [51] the second jump inversion theorem. Both proofs are essentially the same, even though the great differences in the underlying setting make them look quite different.

Vatev [98, 99] extends the jump of a structure to the α-th jump of a structure for arbitrary computable ordinal α. Vatev's approach relies on the notion of *conservative extension*. This notion provides a finer way to compare the relative definability between two structures at arbitrary levels of the Σ^c_α-hierarchy. Given two countable structures $\mathfrak{A}$ and $\mathfrak{B}$ with $|\mathfrak{A}| \subseteq |\mathfrak{B}|$ and α, β computable ordinals the structure $\mathfrak{B}$ is an (α, β) conservative extension of $\mathfrak{A}$ if for every enumeration g of $\mathfrak{B}$ there is an enumeration f of $\mathfrak{A}$ such that $\{\langle x, y\rangle \mid f(x) = g(y)\}$ is Σ^0_β in $g^{-1}(\mathfrak{B})$ and $f^{-1}(\mathfrak{A})^{(\alpha)} \leq_T g^{-1}(\mathfrak{B})^{(\beta)}$, and the opposite, for every enumeration f of $\mathfrak{A}$ there is an enumeration g of $\mathfrak{B}$ such that $\{\langle x, y\rangle \mid f(x) = g(y)\}$ is Σ^0_α in $f^{-1}(\mathfrak{A})$ and $g^{-1}(\mathfrak{B})^{(\beta)} \leq_T f^{-1}(\mathfrak{A})^{(\alpha)}$. He proved that if $\mathfrak{B}$ is an (α, β) conservative extension of $\mathfrak{A}$ then $(\forall X \subseteq |\mathfrak{A}|)(X \in \Sigma^c_\alpha(\mathfrak{A}) \iff X \in \Sigma^c_\beta(\mathfrak{B}))$. He showed furthermore that $\mathfrak{A}^{(\alpha+1)}$ is $(\beta+1, \beta)$ conservative extension of $\mathfrak{A}^{(\alpha)}$ and from here it follows that the $\Sigma^c_{\alpha+1}$ definable in $\mathfrak{A}^*$ subsets of A^* are exactly the Σ^c_α definable sets in $\mathfrak{A}'$. More generally, he shows that for any computable ordinals α, β the $\Sigma^c_{\beta+1}$ definable sets in $\mathfrak{A}^{(\alpha)}$ are exactly the Σ^c_β definable sets in $\mathfrak{A}^{(\alpha+1)}$.

Naturally, once we have a jump of a structure, the question of jump inversion arises: Given a structure $\mathfrak{B}$ with $DS(\mathfrak{B})$ consisting of total degree above $\mathbf{0}'_e$, is there a structure $\mathfrak{C}$ such that $DS_1(\mathfrak{C}) = DS(\mathfrak{B})$. Soskova and Soskov prove an even more general statement. For every structure $\mathfrak{B}$, denote by $DS_t(\mathfrak{B})$ the set of total elements in $DS(\mathfrak{B})$. (In particular, if $\mathfrak{B}$ is total then $DS(\mathfrak{B}) = DS_t(\mathfrak{B})$.)

Theorem 18. [87, 88, 90] *Let* $\mathfrak{A}$ *and* $\mathfrak{B}$ *be structures such that* $DS(\mathfrak{B})_t \subseteq DS_n(\mathfrak{A})$*. Then there exists a structure* $\mathfrak{C}$ *such that* $DS(\mathfrak{C}) \subseteq DS(\mathfrak{A})$ *and* $DS_n(\mathfrak{C}) = DS(\mathfrak{B})_t$*.*

The proof of Theorem 18 uses the method of Marker extensions, which will be discussed in detail in Section 7. This method is also used by Stukachev [93, 94] for similar jump inversion theorem for his notion of the jump of a structure based on Σ-definability. Downey and Knight [14] prove, using a fairly complicated construction, that for every computable ordinal α there exists a structure $\mathfrak{A}$ (a linear ordering, in fact) such that $\mathfrak{A}$ has α-th jump degree equal to $\mathbf{0}^{(\alpha)}$, but no β-th jump degree for any $\beta < \alpha$. Now we can obtain this theorem for the finite ordinals as an application of Theorem 18. Consider a structure $\mathfrak{B}$ such that $DS(\mathfrak{B})$ consists of total elements above $\mathbf{0}^{(n)}_e$ and has no least element, and such that $\mathbf{0}^{(n+1)}_e$ is the least element of $DS_1(\mathfrak{B})$. Let $\mathfrak{A}$ be any total computable structure. Clearly $DS(\mathfrak{B}) \subseteq DS_n(\mathfrak{A})$. By Theorem 18 there exists a structure $\mathfrak{C}$ such that $DS_n(\mathfrak{C}) = DS(\mathfrak{B})$. Therefore $\mathfrak{C}$

[2]Theorem 17 was first announced by Soskov during his LC talk in Münster in 2002.

does not have a n-th jump degree and so no k-th jump degree for $k \leq n$. On the other hand $DS_{n+1}(\mathfrak{C}) = DS_1(\mathfrak{B})$ and hence the $(n+1)$-th jump degree of $\mathfrak{C}$ is $\mathbf{0}_e^{(n+1)}$. Why does such a structure $\mathfrak{B}$ exist? Consider a degree $\mathbf{q}$ that is quasi-minimal relative to $\mathbf{0}_e^{(n)}$ and with $\mathbf{q}' = \mathbf{0}_e^{(n+1)}$. Let $\mathfrak{B} = G$ be the torsion free abelian group of rank 1 such that $\mathbf{s}_G = \mathbf{q}$. Recall that $DS(G) = \{\mathbf{a} \mid \mathbf{s}_G \leq_e \mathbf{a} \text{ and } \mathbf{a} \text{ is total}\}$ and the first jump degree of G is $\mathbf{s}'_G$.

The next natural questions is if one can extend the jump inversion theorem to every constructive ordinal α. Goncharov, Harizanov, Knight, McCoy, Miller and Solomon [32] show that this is true if α is a computable successor ordinal, even though they do not state their result in terms of the jump of a structure. This result was useful later on, for instance Greenberg, Montalbán and Slaman [33] use it to build a structure whose spectrum consists of the non-hyperarithmetic degrees. Vatev [99, 98, 100] proves the α-jump inversion theorem for a computable successor ordinal α based on the construction in [32].

The problem of jump inversion for $\alpha = \omega$, or, in general, any computable limit ordinal remains open for longer. In one of his last papers Soskov [80] finally proves that there is a good reason for that.

Theorem 19. [80] *There is a total structure $\mathfrak{A}$ with $DS(\mathfrak{A}) \subseteq \{\mathbf{b} \mid \mathbf{0}_e^{(\omega)} \leq \mathbf{b}\}$ for which there is no structure $\mathfrak{M}$ with $DS_\omega(\mathfrak{M}) = DS(\mathfrak{A})$.*

The proof relies on an analysis of the ω-jump co-spectrum of a structure. Soskov shows that every member of $\mathbf{a} \in CS_\omega(\mathfrak{M})$ is bounded by a total $\mathbf{b}$, which is also a member of $CS_\omega(\mathfrak{M})$. To see this, let $R \in \mathbf{a}$ and $\mathbf{a} \in CS_\omega(\mathfrak{M})$. It follows from Theorem 7 that the set R is Σ^c_ω definable in $\mathfrak{M}$ and hence there is a computable function γ and parameters $t_1, \ldots, t_m$ of $|\mathfrak{M}|$ such that $x \in R \iff \mathfrak{M} \models F_{\gamma(x)}(t_1, \ldots, t_m)$. Each $F_{\gamma(x)}$ is a computable Σ^c_ω formula, i.e. a c.e. disjunction of computable Σ^c_{n+1} formulas, where $n < \omega$, and so there is a computable function $\delta(n, x)$ such that for all n and x, $\delta(n, x)$ yields a code of some computable Σ^c_{n+1} formula $F_{\delta(n,x)}$ and $x \in R \iff (\exists n)(\mathfrak{M} \models F_{\delta(n,x)}(t_1, \ldots, t_m))$.

Let $R_n = \{x \mid x \in \mathbb{N} \wedge \mathfrak{M} \models F_{\delta(n,x)}(t_1, \ldots, t_n)\}$ and let $\mathbf{b} = d_e(\mathcal{P}_{<\omega}(\{R_n\}))$. Note that $\mathbf{b}$ is a total enumeration degree. It is easy to see that for every enumeration f of $\mathfrak{M}$ we have that $\{R_n\} \leq_e \{f^{-1}(\mathfrak{M})^{(n)}\}$ uniformly in n. It follows that $\mathcal{P}(\{R_n\}) \leq_e \{f^{-1}(\mathfrak{M})^{(n)}\}$ and so $\mathcal{P}_{<\omega}(\{R_n\}) \leq_e f^{-1}(\mathfrak{M})^{(\omega)}$, i.e. $\mathbf{b} \in CS_\omega(\mathfrak{M})$. On the other hand $x \in R \iff (\exists n)(x \in R_n)$ and so $R \leq_e \bigoplus_n R_n \leq_e \mathcal{P}_{<\omega}(\{R_n\})$. Thus $\mathbf{a} \leq_e \mathbf{b}$

To complete the proof of Theorem 19, let $\mathfrak{A}$ be a total structure with co-spectrum $CS(\mathfrak{A}) = \{\mathbf{a} \mid \mathbf{a} \leq_e \mathbf{y}\}$, where $\mathbf{y}$ is some quasi-minimal above $\mathbf{0}_e^{(\omega)}$ degree. We have already seen that such an $\mathfrak{A}$ exists, as every principal ideal is the co-spectrum of a total structure. Then $DS(\mathfrak{A}) \subseteq \{\mathbf{a} \mid \mathbf{0}_e^{(\omega)} \leq_e \mathbf{a}\}$, but $DS(\mathfrak{A})$ cannot be the ω-jump spectrum of any structure $\mathfrak{M}$. If we assume otherwise then $CS_\omega(\mathfrak{M}) = CS(\mathfrak{A})$ and so $\mathbf{y}$ must be bounded by a total enumeration degree $\mathbf{b} \in CS(\mathfrak{A})$. Since $\mathbf{y}$ is the greatest element of $CS(\mathfrak{A})$, $\mathbf{b} = \mathbf{y}$ contradicting the choice of $\mathbf{y}$.

7. Generalized Marker extensions for sequences of structures

The last paper by Soskov [81] settles a series of questions relating to the connections between Turing degree spectra, enumeration degree spectra and spectra of sequences of structures. The main technique is that of Marker extensions. Marker's method [47] is originally used in model theory. The computable content of this construction is established in the work of Goncharov and Khoussainov [31]. Soskov gives a more general version of this approach.

We introduce Soskov's ideas with a simple example. Consider a countable structure $\mathfrak{A}$. A set $Y \subseteq |\mathfrak{A}|$ is relatively intrinsically c.e. in $\mathfrak{A}$ if for every enumeration f of $\mathfrak{A}$ we have that $f^{-1}(Y)$ is c.e. in $f^{-1}(\mathfrak{A})$, or equivalently if Y is definable by some computable infinitary Σ_1^c formula. In this definition $\mathfrak{A}$ is treated as a total object, in particular $f^{-1}(\mathfrak{A})$ is treated as a total oracle. Alternatively, we can consider sets $Y \subseteq |\mathfrak{A}|$, such that Y is (relatively intrinsically) enumeration reducible to $\mathfrak{A}$, i.e. for every enumeration f of $\mathfrak{A}$ we have that $f^{-1}(Y) \leq_e f^{-1}(\mathfrak{A})$, or equivalently if Y is definable by some positive computable infinitary Σ_1^+ formula. In the second case $f^{-1}(\mathfrak{A})$ is treated as a partial oracle. These two notions are in general different, but to what extent? Are there classes of sets that can be characterized as the ones that are enumeration reducible to a fixed structure, but cannot be characterized as the sets that are relatively intrinsically c.e. in any structure. If we move away from computable structure theory and view the analogous question simply in terms of the relations $\leq_e$ and "c.e. in" the question becomes: is it true that for every set A there is a set M such that $\{Y \mid Y \leq_e A\} = \{Y \mid Y \text{ is c.e in M}\}$? The answer to this last question is clearly "no", as there are sets A that are not enumeration equivalent to any total set. So are there truly partial structures in this same sense? Soskov [81] reveals that surprisingly computable structure theory differs from degree theory in this respect: for every structure $\mathfrak{A}$, there is a structure $\mathfrak{M}$, such that for every $Y \subseteq |\mathfrak{A}|$, $Y \leq_e \mathfrak{A}$ if and only if Y is c.e. in $\mathfrak{M}$.

For simplicity let $\mathfrak{A} = (A; R)$ and $R \subseteq A$ is infinite. The 0-th Marker extension $\mathfrak{M}$ of $\mathfrak{A}$ is constructed as follows. Consider an infinite countable set X disjoint from A and a bijection $h : R \to X$. Let $M(a, x)$ be true if and only if $h(a) = x$. Let $\mathfrak{M} = (A \cup X; A, X, M)$, where A and X are unary predicates. Note that R is Σ_1^0 definable in $\mathfrak{M}$ since $R(a) \Leftrightarrow (\exists x \in X)M(a, x)$. Now consider any set $Y \subseteq A$ such that $Y \leq_e \mathfrak{A}$. It is straightforward to check that for every enumeration f of $\mathfrak{M}$ $f^{-1}(Y)$ is c.e. in $f^{-1}(\mathfrak{M})$: Indeed, using a computable in $f^{-1}(\mathfrak{M})$ bijection from $\mathbb{N}$ to $f^{-1}(A)$ we can transform f into an enumeration g of the structure $\mathfrak{A}$. Now we have that $g^{-1}(Y) \leq_e g^{-1}(\mathfrak{A}) = g^{-1}(R)$, and $f^{-1}(R)$ is c.e. in $f^{-1}(\mathfrak{M})$. Since we can pass between f and g using oracle $f^{-1}(\mathfrak{M})$ it follows that $f^{-1}(Y)$ is c.e. in $f^{-1}(\mathfrak{M})$.

For the reverse direction, we show that if $Y \not\leq_e \mathfrak{A}$ then Y is not relatively intrinsically c.e. in $\mathfrak{M}$, i.e. that there is an enumeration f of $\mathfrak{M}$ such that $f^{-1}(Y)$ is not c.e. in $f^{-1}(\mathfrak{M})$. Let g be an enumeration of $\mathfrak{A}$ such that $g^{-1}(Y) \not\leq_e g^{-1}(\mathfrak{A})$. We construct f so that $f(2n) = g(n)$. To fill in $f(2\mathbb{N})$ we construct a bijection $k : f^{-1}(R) \to 2\mathbb{N} + 1$ and complete f by $f(2n + 1) = h(f(k^{-1}(2n + 1)))$. Note that then we will have $f^{-1}(\mathfrak{M}) \equiv_e f^{-1}(M) \equiv_e G_k$ and $f^{-1}(Y) \equiv_e g^{-1}(Y)$. We construct k using forcing so that statements of the form $x \in \Gamma_e(G_k^+)$ are decided at finite stages. For $\sigma : f^{-1}(R) \to 2\mathbb{N} + 1$ we say that $\sigma \Vdash x \in \Gamma_e(G_k^+)$ if there exists v, such that $\langle x, v\rangle \in \Gamma_e$ and for every $u \in D_v$ we have $u = 2\langle a, x\rangle$ and $\sigma(a) = x$ or $u = 2\langle a, x\rangle + 1$ and $\sigma(b) = x$ for some $b \neq a$. Then the set $\{x \mid \exists\sigma \supseteq \tau(\sigma \Vdash x \in \Gamma_e(G_k^+))\}$ is enumeration reducible to $g^{-1}(\mathfrak{A})$. We use this to ensure that $g^{-1}(Y) \neq \Gamma_e(G_k^+)$ and thus Y is not c.e. in $\mathfrak{M}$.

Let $\vec{\mathfrak{A}} = \{\mathfrak{A}_n\}_{n<\omega}$ be a sequence of structures, where $\mathfrak{A}_n = (A_n; R_1^n, R_2^n, \dots R_{m_n}^n)$. An enumeration f of $\vec{\mathfrak{A}}$ is a bijection from $\mathbb{N} \to A = \bigcup_n A_n$. For every n let

$f^{-1}(\mathfrak{A}_n) = f^{-1}(A_n) \oplus f^{-1}(R_1^n) \cdots \oplus f^{-1}(R_{m_n}^n)$ and let $f^{-1}(\vec{\mathfrak{A}})$ be the sequence $\{f^{-1}(\mathfrak{A}_n)\}_{n<\omega}$.

In this setting we can talk about a sequence of sets that is *relatively intrinsically ω-enumeration* reducible to $\vec{\mathfrak{A}}$: a sequence $\{Y_n\}_{n<\omega}$ of subsets of A, such that for every enumeration f of $\vec{\mathfrak{A}}$, $\{f^{-1}(Y_n)\} \leq_\omega f^{-1}(\vec{\mathfrak{A}})$. Soskov and Baleva [76] and A. Soskova [89] show that sequence of this kind also have a syntactic characterization: Y_n is uniformly in n definable by a positive computable infinitary Σ^+_{n+1} formula with predicates only from the first n structures, such that the predicates for the n-th appear for the first time at level $n+1$ positively. We can compare this notion to the following: say that a sequence $\{Y_n\}_{n<\omega}$ of subsets of A is relatively intrinsically c.e. in a structure $\mathfrak{M}$ with $A \subseteq |\mathfrak{M}|$ if for every enumeration f of $\mathfrak{M}$ the set $f^{-1}(Y_n)$ is $\Sigma^0_{n+1}(f^{-1}(\mathfrak{M}))$ uniformly in n.

The key idea is to generalize Marker extensions to the sequence $\vec{\mathfrak{A}}$. First we must define the n-th Marker extension of a predicate. Let $\mathfrak{A} = (A; R_1, \dots, R_k)$ and $R \subseteq A^m$. *The n-th Marker extension of R* is a structure $\mathfrak{M}_n(R)$ defined as follows. Consider new infinite disjoint countable sets $X_0, X_1, \dots X_n$ called *companions.*
Fix bijections: $h_0 : R \to X_0$
$h_1 : (A^m \times X_0) \setminus G_{h_0} \to X_1$
...
$h_n : (A^m \times X_0 \times X_1 \cdots \times X_{n-1}) \setminus G_{h_{n-1}} \to X_n$.

Let $M_n = G_{h_n}$ and $\mathfrak{M}_n(R) = (A \cup X_0 \cup \cdots \cup X_n; X_0, X_1, \dots X_n, M_n)$. Notice, that R is Σ^0_{n+1} definable in $\mathfrak{M}_n(R)$ since for $\bar{a} \in A^m$ we have

$$R(\bar{a}) \iff (\exists x_0 \in X_0)(G_{h_0}(\bar{a}, x_0))$$

and for all $k < n, x_0 \in X_0, \dots, x_k \in X_k$ we have

$$G_{h_k}(\bar{a}, x_0, \dots, x_k) \iff (\forall x_{k+1} \in X_{k+1}) \neg G_{h_{k+1}}(\bar{a}, x_0, \dots, x_k, x_{k+1}).$$

Next we define $\mathfrak{M}(\vec{\mathfrak{A}})$ for the sequence of structures $\vec{\mathfrak{A}} = \{\mathfrak{A}_n\}_{n<\omega}$.

(1) For every n construct the n-th Marker extensions of A_n, R_1^n, $\dots R_{m_n}^n$ with disjoint companions.
(2) For every n let $\mathfrak{M}_n(\mathfrak{A}_n) = \mathfrak{M}_n(A_n) \cup \mathfrak{M}_n(R_1^n) \cup \cdots \cup \mathfrak{M}_n(R_{m_n}^n)$.
(3) Set $\mathfrak{M}(\vec{\mathfrak{A}})$ to be $(\bigcup_n \mathfrak{M}_n(\mathfrak{A}_n))^+$ with additional predicate for $A = \bigcup_n A_n$ and $\overline{A}$.

Note that $\mathfrak{M}(\vec{\mathfrak{A}})$ is a total structure.

Soskov [81] describes the relationship between the enumerations of $\vec{\mathfrak{A}}$ and $\mathfrak{M} = \mathfrak{M}(\vec{\mathfrak{A}})$: It is not too difficult to see that for every enumeration f of $\mathfrak{M}(\vec{\mathfrak{A}})$ there is an enumeration g of $\vec{\mathfrak{A}}$ such that:

(1) the set $\{\langle x, y\rangle \mid f(i) = g(j)\}$ is computable in $f^{-1}(\mathfrak{M})$.
(2) $\mathcal{P}_n(g^{-1}(\vec{\mathfrak{A}})) \leq_e (f^{-1}(\mathfrak{M}))^{(n)}$ uniformly in n.
(3) $\mathcal{P}_{<\omega}(g^{-1}(\vec{\mathfrak{A}})) \leq_T (f^{-1}(\mathfrak{M}))^{(\omega)}$.

The reverse relationship requires an elaborate forcing construction: For every enumeration g of $\vec{\mathfrak{A}}$ and $\mathcal{Y} \not\leq_\omega g^{-1}(\vec{\mathfrak{A}})$ there is an enumeration f of $\mathfrak{M}$:

(1) the set $\{\langle x, y\rangle \mid f(i) = g(j)\}$ is computable.
(2) $\mathcal{P}_{<\omega}(g^{-1}(\vec{\mathfrak{A}})) \equiv_e (f^{-1}(\mathfrak{M}))^{(\omega)}$.
(3) $\mathcal{Y}$ is not c.e. in $f^{-1}(\mathfrak{M})$.

Our simple example is transformed to the following general theorem:

Theorem 20. [81] *A sequence $\mathcal{Y}$ of subsets of A is relatively intrinsically ω-enumeration reducible to $\vec{\mathfrak{A}}$ if and only if $\mathcal{Y}$ is relatively intrinsically c.e. in $\mathfrak{M}(\vec{\mathfrak{A}})$.*

The structure $\mathfrak{M} = \mathfrak{M}(\vec{\mathfrak{A}})$ has very interesting properties. The first one considered in [81] is a characterization its co-spectrum.

Theorem 21. (1) *The n-th co-spectrum of $\mathfrak{M}$ is*

$$CS_n(\mathfrak{M}) = \{d_e(Y) \mid \text{ for every enumeration } g \text{ of } A,\ Y \leq_e \mathcal{P}_n(g^{-1}(\vec{\mathfrak{A}}))\}.$$

(2) *The ω-co-spectrum of $\mathfrak{M}$ is*

$$Ocsp(\mathfrak{M}) = \{d_\omega(\mathcal{Y}) \mid \text{ for every enumeration } g \text{ of } A,\ \mathcal{Y} \leq_\omega g^{-1}(\vec{\mathfrak{A}})\}.$$

Theorem 21 allows us to construct examples of structures with interesting properties in an easy way. Let $\mathcal{R} = \{R_n\}$ be a sequence of sets. Consider the sequence $\vec{\mathfrak{A}}_\mathcal{R}$, where $\mathfrak{A}_0 = (\mathbb{N}; G_S, R_0)$, here G_S is the graph of the successor function, and for all $n \geq 1$, $\mathfrak{A}_n = (\mathbb{N}; R_n)$. Then it is not too hard to see that for every n we have that $CS_n(\mathfrak{M}(\vec{\mathfrak{A}}_\mathcal{R})) = \{d_e(Y) \mid Y \leq_e \mathcal{P}_n(\mathcal{R})\}$ and for each enumeration g of $\mathbb{N}$ $\mathcal{R} \leq_\omega g^{-1}(\vec{\mathfrak{A}}_\mathcal{R})$.

When one takes $\mathcal{R}$ to be an almost zero sequence, we obtain a structure $\mathfrak{M}(\vec{\mathfrak{A}}_\mathcal{R})$ with n-th co-degree $\mathbf{0}_e^{(n)}$, but no n-th jump degree for any n. Indeed, recall that an almost zero sequence $\mathcal{R}$ is one that is not ω-enumeration reducible to $\mathbf{0}_\omega$, but has the property that $\mathcal{P}_n(\mathcal{R}) \equiv_e \emptyset^{(n)}$ for every n. If we assume that the n-th jump degree of $\mathfrak{M} = \mathfrak{M}(\vec{\mathfrak{A}}_\mathcal{R})$ exists, then it must be $\mathbf{0}_e^{(n)}$, so there is an enumeration f of $\mathfrak{M}$ such that $(f^{-1}(\mathfrak{M}))^{(n)} \equiv_e \emptyset^{(n)}$. However this would mean that there is an enumeration g of $\mathbb{N}$ such that for all $k \geq n$, $\mathcal{P}_k(\mathcal{R}) \leq_e \mathcal{P}_k(g^{-1}(\vec{\mathfrak{A}}_\mathcal{R})) \leq_e (f^{-1}(\mathfrak{M}))^{(k)}$ uniformly in k, and for $k \leq n$, $\mathcal{P}_k(\mathcal{R}) \leq_e \emptyset^{(n)}$, contradicting the fact that $d_\omega(\mathcal{R}) \not\leq_\omega \mathbf{0}_\omega$.

Next Soskov [81] turns to investigate the properties of the spectra of Marker extensions. There are two ways in which one can define the spectrum of a sequence of structures. The first one is to treat $\vec{\mathfrak{A}}$ within an underlying structure with domain $\bigcup_n A_n$ and consider enumerations f of A and the sequences $\{f^{-1}(\mathfrak{A}_n)\}_{n<\omega}$. The other possibility is to consider different enumerations: f_n an enumeration of $\mathfrak{A}_n$ for every n, giving rise to a sequence $\{f_n^{-1}(\mathfrak{A}_n)\}$. We can then collect all ω-enumeration degrees of such sequence as a measure of complexity, or better yet, collect all Turing degrees (or total enumeration degrees) in the jump class of one such sequence. For a set C let E_C denote all enumerations of the set C. The *relative spectrum* of a sequence $\vec{\mathfrak{A}}$ is the set

$$RS(\vec{\mathfrak{A}}) = \{d_T(B) \mid \exists g \in E_A(g^{-1}(\mathfrak{A}_n) \in \Sigma^0_{n+1}(B) \text{ uniformly in } n)\}.$$

The *joint spectrum* of the sequence $\vec{\mathfrak{A}}$ is the set

$$JS(\vec{\mathfrak{A}}) = \{d_T(B) \mid \exists \{g_n\}_{n<\omega}(g_n \in E_{A_n} \ \&\ g_n^{-1}(\mathfrak{A}_n) \in \Sigma^0_{n+1}(B) \text{ uniformly in } n)\}.$$

Note that in general $RS(\vec{\mathfrak{A}}) \neq JS(\vec{\mathfrak{A}})$. For example, for the sequence of structures $\vec{\mathfrak{A}}_\mathcal{R}$ obtained from an almost zero sequence $\mathcal{R}$ where $\mathfrak{A}_0 = (\mathbb{N}; G_S, R_0)$ and for all $n \geq 1$, $\mathfrak{A}_n = (\mathbb{N}; R_n)$ we have that $\mathbf{0}_T \in JS(\vec{\mathfrak{A}}_\mathcal{R}) \setminus RS(\vec{\mathfrak{A}}_\mathcal{R})$. However, if the structures in the sequence $\vec{\mathfrak{A}}$ have disjoint domains then the notions coincide.

These two notions can be seen as generalizations of ω-spectra and of joint spectra and relative spectra for finitely many structures. Recall that when these notions were investigated the main unanswered question was wether or not they give rise to new sets of degrees, or if the basic notion of degree spectrum already captures these sets. The next theorem unravels this mystery.

Theorem 22 (Soskov [81])**.** *Let $\vec{\mathfrak{A}} = \{\mathfrak{A}_n\}_{n<\omega}$ be a sequence of structures.*

(1) *There exists a structure $\mathfrak{M}$ such that $DS_T(\mathfrak{M}) = RS(\vec{\mathfrak{A}})$.*
(2) *There exists a structure $\mathfrak{M}$ such that $DS_T(\mathfrak{M}) = JS(\vec{\mathfrak{A}})$.*

The proof of this theorem relies on a generalization of a result by Goncharov and Khoussainov [31].

Lemma 1. *Let R be a Σ^0_{n+1} set of natural numbers possessing an infinite computable subset S. Then there exist functions $\kappa_0, \dots, \kappa_n$ such that the graph of κ_n is computable and κ_0 is a bijection of R onto $\mathbb{N}$; κ_1 is a bijection of $\mathbb{N}^2 \setminus G_{\kappa_0}$ onto $\mathbb{N}$; $\dots \kappa_n$ is a bijection of $\mathbb{N}^{n+1} \setminus G_{\kappa_{n-1}}$ onto $\mathbb{N}$.*

Theorem 4 is a special case of Theorem 22 applied to the sequence $\vec{\mathfrak{A}}$ where $\mathfrak{A}_0 = \mathfrak{A}$ and for every $n > 0$ we have the trivial structure $\mathfrak{A}_n = (A; =)$. To illustrate the main idea consider once again the example that we gave at the beginning of this section. We had a structure $\mathfrak{A} = (A; R)$ for which we built the Marker extension $\mathfrak{M} = (A \cup X; X, A, M)$. Assume that R is infinite, (if not we can instead use the Marker extension of the structure obtained by adding one more element $\perp$ to the domain of A and replace R by a $R_\perp = \{(m,n) \mid R(m) \vee n = \perp\}$). We showed that if f is any enumeration of $\mathfrak{M}$ then we can build an enumeration g of $\mathfrak{A}$, such that $g^{-1}(\mathfrak{A}) \leq_e f^{-1}(\mathfrak{M})^+$. Fix an enumeration g of $\mathfrak{A}$ and a total set Y such that $g^{-1}(\mathfrak{A}) \leq_e Y$. We can use the same trick as before: We construct f so that $f(2n) = g(n)$. To fill in $f(2\mathbb{N})$ we construct a bijection $k : f^{-1}(R) \to 2\mathbb{N}+1$ and complete f by $f(2n+1) = h(f(k^{-1}(2n+1)))$. Then $f^{-1}(\mathfrak{M}) \equiv_e f^{-1}(M) \equiv_e G_k$. To construct G_k we use Lemma 1 relativized to Y. It follows that $DS(\mathfrak{M}^+) = \{\mathbf{y} \mid \mathbf{y}$ is total and $\mathbf{y} \geq \mathbf{x}$ for some $\mathbf{x} \in DS(\mathfrak{A})\}$.

Soskov gives several further applications of Theorem 22. He shows that the ω-enumeration degrees can be embedded into the Muchnick degrees generated by spectra of structures. To see this consider again the sequence $\vec{\mathfrak{A}}_{\mathcal{R}}$ obtained from a sequence of sets $\mathcal{R}$. Recall that for every enumeration g of $\vec{\mathfrak{A}}_{\mathcal{R}}$, we have that $\mathcal{R} \leq_\omega g^{-1}(\vec{\mathfrak{A}}_{\mathcal{R}})$. It follows that $RS(\vec{\mathfrak{A}}_{\mathcal{R}})$ is exactly the jump class of the sequence $\mathcal{R}$ and hence $DS_T(\mathfrak{M}(\vec{\mathfrak{A}}_{\mathcal{R}})) = \{d_T(B) \mid \mathcal{R} \text{ is c.e. in } B\}$. This induces the desired embedding as by definition we have that $\mathcal{R} \leq_\omega \mathcal{Q}$ if and only if $\{d_T(B) \mid \mathcal{R} \text{ is c.e. in } B\} \supseteq \{d_T(B) \mid \mathcal{Q} \text{ is c.e. in } B\}$ and this is true if and only if $DS_T(\mathfrak{M}(\vec{\mathfrak{A}}_{\mathcal{R}})) \supseteq DS_T(\mathfrak{M}(\vec{\mathfrak{A}}_{\mathcal{Q}}))$.

As a final application of these results we show how to build a structure $\mathfrak{M}$ whose spectrum consists of all Turing degrees, which are non-low$_n$ for every n. The previously known related examples are given by Kalimullin [38], who constructs for each low degree $\mathbf{b}$ a structure $\mathfrak{A}$ with $DS_T(\mathfrak{A}) = \{\mathbf{x} \mid \mathbf{x} \not\leq_T \mathbf{b}\}$ and by Goncharov et al., [32] who construct for every n a structure with spectrum consisting of all non-low$_n$ Turing degrees.

The construction relies on Wehner's [101] technique. Let $\mathcal{F}$ be a countable family of sets of natural numbers. An enumeration of $\mathcal{F}$ is a set $U \subseteq \mathbb{N}^2$ such that:

(1) For every a, the set $\{n \mid (a,n) \in U\} \in \mathcal{F}$.
(2) For every $F \in \mathcal{F}$ there is an a such that $\{n \mid (a,n) \in U\} = F$.

Let $\mathfrak{A}_{\mathcal{F}} = (A; S, Z, I)$ where $A = \mathcal{F} \times \mathbb{N}^2$; $Z = \{(F, x, 0) \mid F \in \mathcal{F}, x \in \mathbb{N}\}$, $S = \{((F, x, n), (F, x, n+1)) \mid F \in \mathcal{F}, x, n \in \mathbb{N}\}$ and $I = \{(F, x, n) \mid n \in F\}$. Wehner shows that there is a uniform way to compute an enumeration of $\mathcal{F}$ in any isomorphic copy $\mathcal{B}$ of $\mathfrak{A}_{\mathcal{F}}$ and that there is a uniform way to compute an isomorphic copy $\mathcal{B}$ of $\mathfrak{A}_{\mathcal{F}}$ in any enumeration of $\mathcal{F}$. Consider the relativized version of Wehner's family: $\mathcal{F}^X = \{\{n\} \oplus F \mid F$ is finite and $F \neq W_n^X\}$ for $X \subseteq \mathbb{N}$. No enumeration of $\mathcal{F}^X$ is c.e. in X. Furthermore, if $B \not\leq_T X$ then one can compute uniformly in B and X an enumeration of $\mathcal{F}^X$.

Finally, let $\vec{\mathfrak{A}}$ be the sequence of structures where $\mathfrak{A}_n = \mathfrak{A}_{\mathcal{F}^{\emptyset^{(n)}}}$. Let $\mathfrak{M}$ be such that $DS_T(\mathfrak{M}) = JS(\vec{\mathfrak{A}})$. If $d_T(B) \in DS_T(\mathfrak{M})$ then $B^{(n)}$ computes an enumeration of $\mathcal{F}^{\emptyset^{(n)}}$ and hence $B^{(n)} \not\leq_T \emptyset^{(n)}$. If $B^{(n)} \not\leq_T \emptyset^{(n)}$ for every n then as $\emptyset^{(n)} \leq_T B^{(n)}$ uniformly in n, it follows that $B^{(n)}$ computes an enumeration of $\mathcal{F}^{\emptyset^{(n)}}$.

Theorem 23 (Soskov [81])**.** *There is a structure $\mathfrak{M}$ with*

$$DS_T(\mathfrak{M}) = \{\mathbf{b} \mid \forall n(\mathbf{b}^{(n)} \not\leq \mathbf{0}_T^{(n)})\}.$$

The untimely death of Ivan Soskov left this area not fully explored. We hope that with this exposition, we will attract the interest of researchers who will join us in developing this line of investigation further.

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